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Spectra and energies of iterated line graphs of regular graphs

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Abstract

If G is a graph and $L(G) = L^1(G)$ its line graph, then $L^k(G)$, $k = 2, 3, \dots$, defined recursively via $L^k(G) = L(L^{k-1}(G))$, are the iterated line graphs of G. If G is a regular graph of degree r, $r \geq 3$, then all negative eigenvalues of its iterated line graphs are equal to minus 2. The energy $E(G)$ of a graph G is the sum of absolute values of the eigenvalues of G. If G is a regular graph of order n and of degree $r \geq 3$, then for each $k \geq 2$, $E(L^k(G))$ depends solely on n and r. In particular, $E(L^2(G)) = 2nr(r-2)$. This result enables a systematic construction of pairs of non-cospectral connected graphs of the same order, having equal energies.

Keywords: Spectral graph theory; Line graphs; Iterated line graphs; Regular graphs; Eigenvalues (of graphs); Energy (of graphs)

1. Introduction

Let G be a graph of order n. The eigenvalues of the adjacency matrix of G, denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$, are said to be the eigenvalues of G, and to form the spectrum of G [1].

The line graph of G will be denoted by $L(G)$; the basic properties of line graphs are found in textbooks, e.g., in [2]. The iterated line graphs of G are then defined recursively as $L^2(G) = L(L(G)), L^3(G) = L(L^2(G)), \ldots, L^k(G) = L(L^{k-1}(G)), \ldots$. It is consistent to set $L(G) \equiv L^1(G)$ and $G \equiv L^0(G)$. The basic properties of iterated line graph sequences are summarized in the articles [3,4].

Of the known results on line graphs and iterated line graphs we need the following two.

1° The line graph of a regular graph is a regular graph. In particular, the line graph of a regular graph G of order n_0 and of degree r_0 is a regular graph of order $n_1 = \frac{1}{2}r_0 n_0$ and of degree $r_1 = 2r_0 - 2$. Consequently, the order and degree of $L^k(G)$ are [3,4]:

$$n_k = \frac{1}{2} r_{k-1} n_{k-1} \qquad \text{and} \qquad r_k = 2r_{k-1} - 2$$

where n_{k-1} and r_{k-1} stand for the order and degree of $L^{k-1}(G)$. Therefore,

$$r_k = 2^k r_0 - 2^{k+1} + 2 \tag{1}$$

and

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2). \tag{2}$$

2° If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a regular graph G of order n and of degree r, then the eigenvalues of $L(G)$ are

$$\left.\begin{array}{rll} \lambda_i + r - 2 & i = 1, 2, \ldots, n, & \text{and} \\ -2 & n(r-2)/2 \text{ times} & \end{array}\right\}. \tag{3}$$

Formula (3) was first reported by Sachs [5]. In view of the fact that $L(G)$ is also a regular graph, from (3) we immediately obtain that the eigenvalues of $L^2(G)$ are

$$\left.\begin{array}{rll} \lambda_i + 3r - 6 & i = 1, 2, \ldots, n, & \text{and} \\ 2r - 6 & n(r-2)/2 \text{ times,} & \text{and} \\ -2 & nr(r-2)/2 \text{ times} & \end{array}\right\}. \tag{4}$$

Theorem 1. *If G is a regular graph of order n and of degree $r \geq 3$, then $L^2(G)$ has exactly $nr(r-2)/2$ negative eigenvalues, all being equal to -2.*

Proof. All eigenvalues of a regular graph of degree r belong to the interval $[-r, +r]$. Therefore, if $r \geq 3$, then $\lambda_i + 3r - 6 \geq 0$. Then also $2r_0 - 6 \geq 0$. Theorem 1 follows from Eq. (4). □

Corollary 1.1. *If G is a regular graph of degree $r \geq 3$, then for $k \geq 2$, all negative eigenvalues of $L^k(G)$ are equal to -2.*

Remark. If G is a regular graph of degree $r = 1$, then $L(G)$ consists of isolated vertices, and $L^2(G)$ is the "graph" without vertices. If G is a regular graph of degree $r = 2$, then G and $L(G)$ are isomorphic. Consequently, if $r = 2$, then G and $L^k(G)$ are isomorphic for all $k \geq 1$.

