STOPPING SEMIMARTINGALES ON FOCK SPACE

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Abstract

We define the value, at any non-commutative finite stop time, of some vector semimartingales in Fock space. We apply it to stop a large class of operator processes, including all processes given by Hudson-Parthasarathy quantum stochastic integrals.

1. Introduction and notations

The theory of non-commutative stop times, as an extension of the classical theory of stop times on a probability space, has been mainly developed in [Hud] and [P-S] (revisited in [Me1]). In [P-S] the value at any stop time τ of some Weyl processes in the Fock space is computed. It gives rise to a factorization of the Fock space into a part "before τ" tensor a part "after τ". In this way they generalize the strong Markov property of the quantum Brownian motion proved in [Hud].

In [Me1] it is emphasised that the work of [P-S] gives the value at a quantum stop time of processes of vectors which are constituted of the tensor product of a complete martingale and a process in the future.

The aim of this article is to extend this latter result to processes of vectors which are constituted of the tensor product of a regular semimartingale of vectors and a process in the future. As a byproduct we obtain a method of stopping a large class of operator-valued processes, which contains all the non-commutative stochastic integrals. We recover in this latter case a definition given in [P-S].

Let us now examine the context in which we work. Let Φ be the boson Fock space over $L^2(\mathbb{R}^+)$. Let $\Phi_{t]}$, *resp.* $\Phi_{[t}$, be the Fock space over $L^2([0,t])$, *resp.* $L^2([t,+\infty[)$, for $t \in \mathbb{R}^+$. One then has the continuous tensor product structure: $\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$ for all $t \in \mathbb{R}^+$. Recall that the Fock space Φ is isomorphic to

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the Guichardet space $L^2(\mathcal{P})$ ([Gui]). Indeed, let $\mathcal{P}$ be the set of finite subsets of $\mathbb{R}^+$ that is, $\mathcal{P} = \cup_n \mathcal{P}_n$ where $\mathcal{P}_n$ is the set of subsets of $\mathbb{R}^+$ with cardinal n and $\mathcal{P}_0 = \{\varphi\}$. Each $\mathcal{P}_n$ is equipped with the restriction of the corresponding Lebesgue measure on $\mathbb{R}^n$ and $\mathcal{P}_0$ is equipped with the unit mass. Thus $\mathcal{P}$ is equipped with a σ-finite measure, denoted $d\sigma$, and Φ is isomorphic to $L^2(\mathcal{P})$. Consequently for $f \in \Phi$ we have

$$\|f\|^2 = \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma.$$

For all element $u \in L^2(\mathbb{R}^+)$ define the associated coherent vector $\varepsilon(u) \in \Phi$ by $[\varepsilon(u)](\sigma) = \prod_{s\in\sigma} u(s)$, for $\sigma \in \mathcal{P}$ (with the usual convention $\prod_{s\in\emptyset} u(s) = 1$). For all $s \leq t$ define $u_{t]} = u\mathbb{1}_{[0,t]}, u_{[s,t]} = u\mathbb{1}_{[s,t]}, u_{[t} = u\mathbb{1}_{[t,+\infty[}$; define $\sigma_{t]} = \{r \in \sigma; r \leq t\}$ and $\sigma_{[t} = \{r \in \sigma;\, r \geq t\}$ (note that it makes no difference whether we take strict inequalities in the latter definitions or not, as the set of $\sigma \in \mathcal{P}$ with $t \in \sigma$ is of null measure for every fixed t).

The tensor product structure $\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$ can be seen to correspond to the following :

(i) $f \in \Phi_{t]}$ if and only if $f(\sigma) = 0$ unless $\sigma \subset [0, t]$
(ii) $h \in \Phi_{[t}$ if and only if $h(\sigma) = 0$ unless $\sigma \subset [t, +\infty[$
(iii) $g = f \otimes h$ if and only if $g(\sigma) = f(\sigma_{t]})h(\sigma_{[t})$.

In this way we observe that $\varepsilon(u) = \varepsilon(u_{t]}) \otimes \varepsilon(u_{[t})$ for all u and all t.

Following [H-P] an *adapted process of operators* in Φ is a family $(H_t)_{t\geq 0}$ of operators on Φ, defined on the dense subspace $\mathcal{E} = \text{span}\,\{\varepsilon(u);\, u \in L^2(\mathbb{R}^+)\}$, such that $t \to H_t\, \varepsilon(u)$ is strongly measurable and

$$\begin{cases} H_t\, \varepsilon(u_{t]}) \in \Phi_{t]} \\ H_t\, \varepsilon(u) = [H_t\, \varepsilon(u_{t]})] \otimes \varepsilon(u_{[t}) \end{cases}$$

for all $u \in L^2(\mathbb{R}^+)$, $t \in \mathbb{R}^+$. In this case we say that H_t is *adapted at time* t.

2. Calculus on the Guichardet space

The results of this section can be found in great details in [A-L]. For all $t \in \mathbb{R}^+$, define the operator E_t on Φ by

$$[E_t f](\sigma) = f(\sigma)\mathbb{1}_{\sigma\subset[0,t]},\ \sigma \in \mathcal{P}.$$

The operator E_t is actually the orthogonal projection from Φ onto $\Phi_{t]}$. The operator E_0, also denoted $\mathbb{E}$, is given by

$$\mathbb{E}[f](\sigma) = f(\emptyset)\mathbb{1}_{\sigma=\emptyset},\ \sigma \in \mathcal{P}.$$

For all $f \in \Phi$, all $t \in \mathbb{R}^+$, define

$$[D_t f](\sigma) = f(\sigma \cup \{t\})\mathbb{1}_{\sigma\subset[0,t]},\ \sigma \in \mathcal{P}.$$

It can be easily seen ([A-L]), from the $\not\!\!E$-Lemma ([L-P]) that

$$\int_0^\infty \int_{\mathcal{P}} \left|[D_t f](\sigma)\right|^2 d\sigma\, dt < \infty.$$

Thus, for almost all $t, D_t f$ defines an element of Φ and

$$\int_0^\infty \|D_t f\|^2 dt < \infty.$$

From these definitions one can check that the following properties are satisfied:

