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Transient film profile of thin liquid film flow on a stretching surface

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Abstract. In this paper we have studied a non-planar thin liquid film flow on a planar stretching surface. The stretching surface is assumed to stretch impulsively from rest and the effect of inertia of the liquid is considered. Equations describing the laminar flow on the stretching surface are solved analytically. It is observed that faster stretching causes quicker thinning of the film on the stretching surface. Velocity distribution in the liquid film and the transient film profile as functions of time are obtained.

Mathematics Subject Classification (2000). 76A20, 37E35, 76D05, 76M45.

Keywords. Stretching surfaces, coating, thin liquid films, two dimensional flow, fluid flow, asymptotic solutions.

1. Introduction

Flow dynamics due to the stretching of a boundary is relevant in extrusion process. In particular, in the extrusion of a polymer in a melt-spinning process, the extrudate from the die is generally drawn and simultaneously stretched into a thin sheet, and then solidified through quenching or gradual cooling by direct contact with water. Crane [1] gave an exact similarity solution in closed analytical form for steady two-dimensional boundary layer flow caused by the stretching of a flat sheet which moves in its own plane with velocity varying linearly with distance from a fixed point. The stretching mechanism is also applicable to the elongation of bubbles and pseudopods (Wang [2]). Due to its practical applications, the stretching sheet problem has attracted several researchers for the last three decades and is extensively studied to understand the same, along with, either the sole effects of rotation, heat and mass transfer, chemical reaction, MHD, suction/injection, non-Newtonian fluids or different combinations of these above affects [3–19]. In all these studies, boundary layer equation is considered and the boundary conditions are prescribed at the sheet and on the fluid at infinity. Imposition of similarity transformation reduces the system to a set of ODEs, which are then solved either

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analytically or numerically. Wang [20] further widened its horizon to study the flow of liquid film on an unsteady stretching surface. In this study Wang restricted the motion to a specified family of time dependence and reduced the boundary layer equations to a nonlinear ODE involving by a non-dimensional unsteady parameter by using a special type of similarity transformation. Using this special type of similarity transformation, Andersson et al. [21] have studied the unsteady stretching flow in case of finite thickness for power-law fluid. Later on Andersson et al. [22] and Dandapat et al. [23] extended Wang's unsteady thin film stretching problem to the case of heat transfer and Chen [24] explored the heat transfer in power law fluid. In all the above studies [20–24], tacit assumptions were (i) *the film thickness is uniform at the onset of stretching and* (ii) *the thickness reduces uniformly during the entire process along the stretching direction.*

In this paper we analyze the motion of a film flow with non-planar film thickness at the onset of stretching and the film thickness is allowed to vary with space and time during the process. We have solved the Navier-Stokes equations analytically by *matched asymptotic method* without searching for a similarity transformation.

2. Formulation

We consider a non-planar liquid film on a flat sheet as shown in figure 1. The x-axis is chosen along the plane of the sheet and the z-axis is taken normal to the plane. We assume that the surface at $z = 0$ starts stretching impulsively from rest with stretching rate xf_0, f_0 being constant with dimension of $[\text{time}]^{-1}$. Further we assume that the end effects and gravity are negligible and the film thickness $h(x,t)$ is known at time $t = 0$, u and w are the velocity components along x and z directions respectively and p is the pressure. Due to impulsive stretching, the inertial force causes the fluid to move along its own plane. At the initial stage, this motion is imparted from the plane to the adjacent fluid layer and then gradually spreads out to the entire depth of the film by viscosity. As time increases, the fluid continues to flow in the outward direction and the thickness of the film gradually decreases resulting into an increase of viscous resistance so as to balance the impulsive inertial force. At this stage the Reynolds number Re $(= u_0h_0/v)$ is of $O(1)$ and the balance of aforesaid forces defines a characteristic time scale t_c given by

$$t_c = v/h_0^2 f_0^2, \tag{1}$$

where h_0 and v denote the initial film thickness of the liquid film at $x = 0$ and kinematic viscosity of the fluid respectively. The characteristic velocity u_0 is defined as (L/t_c), where L is the characteristic horizontal length scale of the film. The ratio of the two length scales viz. $h_0/L(= \varepsilon)$ is assumed to be small but finite. Assuming that no extra shear acts on the film surface due to the overlying gas and