2. Energies of iterated line graphs of regular graphs

The *energy of a graph* G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where λ_i, $i = 1, 2, \ldots, n$, are the eigenvalues of G. This graph–spectral invariant was introduced by one of the present authors a long time ago [6], motivated by results obtained in the molecular-orbital theory of organic molecules [7,8]. Only relatively recently have studies of graph energy started to appear also in the mathematical literature [9–16].

Theorem 2. *If G is a regular graph of order n and of degree $r \geq 3$, then $E(L^2(G)) = 2nr(r-2)$.*

Proof. Bearing in mind Theorem 1 and Eq. (4), the energy of $L^2(G)$ is computed as follows:

$$E(L^2(G)) = \sum_{i=1}^{n}(\lambda_i + 3r - 6) + \frac{1}{2}n(r-2) \times (2r-6) + \frac{1}{2}nr(r-2) \times |-2|$$
$$= \sum_{i=1}^{n} \lambda_i + 2nr(r-2).$$

The sum of the eigenvalues of any graph is equal to zero. Theorem 2 follows. □

Another way in which Theorem 2 can be expressed is:

Theorem 2a. *Let G is a regular graph of order n_0, of degree $r_0 \geq 3$, and let for $k \geq 1$ the k-th iterated line graph of G be of degree r_k and possess n_k vertices. Then $E(L^2(G)) = 2n_1(r_1 - 2) = 2n_0r_0(r_0 - 2)$.*

Theorem 2a is directly generalized:

Corollary 2.1. *In the notation specified in Theorem 2a, the equality $E(L^{k+1}(G)) = 2n_k(r_k - 2)$ holds for any $k \geq 1$.*

Corollary 2.2. *In the notation specified in Theorem 2a, for any $k \geq 1$,*

$$E(L^{k+1}(G)) = 2n_0(r_0 - 2)\prod_{i=0}^{k-1}(2^i r_0 - 2^{i+1} + 2). \quad (5)$$

From Corollary 2.2 we see that the energy of any iterated line graph of a regular graph G of degree greater than two, except the energy of $L^1(G)$, is fully determined by the order (n_0) and degree (r_0) of G. The explicit (n_0, r_0)-dependence of the energy of $L^k(G)$ is given by Eq. (5).

Corollary 2.3. *In the notation specified in Theorem 2a, for any $k \geq 2$,*

$$E(L^k(G)) = 4(n_k - n_{k-1}) = 4n_k \frac{r_k - 2}{r_k + 2}.$$

3. An application: constructing equienergetic graphs

It is not difficult to find non-cospectral graphs that have equal energies, which we refer to as *equienergetic graphs*. The simplest pair of connected equienergetic graphs are the triangle and the quadrangle, both having $E = 4$. If we additionally require that such graphs be of the same order and have equal number of edges (which would be of value in chemical applications), then the problem becomes much less easy. Anyway, until now no systematic method for constructing pairs (or larger families) of equienergetic graphs was known. We are now able to offer one. It is based on the following elementary

Lemma 3. *Let G_1 and G_2 be two regular graphs of the same order and of the same degree. Then for any $k \geq 1$ the following holds:* (a) *$L^k(G_1)$ and $L^k(G_2)$ are of the same order, and have the same number of edges.* (b) *$L^k(G_1)$ and $L^k(G_2)$ are cospectral if and only if G_1 and G_2 are cospectral.*

Proof. Statement (a) follows from Eqs. (1) and (2), and the fact that the number of edges of $L^k(G)$ is equal to the number of vertices of $L^{k+1}(G)$. Statement (b) follows from relation (3), applied a sufficient number of times. □

Combining Lemma 3 with Corollary 2.2 we arrive at:

Theorem 4. *Let G_1 and G_2 be two non-cospectral regular graphs of the same order and of the same degree $r \geq 3$. Then for $k \geq 2$ the iterated line graphs $L^k(G_1)$ and $L^k(G_2)$ form a pair of non-cospectral equienergetic graphs of equal order and with the same number of edges. If, in addition, G_1 and G_2 are chosen to be connected, then also $L^k(G_1)$ and $L^k(G_2)$ are connected.*

It is now easy to generate large families of equienergetic graphs, satisfying the requirements given in Theorem 4. For instance, there are 2, 5, 19, and 85 connected regular graphs of degree 3 of order 6, 8, 10, and 12, respectively. No two of these are cospectral [1, pp. 268–269]. Their second and higher iterated line graphs form families consisting of 2, 5, 19, 85, ..., equienergetic graphs.

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