(i) for all $f \in \Phi$, $\quad D_u E_t f = \begin{cases} 0 & \text{for a.a. } u > t \\ D_u f & \text{for a.a. } u < t \end{cases}$

(ii) for all $u \in L^2(\mathbb{R}^+)$, $D_t\, \varepsilon(u) = u(t)\varepsilon(u_{t]})$ for a.a. t

(iii) if $g = f \otimes h$ in the structure $\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$, then for almost all $u \geq t$

$$D_u\, g = f \otimes D_u h.$$

For a strongly measurable family $g. = (g_t)_{t\geq 0}$ of elements of Φ such that $g_t \in \Phi_{t]}$ for all t and $\int_0^\infty \|g_t\|^2 dt$ is finite, define

$$[I(g.)](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ g_{\vee\sigma}(\sigma-) & \text{otherwise} \end{cases}$$

where $\vee\sigma = \max\,\{s \in \sigma\}$ and $\sigma- = \sigma\backslash\{\vee\sigma\}$. It can be easily seen ([A-L]) that $I(g.)$ defines an element of $L^2(\mathcal{P})$, thus an element of Φ. From now on $I(g.)$ is denoted $\int_0^\infty g_t\, dx_t$. This notation is justified by the following. Let $f \in \Phi$, we can compute $\int_0^\infty D_t f\, dx_t$ and observe that

$$[\int_0^\infty D_t f\, dx_t](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ f(\sigma) & \text{otherwise} \end{cases}$$

Thus, we have the following *Fock space predictable representation.*

Theorem 1 – *For all $f \in \Phi$ one has the representation*

$$f = \mathbb{E}[f] + \int_0^\infty D_t f\, dx_t$$

and

$$\langle f,\, g\rangle = \overline{\mathbb{E}[g]}\, \mathbb{E}[f] + \int_0^\infty \langle D_t\, g,\, D_t f\rangle\, dt$$

for all $g \in \Phi$. ■

This result is an analogue of the probabilistic predictable representation property of Brownian motion, compensated Poisson process, Azéma's martingales... (for which the notation x_t stands), but here it is purely intrinsic to the Fock space structure and has nothing to do with probabilistic interpretations of Φ.

A *process of vectors* is a strongly measurable family $(z_t)_{t\geq 0}$ of elements of Φ. A process of vectors is *adapted* if $z_t \in \Phi_{t]}$ for all t. An adapted process $(m_t)_{t\geq 0}$ is a *martingale* if $E_s\, m_t = m_s$ for all $s \leq t$. A martingale $(m_t)_{t\geq 0}$ is *complete* if there exists a $m \in \Phi$ such that $m_t = E_t\, m$ for all t (or equivalently, if m_t converges in Φ to a vector m when t tends to $+\infty$).

If $(g_t)_{t\geq 0}$ is an adapted process of vectors such that $\int_a^b \|g_t\|^2 dt < \infty$ for every $0 \leq a < b \leq +\infty$, then define

$$\int_a^b g_t\, dx_t = \int_0^\infty g_t\, 1\!\!1_{[a,b]}(t)\, dx_t.$$

Lemma 2 – *Let $(m_t)_{t\geq 0}$ be a martingale in Φ. Then there exists a unique adapted process $(\xi_t)_{t\geq 0}$ in Φ such that for all $t \in I\!R^+$ one has $\int_0^t \|\xi_u\|^2 du < \infty$ and*

$$m_t = m_0 + \int_0^t \xi_u\, dx_u.$$

The process $(\xi_t)_{t\geq 0}$ is given by $\xi_t = D_t m_{t+} = D_t m_{t+h}$ for almost all $t, h > 0$.

Proof

From Theorem 1 one has $m_t = m_0 + \int_0^\infty D_u m_t\, dx_u$. But we have seen that for all $f \in \Phi$, almost all $u > t$ one has $D_u E_t f = 0$. Thus one has in fact $m_t = m_0 + \int_0^t D_u m_t\, dx_u$. Furthermore, as $D_u E_t = D_u$ for almost all $u < t$, we have for all $t < t'$, almost all $u < t, D_u m_{t'} = D_u E_t m_{t'} = D_u m_t$. Consequently the vector $D_u m_{u+h}$ does not depend on $h > 0$, one can choose ξ_u to be $D_u m_{u+h}$ for any $h > 0$. ■

Lemma 3 – *Let H be a bounded operator on Φ, time t adapted. Let $(g_s)_{s\geq 0}$ be an adapted process of vectors in Φ such that $\int_t^\infty \|g_s\|^2 ds < \infty$. Then one has*

$$H \int_t^\infty g_s\, dx_s = \int_t^\infty H g_s\, dx_s.$$

Proof

First of all notice that the boundedness and adaptedness of H implies that $(Hg_t)_{t\geq 0}$ is strongly measurable and $\int_t^\infty Hg_s\, dx_s$ is well-defined. For all $u \in L^2(I\!R^+)$, one has

$$\begin{aligned}
&\langle \varepsilon(u),\, H \int_t^\infty g_s\, dx_s\rangle = \\
&\quad = \langle H^* \varepsilon(u),\, \int_t^\infty g_s\, dx_s\rangle = \int_t^\infty \langle D_s H^* \varepsilon(u), g_s\rangle\, ds \\
&\quad = \int_t^\infty \langle D_s(H^*\varepsilon(u_{t]}) \otimes \varepsilon(u_{[t})),\, g_s\rangle\, ds = \int_t^\infty \langle H^*\varepsilon(u_{t]}) \otimes D_s \varepsilon(u_{[t}), g_s\rangle\, ds \\
&\quad = \int_t^\infty \langle u(s) H^* \varepsilon(u_{t]}) \otimes \varepsilon(u_{[t,s]}),\, g_s\rangle\, ds = \int_t^\infty \langle u(s) H^* \varepsilon(u_{s]}),\, g_s\rangle\, ds \\
&\quad = \int_t^\infty \langle u(s) \varepsilon(u_{s]}),\, H g_s\rangle\, ds = \int_t^\infty \langle D_s \varepsilon(u), H\, g_s\rangle\, ds = \langle \varepsilon(u),\, \int_t^\infty H g_s\, dx_s\rangle.
\end{aligned}$$

One concludes by density of the space $\mathcal{E}$ in Φ. ■

3. Non-commutative stop times on Fock space

Let us recall the main definitions of [P-S].