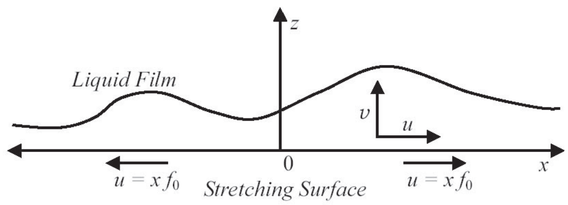

Figure 1. Schematic flow diagram

using the dimensionless variables as

$$X = x/L,\ Z = z/h_0,\ H = h/h_0,\ T = t/t_c,\ U = u/u_0, \\ W = w/\varepsilon u_0,\ P = ph_0^2 t_c/\rho v L^2, \tag{2}$$

we obtain the non-dimensional equations of motion and continuity as:

$$\varepsilon \mathrm{Re}\left(\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + W\frac{\partial U}{\partial Z}\right) = -\frac{\partial P}{\partial X} + \varepsilon^2\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2}, \tag{3a}$$

$$\varepsilon^3 \mathrm{Re}\left(\frac{\partial W}{\partial T} + U\frac{\partial W}{\partial X} + W\frac{\partial W}{\partial Z}\right) = -\frac{\partial P}{\partial Z} + \varepsilon^4\frac{\partial^2 W}{\partial X^2} + \varepsilon^2\frac{\partial^2 W}{\partial Z^2}, \tag{3b}$$

$$\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} = 0. \tag{3c}$$

The corresponding initial and boundary conditions are:

At the initial stage ($T = 0$),

$$U(0, X, Z) = W(0, X, Z) = 0,\ H(0, X) = \delta(X),\ H_T(0, X) = 0. \tag{4}$$

On the plane ($Z = 0$),

$$U(T, X, 0) = aX,\ W(T, X, 0) = 0, \tag{5}$$

where $a(= f_0 t_c)$ is the measure of the impulsive stretching strength.

At the free surface ($Z = H(X, T)$),

$$-\left\{1 + \varepsilon^2\left(\frac{\partial H}{\partial X}\right)^2\right\}P \\ + 2\varepsilon^2\left\{\varepsilon^2\frac{\partial U}{\partial x}\left(\frac{\partial H}{\partial X}\right)^2 + \frac{\partial W}{\partial Z} - \frac{\partial U}{\partial Z}\frac{\partial H}{\partial X} - \varepsilon^2\frac{\partial W}{\partial X}\frac{\partial H}{\partial X}\right\} = 0, \tag{6a}$$

$$2\varepsilon^2\frac{\partial H}{\partial X}\left(\frac{\partial W}{\partial Z} - \frac{\partial U}{\partial X}\right) + \left(\frac{\partial U}{\partial Z} + \varepsilon^2\frac{\partial W}{\partial X}\right)\left\{1 - \varepsilon^2\left(\frac{\partial H}{\partial X}\right)^2\right\} = 0, \tag{6b}$$

$$\frac{\partial H}{\partial T} = -U\frac{\partial H}{\partial X} + W. \tag{6c}$$

Equations (6a), (6b) and (6c) represent the conditions for vanishing of normal stress, shear stress and the kinematic condition at the free surface respectively.

3. Asymptotic analysis

In order to obtain an asymptotic solution of the problem described by the equations (3)-(6), the dependent variables in the system are expanded in powers of ε as follows:

$$(U_L, W_L, P_L) \approx (U_{L_0}, W_{L_0}, P_{L_0}) + \varepsilon(U_{L_1}, W_{L_1}, P_{L_1}) + O(\varepsilon^2), \tag{7}$$

were subscript 'L' stands for long-time-scale analysis. Inserting (7) into the system of equations (3)–(6b) and equating terms of like orders, we obtain several sets of linear equations as usual. Solving the same up to $O(\varepsilon)$, we get

$$U_L = aX + \varepsilon \mathrm{Re} a^2 X\left(\frac{1}{2}Z^2 - HZ\right), \tag{8}$$

$$W_L = -aZ + \varepsilon \mathrm{Re} a^2\left(-\frac{1}{6}Z^3 + \frac{1}{2}\frac{\partial(XH)}{\partial X}Z^2\right), \tag{9}$$

and $P_L = 0$.

It is to be noted here that U_L and W_L represented by (8) and (9) respectively do not satisfy the initial conditions (4) as they represent the long time solution.

