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Cesàro α-Integrability and Laws of Large Numbers I

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For a sequence of integrable random variables, we introduce a new set of conditions called Cesàro α-Integrability and Strong Cesàro α-Integrability and show that, for $\alpha < \frac{1}{2}$, these conditions that are strictly weaker than Cesàro Uniform Integrability and Strong Cesàro Uniform Integrability respectively, are sufficient for WLLN and SLLN to hold for a sequence of pairwise independent random variables. For some special kinds of dependent sequences of random variables also, Cesàro α-integrability for appropriate α is shown to be sufficient for WLLN to hold.

KEY WORDS: Uniform integrability; pairwise independent; ϕ-mixing sequence.

1. INTRODUCTION

In recent years, a uniform integrability condition of some kind has played an increasingly important role as a key condition in proving laws of large numbers for a sequence of random variables. In Landers–Rogge,(6) for example, the authors considered a sequence of pairwise independent random variables and proved that if the sequence is uniformly integrable, then WLLN (weak law of large numbers) holds and that if the sequence is strongly uniformly integrable, then SLLN (strong law of large numbers) holds. Chandra(2) introduced the notion of uniform integrability in the Cesàro sense or what has now come to be known as Cesàro Uniform Integrability (CUI, in short). Since then, there has been a series of papers establishing that CUI, instead of the stronger condition of uniform

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integrability, is just the correct condition in the context of laws of large numbers. Chandra[2] and later Chandra–Goswami[3] improved the above-mentioned result of Landers–Rogge.[6] They showed that for a sequence of pairwise independent random variables, CUI is sufficient for WLLN to hold and Strong Cesàro Uniform Integrability (SCUI, in short) is sufficient for SLLN to hold. Recently, Landers–Rogge[7] have obtained a slight improvement over the results of Chandra[2] and Chandra–Goswami[3] for the case of non-negative random variables. They have shown that in this case, the condition of paiwise independence can be replaced by the weaker assumption of pairwise non-positive correlation.

In this paper, we introduce a new set of conditions to be called Cesàro α-Integrability (CI(α), in short) and Strong Cesàro α-Integrability (SCI(α), in short) for a sequence of random variables. We show that CI(α) for *any* α is strictly weaker than CUI, while SCI(α) for *any* α is strictly weaker than SCUI (Lemma 2.1 and Example 2.2).

Our main result in Section 2 is Theorem 2.2, which shows that, for a sequence of pairwise independent random variables, the conditions CI(α) and SCI(α) for *some* $\alpha < \frac{1}{2}$, are sufficient respectively for WLLN and for SLLN to hold. Theorem 2.1, where we consider the special case of non-negative random variables, comes as an intermediate step, but, in its own right, it is an improvement over Theorem 2 of Landers–Rogge.[7]

In Section 3, we consider certain special kinds of dependent sequences of random variables with suitable dependence structure and prove the condition CI(α) for appropriate α to be sufficient for WLLN to hold for such sequences as well. Theorems 3.1 and 3.4 here are improvements over Theorems 4 and 5 of Chandra[2].

2. DEFINITIONS AND MAIN THEOREM IN THE INDEPENDENT CASE

For the convenience of the reader, we start with the definitions of the various existing concepts of uniform integrability.

Definition 1. Let $\Phi = \{\phi: (0, \infty) \to (0, \infty) \mid \frac{\phi(t)}{t} \uparrow \infty \text{ as } t \uparrow \infty\}$ and $\Phi_s = \{\phi \in \Phi : \sum_{n=1}^{\infty} \frac{1}{\phi(n)} < \infty\}$. For $\phi \in \Phi$, we put $\phi(0) = 0$.

A sequence $\{X_n, n \geq 1\}$ of random variables is called

(i) *Uniformly Integrable* (UI, in short), respectively *Strongly Uniformly Integrable* (SUI, in short), if there exists $\phi \in \Phi$, respectively $\phi \in \Phi_s$, such that

$$\sup_{n \geq 1} E[\phi(|X_n|)] < \infty. \tag{1}$$

(ii) *Cesàro Uniformly Integrable* (CUI, in short), respectively *Strongly Cesàro Uniformly Integrable* (SCUI, in short), if there exists $\phi \in \Phi$, respectively $\phi \in \Phi_s$, such that

$$\sup_{n \geq 1} \frac{1}{n} \sum_{i=1}^{n} E[\phi(|X_i|)] < \infty. \tag{2}$$

The notion of Uniform Integrability is classical and is usually defined through the following equivalent condition

$$\sup_{n \geq 1} E[|X_n| \, I_{\{|X_n| > \lambda\}}] \to 0 \qquad \text{as} \quad \lambda \to \infty. \tag{3}$$