A *stop time* τ on Φ is a spectral measure on $\mathbb{R}^+ \cup \{+\infty\}$ with values in the space of orthogonal projections on Φ and such that, for all t, the operator $\tau([0,t])$ is adapted at time t.

In the following we adopt a probabilistic-like notation: for any Borel subset $A \subset \mathbb{R}^+ \cup \{+\infty\}$, the operator $\tau(A)$ is denoted $\mathbb{1}_{\tau \in A}$; in the same way $\tau(\{t\})$ is denoted $\mathbb{1}_{\tau=t}$, $\tau([0,t])$ is denoted by $\mathbb{1}_{\tau \leq t}$, etc.

A stop time τ is *finite* if $\mathbb{1}_{\tau=+\infty} = 0$. It is bounded by T if $\mathbb{1}_{\tau \leq T} = I$ for some $T \in \mathbb{R}^+$.

A point t in $\mathbb{R}^+$ is a *continuity point* for τ if $\mathbb{1}_{\tau=t} = 0$. Note that, unless $\tau \equiv 0$, the point 0 is always a continuity point for τ. It is also easy to check that the set of points $t \in \mathbb{R}^+$ which are not continuity points for τ is at most countable. If τ and τ' are two stop times on Φ, one says that $\tau \leq \tau'$ if, for all $t \in \mathbb{R}^+$ one has $\mathbb{1}_{\tau \leq t} \geq \mathbb{1}_{\tau' \leq t}$ (in the usual sense of comparison of two projections).

A stop time τ is *discrete* if there exists a finite set $E = \{0 \leq t_1 < t_2 < \cdots < t_n < +\infty\}$ such that $\mathbb{1}_{\tau \in E} = I$.

A sequence of stop times $(\tau_n)_n$ is said to *converge* to a stop time τ if, for all continuity point t for τ, the operators $\mathbb{1}_{\tau_n \leq t}$ converge strongly to $\mathbb{1}_{\tau \leq t}$.

A *sequence of refining τ-partitions* is a sequence $(E_n)_n$ of partitions $E_n = \{0 \leq t_1^n < t_2^n < \cdots < t_{i_n}^n < +\infty\}$ of partitions of $\mathbb{R}^+$ such that

(i) all the t_j^i are continuity points for τ;

(ii) $E_n \subseteq E_{n+1}$ for all n;

(iii) the diameter, $\max\{t_{i+1}^n - t_i^n\,; i = 1, ..., i_n\}$, of E_n tends to 0 as n tends to $+\infty$;

(iv) $t_{i_n}^n$ tends to $+\infty$ when n tends to $+\infty$.

The following result is taken from [P-S], Proposition 3.3 and [Me1].

Proposition 4 – *Let τ be any stop time. Then there exists a sequence $(\tau_n)_n$ of discrete stop times such that $\tau_1 \geq \tau_2 \geq \cdots \geq \tau$ and $(\tau_n)_n$ converges to τ.*

Proof

Let $E = \{0 \leq t_1 < t_2 < \cdots < t_n < +\infty\}$ be a partition of $\mathbb{R}^+$. Define a spectral measure τ_E by

$$\tau_E(\{t_i\}) = \begin{cases} \mathbb{1}_{\tau < t_1} & \text{if } i = 1 \\ \mathbb{1}_{\tau \in [t_{i-1}, t_i[} & \text{if } 1 < i \leq n-1, \end{cases}$$
$$\tau_E(\{t_n\}) = \mathbb{1}_{\tau \geq t_{n-1}}.$$

The spectral measure τ_E clearly defines a discrete stop time on Φ and $\tau_E \geq \tau$. Taking a sequence $(E_n)_n$ of refining τ-partitions of $\mathbb{R}^+$ gives the required sequence $(\tau_n)_n = (\tau_{E_n})_n$. ■

4. Stopping vectors processes

Our aim is to define the value z_τ at a finite stop time τ of a large class of processes of vectors $(z_t)_{t\geq 0}$ in Φ.

An adapted process of vectors $(z_t)_{t\geq 0}$ in Φ is a *regular semimartingale of vectors* if $(z_t)_{t\geq 0}$ admits a decomposition (always unique) as $z_t = m_t + a_t$ where m_t is a martingale and $a_t = \int_0^t h_s\, ds$ with $h_t \in \Phi_{t]}$ and $\int_0^t \|h_s\| ds < \infty$ for all t. The integral $\int_0^t h_s\, ds$ is understood in the usual hilbertian sense that is, $\langle f, \int_0^t h_s\, ds\rangle = \int_0^t \langle f, h_s\rangle ds$ which defines a vector in Φ for

$$|\langle f, \int_0^t h_s\, ds\rangle| \leq \int_0^t |\langle f, h_s\rangle| ds \leq \|f\| \int_0^t \|h_s\| ds.$$

It is interesting to recall a characterisation of the regular semimartingales of vectors in Φ.

Theorem 5 – *An adapted process of vectors $(z_t)_{t\geq 0}$ in Φ is a regular semimartingale of vectors if and only if there exists a locally integrable fucntion g on $\mathbb{R}^+$ such that, for all $s \leq t$, one has*

$$\|E_s z_t - z_s\| \leq \int_s^t g(u) du.$$

Proof

If $(z_t)_{t\geq 0}$ is a regular semimartingale of vectors the estimate is trivial. The converse is a simple consequence of Enchev's characterization of Hilbertian quasi-martingales in [Enc] (see also [Me2]). ∎

A process of vectors $(y_t)_{t\geq 0}$ in Φ is said to be *adapted to the future* if $y_t \in \Phi_{[t}$ for all t.