Using equations (8) and (9) in condition (6c), we can obtain the long time evolution equation for H correct up to $O(\varepsilon)$ as

$$(H_L)_T + \left[aXH_L - \frac{\varepsilon \mathrm{Re} a^2}{3}XH_L^3\right]_X = 0. \tag{10}$$

To solve the equation (10), we expand H_L in powers of ε as in equation (7) and then using the method of characteristics, we obtain

$$H_L = C_0 e^{-aT} + \varepsilon \mathrm{Re} a\left(C_1 e^{-aT} + \frac{1}{3}C_0^3 e^{-3aT}\right), \tag{11}$$

along with,

$$X = Ce^{aT}. \tag{12}$$

The constants C, C_0 and C_1 are to be determined by matching the short time scale solutions of the transient film profile with the long-term solution mentioned above. The short time scale analysis can be done by just stretching the temporal coordinates as $\tau = T/\varepsilon$ and keeping other variables same as stated in equation (2). Under this temporal transformation, equations (3a) and (6c) respectively reduce to

$$\mathrm{Re}\frac{\partial U}{\partial \tau} + \varepsilon \mathrm{Re}\left(U\frac{\partial U}{\partial X} + W\frac{\partial U}{\partial Z}\right) = -\frac{\partial P}{\partial X} + \varepsilon^2\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2}, \tag{13a}$$

and

$$\frac{\partial H}{\partial \tau} = \varepsilon \left(-U \frac{\partial H}{\partial X} + W \right). \tag{13b}$$

Continuity equation (3c) and all the associated initial and boundary conditions (4)–(6b) remain the same under this transformation. Expanding the newly scaled dependent variables again in powers of ε (cf. (7)) and substituting them in short time scale equations (13) and associated initial and boundary conditions, we get separate set of equations of like orders. At the leading order, solutions of equations (13b) and (13a) give

$$H_{s_0} = \delta(X), \tag{14a}$$

$$U_{s_0} = aX \left[1 - 2 \sum_{n>0} \frac{\sin(\lambda_n Z/\delta)}{\lambda_n} \exp(-\lambda_n^2 \tau / \mathrm{Re}\delta^2) \right]. \tag{14b}$$

Inserting (14b) into the continuity equation (3c) and integrating with respect to Z and using the boundary condition (5) on W, we obtain .

$$W_{s_0} = -aZ - 2a \frac{\partial}{\partial X} \left[\delta X \sum_{n>0} \frac{\cos(\lambda_n Z/\delta) - 1}{\lambda_n^2} \exp(-\lambda_n^2 \tau / \mathrm{Re}\delta^2) \right]. \tag{14c}$$

It is to be noted here that transformed version of equation (3b) along with (6a) gives pressure $P = 0$ up to $O(\varepsilon)$. Finally we obtain the transient film profile up to $O(\varepsilon)$ as

$$\begin{aligned} H_s = {} & \delta(X) - a \frac{\partial (X\delta)}{\partial X} T \\ & - 2a\varepsilon \mathrm{Re} \left\{ \frac{\partial}{\partial X} \left[\delta^3 X \sum_{n>0} \frac{1}{\lambda_n^4} (\exp(-\lambda_n^2 \tau / \mathrm{Re}\delta^2) - 1) \right] \right. \\ & \left. + \frac{\partial \delta}{\partial X} X \delta^2 \sum_{n>0} \frac{(-1)^{n-1}}{\lambda_n^3} (\exp(-\lambda_n^2 \tau / \mathrm{Re}\delta^2) - 1) \right\} \end{aligned} \tag{15}$$

where, $\lambda_n = (2n-1)\pi/2$. Here subscript s stands for the short-time scale solution.

The matching condition that is derived from the requirement that the flow is continuous from the start of stretching to all succeeding time suggests

$$\lim_{\tau \to \infty} H_s(\tau) = \lim_{T \to 0} H_L(T),$$

which implies

$$C = X(0) = \xi \text{ (Say)},\ C_0 = H_L(\xi, 0) = \delta(\xi) \text{ and } C_1 = \frac{3}{2} \xi \delta^2 \delta_\xi. \tag{16}$$

The composite solution that possesses features of both the time scales is given by

$$
\begin{aligned}
H^C &= \delta(\xi)e^{-aT} - 4a\xi\delta_\xi T\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}e^{-\lambda_n^2 T/\delta^2\varepsilon\mathrm{Re}} \\
&+ \varepsilon\mathrm{Re}a\left[\frac{3}{2}\xi\delta^2\delta_\xi\left(e^{-aT}-1\right)+\frac{\delta^3}{3}\left(e^{-3aT}-1\right)\right] \\
&- 2\varepsilon\mathrm{Re}a\sum_{n>0}\left[\frac{1}{\lambda_n^4}\left(e^{-\lambda_n^2 T/\varepsilon\mathrm{Re}\delta^2}\right)\left\{\delta^3+\left(3+\lambda_n(-1)^{n-1}\right)\xi\delta^2\delta_\xi\right\}\right], \qquad (17)
\end{aligned}
$$

where $\delta(\xi)$ is the initial distribution of the liquid film and ξ is the initial lateral position. In arriving at equation (16) we have used

$$2\sum_{n=1}\frac{1}{\lambda_n^4}=\frac{1}{3} \text{ and } 2\sum_{n=1}\frac{(-1)^{n-1}}{\lambda_n^3}=\frac{1}{2}.$$

Here we like to point out that Dandapat et al.[25] have also used the same method to derive the composite height in connection with the development of thin film on a rotating disk under axisymmetric case. It is to be noted here that the present result is derived for asymmetrical profile.

4. Results and discussion

We begin our presentation first by focusing on the general features that can be drawn from the present analysis. For large T, the short time solutions for U_s and W_s give the leading order solutions of the long time solutions for U_L and W_L, respectively, as aX and $-aZ$. From equation (8) it can be seen that U depends on X linearly up to the first order term, but does not depend on Z for the leading order term. The fact that $H_{L_0}X$ = constant along $X = \xi e^{aT}$ gives the equation (9) in more simple form as

$$W_L = -aZ - \varepsilon\mathrm{Re}a^2\frac{1}{6}Z^3, \qquad (18)$$

for the first order approximation and is independent of X along the curve $X = \xi e^{aT}$.

Again, the short time solution (15) indicates that the inertial effect resists film from stretching, because the terms proportional to $e^{-\lambda_n^2 T/H_0^2\varepsilon\mathrm{Re}}-1$ are very small at the beginning. The composite solution has both the features viz. initially it behaves as short time solution whereas at subsequent times it behaves as long time solution.

In figures from 2a to 2d, we have plotted the transient liquid film thickness for various kinds of initial film profiles taking $a = 1$. As it can be seen from these figures, in most cases, the film profile becomes planar around $T = 2.5$ (This is true for the asymmetrical case, figures 2b as well). The transient film thickness

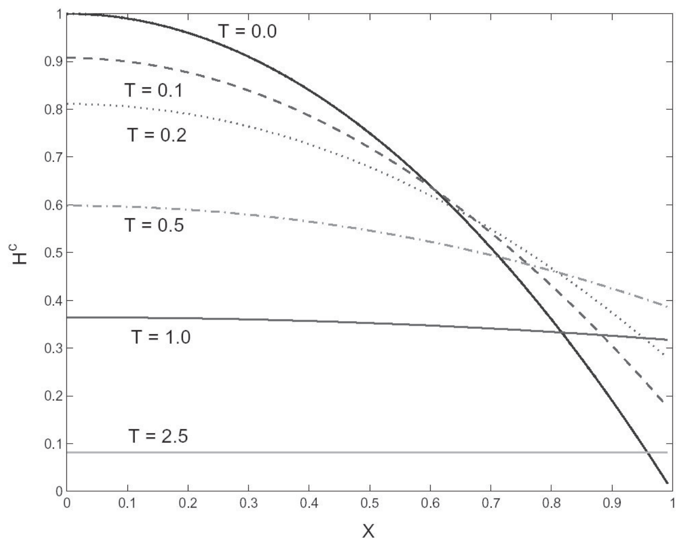

Figure 2. a) Variation of composite height H^C with X for several time steps. Here $\delta = 1 - \xi^2$, $\varepsilon\mathrm{Re} = 0.1$. $a = 1$.