For any process of vectors $(w_t)_{t\geq 0}$ in Φ and for any discrete stop time τ one can obviously define, following the case of classical stop times, w_τ by

$$w_\tau = \sum_i \mathbb{1}_{\tau = t_i} w_{t_i}. \tag{1}$$

But when τ is any finite stop time we wish to pass to the limit on the expression (1) for a sequence $(\tau_n)_n$ of discrete stop times converging to τ (Proposition 4).

In [P-S] and [Me1] it is shown that this convergence can be obtained when $(w_t)_{t\geq 0}$ is of the form $(m_t \otimes y_t)_{t\geq 0}$ where $(m_t)_{t\geq 0}$ is a complete martingale and $(y_t)_{t\geq 0}$ is a process of vectors adapted to the future. We are going to extend this result to processes $(w_t)_{t\geq 0}$ of the form $(z_t \otimes y_t)_{t\geq 0}$ where $(z_t)_{t\geq 0}$ is a regular semimartingale of vectors and $(y_t)_{t\geq 0}$ is adapted to the future and bounded in norm. We first need some preliminary results.

Remark : If c is an element of $\Phi_{0]} \simeq \mathbb{C}\,\mathbb{1}$ we have

$$\sum_i \mathbb{1}_{\tau = t_i}\, c = c.$$

Thus, in the following we assume that all our martingales $(m_t)_{t\geq 0}$ are such that $m_0 = 0$. Consequently, by Lemma 2, every regular semimartingale is of the form $z_t = \int_0^t \xi_s\, dx_s + \int_0^t h_s\, ds, \quad t \geq 0$.

Proposition 6 – *Let*

$$z_t = \int_0^t \xi_s\, dx_s + \int_0^t h_s\, ds, \quad t \geq 0$$

be a regular semimartingale of vectors. Let τ be a bounded stop time with bound T. Let $(E_n)_n$ be a sequence of refining τ-partitions of $\mathbb{R}^+$. Put $\tau_n = \tau_{E_n}$, for all $n \in \mathbb{N}$. Then the sequence $(z_{\tau_n})_n$ converges to a vector z_τ in Φ which is given by

$$z_\tau = \int_0^T 1\!\!1_{\tau > s}\, \xi_s\, dx_s + \int_0^T 1\!\!1_{\tau > s}\, h_s\, ds.$$

Proof

One has

$$\begin{aligned} z_{\tau_n} &= \sum_i 1\!\!1_{\tau_n = t_i}\, z_{t_i} = \sum_i 1\!\!1_{\tau_n = t_i} \Big[\int_0^{t_i} \xi_s\, dx_s + \int_0^{t_i} h_s\, ds\Big] \\ &= \sum_i \sum_{j<i} 1\!\!1_{\tau_n = t_i} \Big[\int_{t_j}^{t_{j+1}} \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} h_s\, ds\Big] \\ &= \sum_j \sum_{i>j} 1\!\!1_{\tau_n = t_i} \Big[\int_{t_j}^{t_{j+1}} \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} h_s\, ds\Big] \\ &= \sum_j 1\!\!1_{\tau_n > t_j} \Big[\int_{t_j}^{t_{j+1}} \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} h_s\, ds\Big] \\ &= \sum_j \Big[\int_{t_j}^{t_{j+1}} 1\!\!1_{\tau_n > t_j}\, \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} 1\!\!1_{\tau_n > t_j}\, h_s\, ds\Big] \end{aligned}$$

(by boundedness and t_j-adaptedness of $1\!\!1_{\tau_n > t_j}$, and by Lemma 3)

$$\begin{aligned} &= \sum_j [\int_{t_j}^{t_{j+1}} 1\!\!1_{\tau_n > s}\, \xi_s\, dx_s + \int_{t_j}^{t_{j+1}} 1\!\!1_{\tau_n > s} h_s\, ds] \\ &= \int_0^T 1\!\!1_{\tau_n > s}\, \xi_s\, dx_s + \int_0^T 1\!\!1_{\tau_n > s} h_s\, ds. \end{aligned}$$

Now, one has

$$\begin{aligned} &\| z_{\tau_n} - [\int_0^T 1\!\!1_{\tau > s}\, \xi_s\, dx_s + \int_0^T 1\!\!1_{\tau > s} h_s\, ds] \|^2 \\ &\quad \leq 2 \int_0^T \|(1\!\!1_{\tau_n > s} - 1\!\!1_{\tau > s})\xi_s\|^2 ds + 2[\int_0^T \|(1\!\!1_{\tau_n > s} - 1\!\!1_{\tau > s}) h_s\| ds]^2. \end{aligned}$$

The quantities inside the integrals converge to 0 when n tends to $+\infty$ and are respectively dominated by $4\|\xi_s\|^2$ and $2\|h_s\|$ which are integrable on $[0, T]$. Thus, one concludes by the dominated convergence Theorem. ■

For any finite stop time τ and $n \in I\!N$ one can define the stop time $\tau \wedge n$ by

$$\mathbb{1}_{\tau\wedge n\leq t} = \begin{cases} 0 & \text{if } n > t \\ \mathbb{1}_{\tau\leq t} & \text{if } n \leq t. \end{cases}$$

It is clear that $\tau \wedge n$ is a bounded stop time with bound n and that $(\tau \wedge n)_n$ converges to τ. Thus, we easily deduce the following result.

Proposition 7 – *Let $z_t = \int_0^t \xi_s\, dx_s + \int_0^t h_s\, ds$, $t \in I\!R^+$, be a regular semimartingale of vectors. Let τ be a finite stop time such that*

$$\int_0^\infty \|\mathbb{1}_{\tau>s}\, \xi_s\|^2 ds < \infty \text{ and } \int_0^\infty \|\mathbb{1}_{\tau>s}\, h_s\| ds < \infty$$

then the sequence $(z_{\tau\wedge n})_n$ converges to a vector z_τ in Φ given by

$$z_\tau = \int_0^\infty \mathbb{1}_{\tau>s}\, \xi_s\, dx_s + \int_0^\infty \mathbb{1}_{\tau>s}\, h_s\, ds. \qquad \blacksquare$$

If τ is a finite stop time, the mapping $A \to \|\mathbb{1}_{\tau\in A} f\|^2$ defines a measure on $(I\!R^+, \mathcal{B}(I\!R^+))$ for any $f \in \Phi$. We denote this measure by $\|\mathbb{1}_{\tau\in ds} f\|^2$.

The following result is an improvement of [P-S]'s results, together with a shortening of their proofs as we are dealing with a slightly simpler case.

Proposition 8 – *Let $(m_t)_{t\geq 0}$ be a complete martingale. Let $(y_t)_{t\geq 0}$ be a process of vectors adapted to the future. Put $w_t = m_t \otimes y_t$, $t \in I\!R^+$. Let τ be a finite stop time such that*

$$\int_0^\infty \|y_s\|^2 \|\mathbb{1}_{\tau\in ds}\, m\|^2 < \infty$$

where $m = \lim_{t\to+\infty} m_t$. Let $(E_n)_n$ be a sequence of refining τ-partitions. Put $\tau_n = \tau_{E_n}$, $n \in I\!N$. Then the sequence (w_{τ_n}) converges in Φ to a vector w_τ which is independent of the chosen sequence $(E_n)_n$.

Proof

Consider $p \leq q \in I\!N$. As $E_q \supset E_p$ we can assume that E_p is of the form $\{0 \leq t_1 < \cdots < t_n\}$ and E_q is of the form $\{0 \leq \cdots < t_i = t_i^0 < t_i^1 < \cdots < t_i^{n_i} = t_{i+1} < \cdots\}$. So it is sufficient to prove that the expression

$$\|\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]} w_{t_{i+1}} - \sum_{i,j} \mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} w_{t_i^{j+1}}\|^2$$

converges to 0 when the diameter δ of E_p tends to 0.

As the space $\mathcal{E}$ is dense in Φ there exists a $\tilde{m} \in \mathcal{E}$ such that $\|m - \tilde{m}\|$ is small. Suppose $\tilde{m}$ is of the form

$$\tilde{m} = \sum_{k=1}^K \lambda_k\, \varepsilon(u^k).$$

In [P-S], Proposition 4.9, it is proved that there exists a sequence $(y^n)_n$ of processes of vectors adapted to the future such that, for all n, the mapping $t \to y_t^n$ is strongly continuous, $\int_0^\infty \|y_s^n\|^2 \|\mathbb{1}_{\tau\in ds} m\|^2 < \infty$ and

$$\int_0^\infty \|y_s - y_s^n\|^2 \, \|\mathbb{1}_{\tau\in ds} m\|^2 \to 0, \quad n \to +\infty.$$

To simplify the notation choose a $(\tilde{y}_t)_{t\geq 0}$ strongly continuous process of vectors adapted to the future such that $\int_0^\infty \|\tilde{y}_s\|^2 \|\mathbb{1}_{\tau\in ds} m\|^2 < \infty$ and $\int_0^\infty \|y_s - \tilde{y}_s\|^2 \|\mathbb{1}_{\tau\in ds} m\|^2$ is small.

Finally, notice that an operator H which is adapted at time t always satisfies $HE_u = E_u H$ for all $u \geq t$. One has

$$\begin{aligned}
\|\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]} w_{t_{i+1}} - \sum_{i,j} \mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} w_{t_i^{j+1}}\|^2 \\
&= \|\sum_{i,j} \mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} (w_{t_{i+1}} - w_{t_i^{j+1}})\|^2 \\
&= \sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} (m_{t_{i+1}} \otimes y_{t_{i+1}} - m_{t_i^{j+1}} \otimes y_{t_i^{j+1}})\|^2 \\
&\leq 3\sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} (m_{t_{i+1}} \otimes y_{t_{i+1}} - m_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}})\|^2 \\
&+ 3\sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} (m_{t_i^{j+1}} \otimes y_{t_i^{j+1}} - m_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}})\|^2 \\
&+ 3\sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} (m_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}} - m_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}})\|^2. \quad (2)
\end{aligned}$$

We now concentrate on the last term of the right hand side of (2). For any fixed $b > 0$, it is equal to

$$\begin{aligned}
&3 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} (m_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}} - m_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}})\|^2 \\
&+ 6 \sum_{i,j;\, t_i^{j+1}>b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} m_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}}\|^2 + 6 \sum_{i,j;\, t_i^{j+1}>b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} m_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}}\|^2 \quad (3)
\end{aligned}$$

We now concentrate on the first term of (3). It is dominated by

$$\begin{aligned}
&9 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} (m_{t_{i+1}} - \tilde{m}_{t_{i+1}}) \otimes \tilde{y}_{t_{i+1}}\|^2 \\
&+ 9 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} (m_{t_i^{j+1}} - \tilde{m}_{t_i^{j+1}}) \otimes \tilde{y}_{t_i^{j+1}}\|^2 \\
&+ 9 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} (\tilde{m}_{t_{i+1}} \otimes \tilde{y}_{t_{i+1}} - \tilde{m}_{t_i^{j+1}} \otimes \tilde{y}_{t_i^{j+1}})\|^2 \\
&\leq 9 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} E_{t_{i+1}} (m - \tilde{m})\|^2 \|\tilde{y}_{t_{i+1}}\|^2 \\
&+ 9 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} E_{t_i^{j+1}} (m - \tilde{m})\|^2 \|\tilde{y}_{t_i^{j+1}}\|^2
\end{aligned}$$

$$+ 9K\sum_{k=1}^{K}\lambda_k^2 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\big(\varepsilon(u^k_{t_{i+1}]}) \otimes \tilde y_{t_{i+1}} - \varepsilon(u^k_{t_i^{j+1}]}\big) \otimes \tilde y_{t_i^{j+1}})\|^2$$

$$\leq 18 \max_{s\leq b}\|\tilde y_s\|^2 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}(m-\tilde m)\|^2$$

$$+ 9K\sum_{k=1}^{K}\lambda_k^2 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\varepsilon(u^k_{t_i^{j+1}]})\|^2 \|\varepsilon(u^k_{[t_i^{j+1},t_{i+1}]}) \otimes \tilde y_{t_{i+1}} - 1 \otimes \tilde y_{t_i^{j+1}}\|^2$$