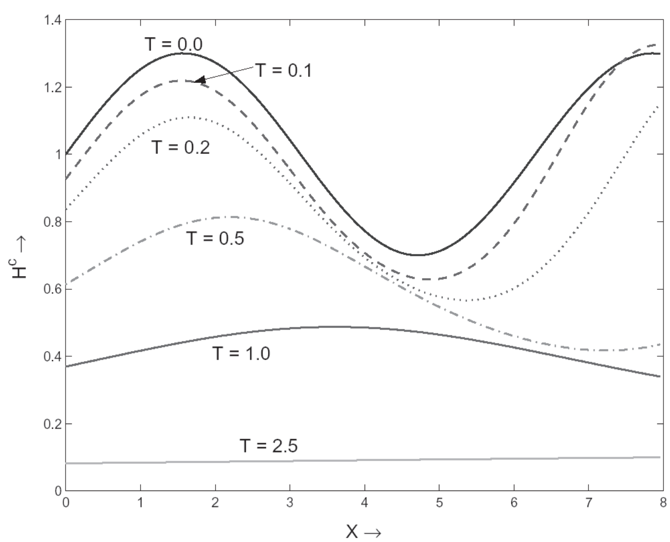

Figure 2. b) Variation of composite height H^C with X for several time steps. Here $\delta = 1 + 0.3\sin(\xi)$, $\varepsilon\mathrm{Re} = 0.1$. $a = 1$.

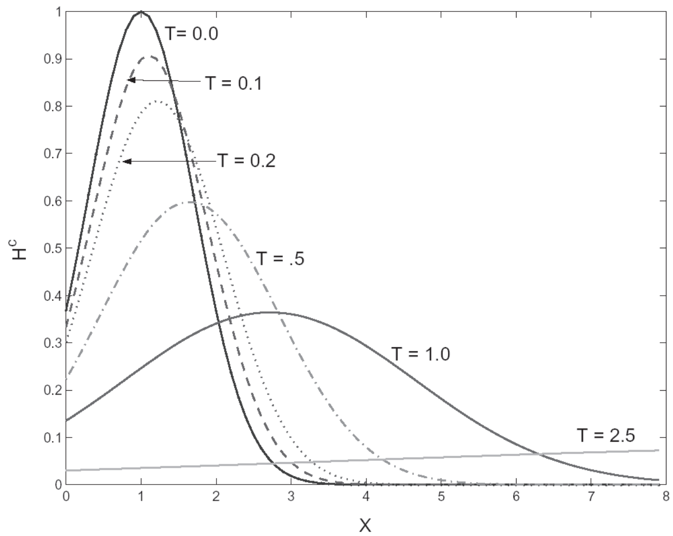

Figure 2. c) Variation of composite height H^C with X for several time steps. Here $\delta = \exp(-(\xi - 1)^2)$, $\varepsilon\mathrm{Re} = 0.1$. $a = 1$.

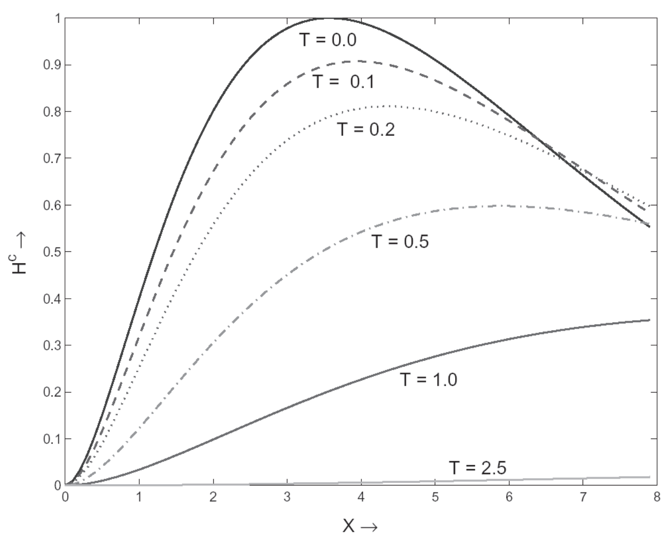

Figure 2. d) Variation of composite height H^C with X for several time steps. Here $\delta = 1.089\xi^2 \exp(-\xi^{0.76})$, $\varepsilon\mathrm{Re} = 0.1$. $a = 1$.

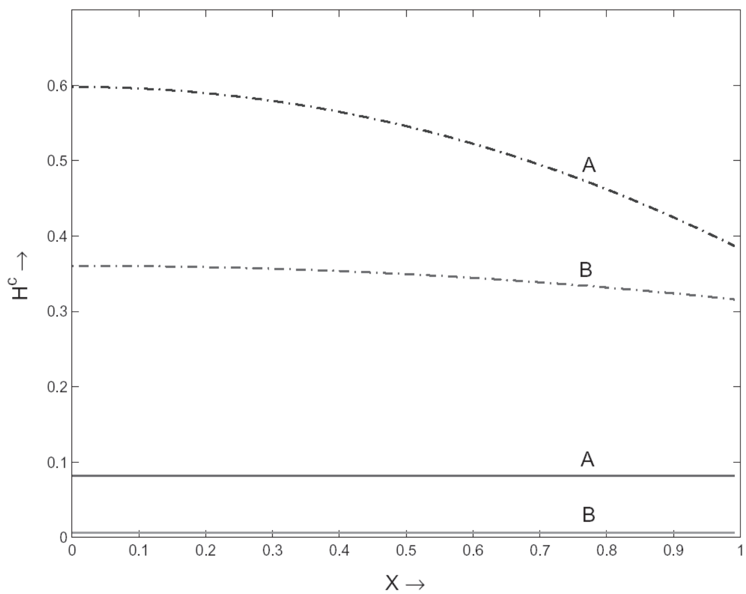

Figure 3. Variation of composite height H^C with X at different times ($T = 0.5$ dash-dot and $T = 2.5$, solid lines) for $a = 1$ (marked A) and for $a = 1.5$ (marked B) . $\varepsilon\mathrm{Re} = 0.1$.

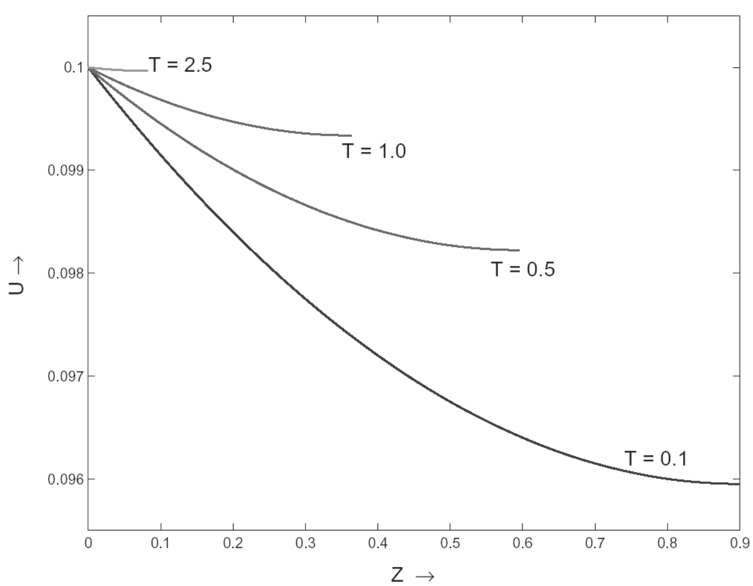

Figure 4. U Versus Z at $X = 0.1$ for different time steps. $\delta = 1 - \xi^2$, $\varepsilon\mathrm{Re} = 0.1$. $a = 1$.

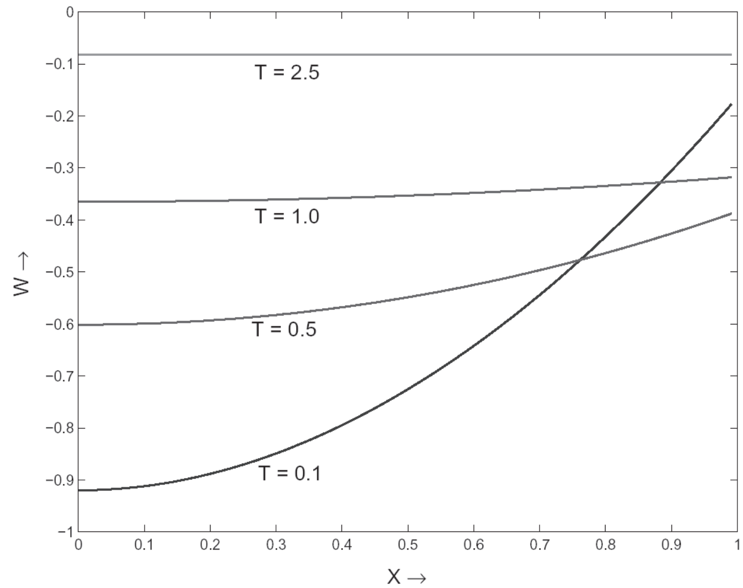

Figure 5. W Versus X at $Z = H$ for various time steps. $\varepsilon\mathrm{Re} = 0.1$. $a = 1$.