$$\leq 18 \max_{s\leq b}\|\tilde y_s\|^2 \|m-\tilde m\|^2$$

$$+ 18K\sum_{k=1}^{K}\lambda_k^2 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\varepsilon(u^k_{t_i^{j+1}]})\|^2 \|\varepsilon(u^k_{[t_i^{j+1},t_{i+1}]}) - 1\|^2 \|\tilde y_{t_{i+1}}\|^2$$

$$+ 18K\sum_{k=1}^{K}\lambda_k^2 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\varepsilon(u^k_{t_i^{j+1}]})\|^2 \|\tilde y_{t_{i+1}} - \tilde y_{t_i^{j+1}}\|^2$$

$$\leq 18 \max_{s\leq b}\|\tilde y_s\|^2 \|m-\tilde m\|^2$$

$$+ 18K\sum_{k=1}^{K}\lambda_k^2 \max_{s\leq b}\|\tilde y_s\|^2 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\varepsilon(u^k_{t_i^{j+1}]})\|^2 \times$$

$$\times \Big(\int_{t_i^{j+1}}^{t_{i+1}} |u^k(s)|^2 \|\varepsilon(u^k_{[t_i^{j+1},s]})\|^2 ds\Big)$$

$$+ 18\, K\sum_{k=1}^{K}\lambda_k^2 \sup_{\substack{s,t\leq b\\ |s-t|\leq\delta}} \|\tilde y_s - \tilde y_t\|^2 \sum_{i,j;\, t_i^{j+1}\leq b} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\varepsilon(u^k_{t_i^{j+1}]})\|^2$$

$$\leq 18 \max_{s\leq b}\|\tilde y_s\|^2 \|m-\tilde m\|^2$$

$$+ 18K\sum_{k=1}^{K}\lambda_k^2 \max_{s\leq b}\|\tilde y_s\|^2 \sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\varepsilon(u^k)\|^2 \sup_{\substack{s,t\leq b\\ |s-t|\leq\delta}} \int_s^t |u^k(v)|^2 dv$$

$$+ 18K\sum_{k=1}^{K}\lambda_k^2 \sup_{\substack{s,t\leq b\\ |s-t|\leq\delta}} \|\tilde y_s - \tilde y_t\|^2 \sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]}\varepsilon(u^k)\|^2.$$

Inserting this in (3) and then in (2) we get

$$\|w_{\tau_p} - w_{\tau_q}\|^2$$

$$\leq 3\sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} m\|^2 \|y_{t_{i+1}} - \tilde y_{t_{i+1}}\|^2 + 3\sum_{i,j} \|\mathbb{1}_{\tau\in[t_i^j,t_i^{j+1}]} m\|^2 \|y_{t_i^{j+1}} - \tilde y_{t_i^{j+1}}\|^2$$

$$+ 18 \max_{s\leq b}\|\tilde y_s\|^2 \|m-\tilde m\|^2$$

$$+ 18\,K \sum_{k=1}^{K} \lambda_k^2 \|\varepsilon(u^k)\|^2 \max_{s\leq b}\|\tilde{y}_s\|^2 \sup_{\substack{s,t\leq b\\ |s-t|\leq\delta}} \int_s^t |u^k(v)|^2 dv$$

$$+ 18\,K \sum_{k=1}^{K} \lambda_k^2 \|\varepsilon(u^k)\|^2 \sup_{\substack{s,t\leq b\\ |s-t|\leq\delta}} \|\tilde{y}_s - \tilde{y}_t\|^2$$

$$+ 12 \sum_{i,j;t_i^{j+1}>b} \|1\!\!1_{\tau\in[t_i^j,t_i^{j+1}]} m\|^2 \|\tilde{y}_{t_{i+1}}\|^2.$$

When δ tends to 0 the fourth and the fifth terms converge to 0 and the expression above converges to

$$6\int_0^\infty \|y_s-\tilde{y}_s\|^2\|1\!\!1_{\tau\in ds}m\|^2 + 18\max_{s\leq b}\|\tilde{y}_s\|^2\|m-\tilde{m}\|^2 + 12\int_b^\infty \|\tilde{y}_s\|^2\|1\!\!1_{\tau\in ds}m\|^2.$$

This latter expression converges to $6\int_b^\infty \|y_s\|^2\|1\!\!1_{\tau\in ds}m\|^2$ when $\tilde{y}$ tends to y and $\tilde{m}$ tends to m. This finally tends to 0 when b tends to $+\infty$. We have thus proved the convergence of $(w_{\tau_n})_n$ to a limit $w_\tau \in \Phi$.

If $(E_n)_n$ and $(F_n)_n$ are two sequences of refining τ-partitions, denote by $E_n \vee F_n$ the τ-partition made of $E_n \cup F_n$. We then have

$$\|w_{\tau_{E_n}} - w_{\tau_{F_n}}\|^2 \leq 2\|w_{\tau_{E_n}} - w_{\tau_{E_n\vee F_n}}\|^2 + 2\|w_{\tau_{E_n\vee F_n}} - w_{\tau_{F_n}}\|^2.$$

From the estimate obtained above we see that, as $E_n\vee F_n \subset E_n$ and $E_n \vee F_n \subset F_n$ that $\|w_{\tau_{E_n}} - w_{\tau_{E_n\vee F_n}}\|^2$ (respectively $\|w_{\tau_{E_n\vee F_n}} - w_{\tau_{F_n}}\|^2$) is dominated by an expression which depends only on the diameter of E_n (respectively F_n) and converges to 0 with it. Thus, the limit w_τ does not depend on the choice of the sequence $(E_n)_n$. ■

Remark : The vector w_τ obtained from this proposition is denoted

$$\int 1\!\!1_{\tau\in ds}(E_s m)\otimes y_s.$$

Let $w_t = m_t \otimes y_t$ and $w'_t = m'_t \otimes y'_t$ with integrability condition:

$$\int_0^\infty \|y_s\|^2\|1\!\!1_{\tau\in ds}m\|^2 < \infty \quad \text{and} \quad \int_0^\infty \|y'_s\|^2\|1\!\!1_{\tau\in ds}m'\|^2 < \infty.$$