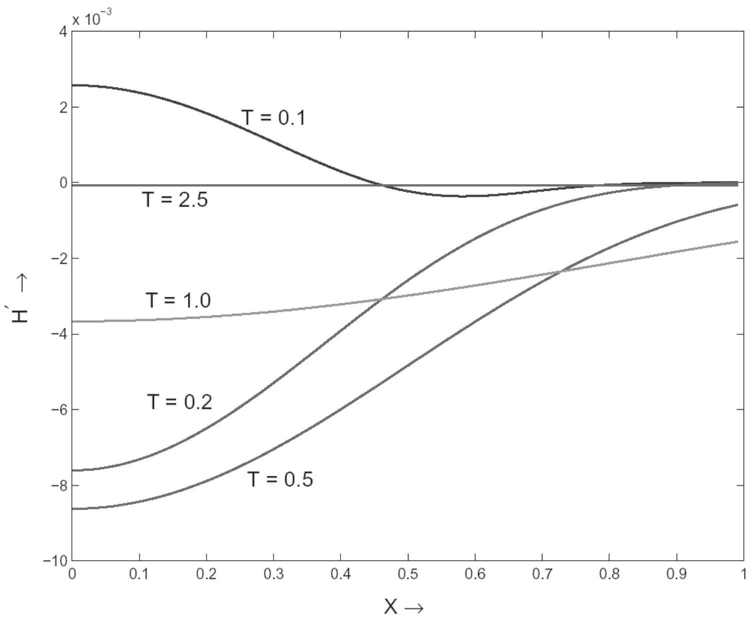

Figure 6. $H'(= H^C - \delta e^{-aT})$ Versus X at different time steps. $a = 1$

is about 0.08 at this time. This can be estimated by inserting $T = 2.5$ into the leading order solution (equation (11)), i.e. e^{-aT} (note, here we have taken $a = 1$), which give 0.0821. For those that possess very thin initial film thickness at $X = 0$ (Figures 2c and 2d), small slopes can be seen even at $T = 2.5$, but we can expect from the figures that these slopes eventually diminish as well.

Figure 3 shows the variation of H^C with X for different values of a at different time stage. It is clear from the graph that as a increases H^C decreases, implying faster stretching causes quicker thinning of the film.

In order to see the dynamics occurring in these film flows, we take $\delta(\xi) = 1 - \xi^2$ as a representative initial film thickness and we take a close look for its velocity distributions and additional effects on transient film thickness. In figure 4, U is plotted for various time steps against Z at $X = 0.1$ for the initial distribution $\delta(\xi) = 1 - \xi^2$. The inertial effects evident from the figure are as follows. As Z increases the rate of change in U decreases in the initial stages. U eventually becomes constant throughout all the regions of the film at $T = 2.5$ and the film thickness reduces to $Z = 0.0821$.

Figure 5, depicts W against X for various time steps at the free surface $Z = H$. The velocity distribution becomes flattened as time elapses and the film profile becomes flattened as well. From equation (9) we obtain $Z = C_0 e^{-aT}$ at the interface, and comparing the result with equation (11), we see that at the leading order, we can estimate $W \approx -aH$ This is also evident by comparing the figure 5 with fig. 2a for $a = 1$.

In figure 6, $H' \equiv H^C - \delta e^{aT}$ is plotted as a function of time taking $a = 1$. This result reveals how the secondary effects affect the transient film thickness. Inertial effect is evident from this figure as H' takes positive value initially, implying resistance on film thinning. The contribution of this term eventually diminishes as time elapses (around $T = 2.5$) and ultimately we obtain a uniform thin film.

5. Conclusion

Previous works on stretching sheet problem were solved by the use of similarity transformation on boundary layer equations, but in the present study, we have employed both the method of matched asymptotic expansions and the method of characteristics to solve the full N-S equations. It has been shown, how a transient non-planar film thickness changes subsequently to a planar film in due course of time. Further it is observed that a faster stretching causes quicker thinning of the film on the stretching surface.

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