Then it follows from the above that

$$\langle w_\tau, w'_\tau\rangle = \int_0^\infty \langle y_s, y'_s\rangle\langle \mathbb{E}_s 1\!\!1_{\tau\in ds} m, m'\rangle.$$

Theorem 9 – *Let $z_t = \int_0^t \xi_s dx_s + \int_0^t h_s\, ds$, $t\geq 0$ be a regular semimartingale of vectors. Let $(y_t)_{t\geq 0}$ be a process of vectors, adapted to the future and bounded in norm. Let τ be a finite stop time such that*

$$\int_0^\infty \|1\!\!1_{\tau>s}\,\xi_s\|^2 ds < \infty \quad \textit{and} \quad \int_0^\infty \|1\!\!1_{\tau>s}\,h_s\|ds < \infty.$$

Let $w_t = z_t \otimes y_t$, $t\geq 0$. Let $(E_n)_n$ be a sequence of refining τ-partitions of $\mathbb{R}^+$. Put $\tau_n = \tau_{E_n}, n \in \mathbb{N}$. Then the sequence $(w_{\tau_n})_n$ converges to a vector w_τ which *is given by*

$$w_\tau = \int \mathbb{1}_{\tau\in ds}[E_s z_\tau]\otimes y_s.$$

Proof

One has

$$\begin{aligned}
z_{\tau_n} &= \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}\left[\int_0^{t_{i+1}} \xi_s dx_s + \int_0^{t_{i+1}} h_s\,ds\right]\otimes y_{t_{i+1}}\\
&= \sum_i\sum_{j\leq i} \mathbb{1}_{\tau\in[t_i,t_{i+1}]}\left[\int_{t_j}^{t_{j+1}} \xi_s\,dx_s + \int_{t_j}^{t_{j+1}} h_s\,ds\right]\otimes y_{t_{i+1}}\\
&= \sum_i\sum_{j\leq i} \mathbb{1}_{\tau\in[t_i,t_{i+1}]}\mathbb{1}_{\tau\geq t_j}\left[\int_{t_j}^{t_{j+1}} \xi_s\,dx_s + \int_{t_j}^{t_{j+1}} h_s\,ds\right]\otimes y_{t_{i+1}}\\
&= \sum_i\sum_{j\leq i} \mathbb{1}_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}\mathbb{1}_{\tau\geq t_j}\left[\int_{t_j}^{t_{j+1}} \xi_s\,dx_s + \int_{t_j}^{t_{i+1}} h_s\,ds\right]\otimes y_{t_{i+1}}\\
&= \sum_i\sum_{j} \mathbb{1}_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}\mathbb{1}_{\tau\geq t_j}\left[\int_{t_j}^{t_{j+1}} \xi_s\,dx_s + \int_{t_j}^{t_{j+1}} h_s\,ds\right]\otimes y_{t_{i+1}}
\end{aligned}$$

(as for $j\geq i+1$ the terms inside the sum vanish)

$$\begin{aligned}
&= \sum_i\sum_{j} \mathbb{1}_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}\left[\int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau\geq t_j}\,\xi_s\,dx_s + \int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau\geq t_j} h_s\,ds\right]\otimes y_{t_{i+1}}\\
&= \sum_i\sum_{j} \mathbb{1}_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}\left[\int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau_n> s}\,\xi_s\,dx_s + \int_{t_j}^{t_{j+1}} \mathbb{1}_{\tau_n\geq s}\, h_s\,ds\right]\otimes y_{t_{i+1}}\\
&= \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}\left[\int_0^\infty \mathbb{1}_{\tau_n> s}\,\xi_s\,dx_s + \int_0^\infty \mathbb{1}_{\tau_n> s}\, h_s\,ds\right]\otimes y_{t_{i+1}}\\
&= \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}(z_{\tau_n})\otimes y_{t_{i+1}}\\
&= \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}(z_\tau)\otimes y_{t_{i+1}}\\
&+ \sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}(z_\tau - z_{\tau_n})\otimes y_{t_{i+1}}.
\end{aligned}$$

The first term of the right hand side converges to

$$\int \mathbb{1}_{\tau\in ds}\,E_s(z_\tau)\otimes y_s$$

by Proposition 8 (as the condition $\int_0^\infty \|y_s\|^2\|\mathbb{1}_{\tau\in ds}z_\tau\|^2<\infty$ is trivial since $s\to\|y_s\|$ is bounded). The second term has the square of its norm dominated by

$$\sum_i \|\mathbb{1}_{\tau\in[t_i,t_{i+1}]}E_{t_{i+1}}(z_\tau - z_{\tau_n})\|^2\sup_s\|y_s\|^2 \leq \sup_s\|y_s\|^2\|z_\tau - z_{\tau_n}\|^2$$

which converges to 0 by Proposition 7. ■

5. Stopping operators processes

We now consider a process $(X_t)_{t\geq 0}$ of adapted operators on Φ. As for vectors in the previous section, we want to define the value X_τ of $(X_t)_{t\geq 0}$ at a finite stop time τ. In the case of discrete stop times, three non-equivalent definitions appear:

left-stopping : $\tau \circ X = \sum_i \mathbb{1}_{\tau=t_i} X_{t_i}$

right-stopping : $X \circ \tau = \sum_i X_{t_i}\, \mathbb{1}_{\tau=t_i}$

two-sided-stopping : $\tau \circ X \circ \tau = \sum_i \mathbb{1}_{\tau=t_i}\, X_{t_i}\, \mathbb{1}_{\tau=t_i}$

As previously we wish to pass to the limit of discrete stop times converging to a finite stop time, for a large class of processes of operators $(X_t)_{t\geq 0}$.

Let τ be a finite stop time. A regular semimartingale of vectors $z_t = \int_0^t \xi_s dx_s + \int_0^t h_s\, ds$ is said to be τ-integrable if

$$\int_0^\infty \|\mathbb{1}_{\tau>s}\xi_s\|^2 ds < \infty \text{ and } \int_0^\infty \|\mathbb{1}_{\tau>s} h_s\| ds < \infty.$$

Proposition 10 – *Let $(X_t)_{t\geq 0}$ be an adapted process of operators on Φ. Let $u \in L^2(\mathbb{R}^+)$ be such that $(X_t\, \varepsilon(u_{t]}))_{t\geq 0}$ is a regular semimartingale of vectors. Let τ be a finite stop time such that $(X_t\, \varepsilon(u_{t]}))_{t\geq 0}$ is τ-integrable. Let $(E_n)_n$ be a sequence of refining τ-partitions of $\mathbb{R}^+$. Then the sequence $(X_{\tau_{E_n}} \varepsilon(u))_n$ converges to a vector $X_\tau\, \varepsilon(u)$.*

Proof

As $(X_t)_{t\geq 0}$ is an adapted process of operators, one has

$$X_t\, \varepsilon(u) = X_t\, \varepsilon(u_{t]}) \otimes \varepsilon(u_{[t}).$$

The process of vectors $(\varepsilon(u_{[t}))_{t\geq 0}$ is clearly adapted to the future and bounded in norm. By hypothesis the process $(X_t\, \varepsilon(u_{t]}))_{t\geq 0}$ is a τ-integrable regular semimartingale of vectors. Thus we can apply Theorem 9 to the process $(w_t)_{t\geq 0} = (X_t\, \varepsilon(u))_{t\geq 0}$. ■

Theorem 11 – *Let $(X_t)_{t\geq 0}$ be an adapted process of operators on Φ. Suppose that for all $u \in L^2(\mathbb{R}^+)$ the process $(X_t\, \varepsilon(u_{t]}))_{t\geq 0}$ is a regular semimartingale of vectors. Let τ be a finite stop time such that, for all $u \in L^2(\mathbb{R}^+)$, the process $(X_t\, \varepsilon(u_{t]}))_{t\geq 0}$ is τ-integrable.*

Then the left stopping $\tau \circ X$ converges strongly on $\mathcal{E}$.

Proof

By Proposition 10 we have that the quantity

$$\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}(X_{t_{i+1}}\varepsilon(u_{t_{i+1}]})) \otimes \varepsilon(u_{[t_{i+1}})$$

admits a limit when the diameter δ of the τ-partition $\{t_i;\ i=1,...,n\}$ tends to 0. But this quantity is also equal to

$$\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}(X_{t_{i+1}}\varepsilon(u)) = \left[\sum_i \mathbb{1}_{\tau\in[t_i,t_{i+1}]}X_{t_{i+1}}\right]\varepsilon(u).$$

This proves that the Riemann sums associated to the left stopping of X converge. ■

One can wonder what is this class of operator processes such that $(X_t\varepsilon(u_{t]}))_{t\geq 0}$ is a regular semimartingale of vectors, and what are the stop time τ such that $(X_t\varepsilon(u_{t]}))_{t\geq 0}$ is τ-integrable.

We now recall the definitions of the non-commutative stochastic integrals ([H-P]) of adapted processes of operators with respect to the creation $(A_t^+)_{t\geq 0}$, annihilation $(A_t)_{t\geq 0}$, conservation $(\Lambda_t)_{t\geq 0}$ and time $(tI)_{t\geq 0}$ processes, and also their extension as defined in [A-M].

Let H, K, L, M be adapted processes of operators defined on a domain $\mathcal{D}$ containing $\mathcal{E}$. Assume that the following integrals are meaningful (from the point of view of domains) and finite for all $f\in\mathcal{D}$, $t\in\mathbb{R}^+$:

$$\int_0^t \|H_sD_sf\|^2ds,\ \int_0^t \|K_sE_sf\|^2ds,\ \int_0^t \|L_sD_sf\|ds,\ \int_0^t \|M_sE_sf\|ds. \qquad (4)$$

According to [A-M], we say that an adapted process of operators $(T_t)_{t\geq 0}$ defined on $\mathcal{D}$ has the integral representation

$$T_t = \int_0^t H_s d\Lambda_s + \int_0^t K_s dA_s^+ + \int_0^t L_s dA_s + \int_0^t M_s ds$$

on *the domain* $\mathcal{D}$ if for all $f\in\mathcal{D}$ one has that $\int_0^t \|T_sD_sf\|^2ds$ is well-defined meaningful and finite, and

$$T_tE_tf = \int_0^t T_sD_sf dx_s + \int_0^t H_sD_sf dx_s + \int_0^t K_sE_sf dx_s + \int_0^t L_sD_sf ds + \int_0^t M_sE_sf\, ds. \qquad (5)$$

Theorem 12 ([A-M], Theorem 1, and [Me3] p. 123) – *On the domain $\mathcal{D}=\mathcal{E}$, this definition is equivalent to Hudson-Parthasarathy's definition of non-commutative stochastic integrals.* ■

A consequence of Theorem 12 is that if $(X_t)_{t\geq 0}$ is any process of the form (in [H-P]'s sense)

$$X_t = \int_0^t H_s d\Lambda_s + \int_0^t K_s dA_s^+ + \int_0^t L_s dA_s + \int_0^t M_s\, ds$$

then for all $f\in\mathcal{E}$, the process $(X_tE_tf)_{t\geq 0}$ is a regular semimartingale of vectors.

Now, if τ is a finite stop time such that the integral

$$X_\tau = \int_0^\infty \mathbb{1}_{\tau>s}H_s\, d\Lambda_s + \int_0^\infty \mathbb{1}_{\tau>s}K_s dA_s^+ + \int_0^\infty \mathbb{1}_{\tau>s}L_s dA_s + \int_0^\infty \mathbb{1}_{\tau>s}M_s ds \quad (6)$$

is well-defined in [H-P]'s sense we have by Theorem 12 and (4) that $(X_t\varepsilon(u_{t]}))_{t\geq 0}$ is a τ-integrable regular semimartingale of vectors. The left stopping $\tau\circ X$ given by Theorem 11 is then the operator X_τ given by (6). By this way we have seen that *the set of processes of operators concerned by Theorem 11 at least contains all the Hudson-Parthasarathy stochastic integrals.*

Note that, since $(\tau\circ X)^* = X^*\circ\tau$ for discrete stop times, we get some obvious extensions of the results of this section in the case of right-stopping.

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