The notion of Cesàro Uniform Integrability was first introduced in Chandra[2] and was defined through the following equivalent condition

$$\sup_{n \geq 1} \frac{1}{n} \sum_{i=1}^{n} E[|X_i| \, I_{\{|X_i| > \lambda\}}] \to 0 \qquad \text{as} \quad \lambda \to \infty. \tag{4}$$

The equivalence of (1) for $\phi \in \Phi$ and (3) is a well-known result of La Vallée Poussin, while the equivalence of (2) for $\phi \in \Phi$ and (4) was proved in Chandra–Goswami.[3] The advantage of formulations (3) and (4) for UI and CUI respectively, is that they are stated explicitly in terms of the marginal distributions of the random variables X_n and do not depend on looking for an appropriate function $\phi \in \Phi$ such that condition (1), respectively condition (2), holds. In general, one may not have any clue as to how to go about getting hold of such a function ϕ.

For the notions of SUI and SCUI, however, there does not seem to exist any equivalent formulations stated directly in terms of the marginal distributions of the X_n. The only way, therefore, to verify that a given sequence $\{X_n, n \geq 1\}$ is SUI, respectively SCUI, is to prove that condition (1), respectively condition (2), holds for some function $\phi \in \Phi_s$.

We now introduce two new concepts which, just like the above concepts, are again related to the tail probabilities of the random variables $\{X_n\}$, but in a different way.

Definition 2. Let $\alpha \in (0, \infty)$. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be *Cesàro α-Integrable* (CI(α), in short) if the following two conditions hold:

$$\sup_{n \geq 1} \frac{1}{n} \sum_{i=1}^{n} E[|X_i|] < \infty \qquad \text{and} \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E[|X_i| \, I_{\{|X_i| > i^\alpha\}}] = 0. \tag{5}$$

The sequence $\{X_n, n \geqslant 1\}$ is said to be *Strongly Cesàro α-Integrable* (SCI(α), in short) if

$$\sup_{n\geqslant 1} \frac{1}{n}\sum_{i=1}^{n} E[|X_i|] < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n} E[|X_n|\, I_{\{|X_n| > n^\alpha\}}] < \infty. \tag{6}$$

An easy application of Kronecker's Lemma shows that (6) is indeed stronger than (5). It is also clear from the definition that CI(α), respectively SCI(α), for some $\alpha > 0$ implies CI(β), respectively SCI(β), for all $\beta > \alpha$. We now show that CUI implies CI(α) for all $\alpha > 0$ and that similarly SCUI implies SCI(α) for all $\alpha > 0$.

Lemma 2.1. A CUI sequence $\{X_n, n \geqslant 1\}$ of random variables is CI(α) for all $\alpha > 0$. If, moreover, the sequence is SCUI, then it is SCI(α) for all $\alpha > 0$.

Proof. If $\{X_n, n \geqslant 1\}$ is CUI, then using (4), $\sup_{n\geqslant 1} \frac{1}{n}\sum_{i=1}^{n} E[|X_i|\, I_{\{|X_i| > \lambda\}}] < 1$ for some λ with $0 < \lambda < \infty$. It follows that $\sup_{n\geqslant 1} \frac{1}{n}\sum_{i=1}^{n} E[|X_i|] \leqslant \lambda + 1 < \infty$. Further, if $\{X_n, n \geqslant 1\}$ is CUI, then there exists a function $\phi \in \Phi$, such that (2) holds. Using the fact that $\frac{\phi(t)}{t}$ is increasing in t, we obtain that for every $n \geqslant 1$ and for $\alpha > 0$,

$$E[|X_n|\, I_{\{|X_n| > n^\alpha\}}] \leqslant \frac{n^\alpha}{\phi(n^\alpha)} E[\phi(|X_n|)].$$

It follows that, for every $n \geqslant 1$ and $\alpha > 0$,

$$\frac{1}{n}\sum_{i=1}^{n} E[|X_i|\, I_{\{|X_i| > i^\alpha\}}] \leqslant \frac{1}{n}\sum_{i=1}^{n} \frac{i^\alpha}{\phi(i^\alpha)} E[\phi(|X_i|)].$$

Since $\sup_{n\geqslant 1} \frac{1}{n}\sum_{i=1}^{n} E[\phi(|X_i|)] < \infty$ and $\frac{n^\alpha}{\phi(n^\alpha)} \to 0$ as $n \to \infty$, one gets [see Remark (i) below]

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E[|X_i|\, I_{\{|X_i| > i^\alpha\}}] = 0.$$

This proves that $\{X_n, n \geqslant 1\}$ is CI(α).

In case $\{X_n, n \geqslant 1\}$ is SCUI, then (2) actually holds for a function $\phi \in \Phi_s$. But if $\phi \in \Phi_s$, then it is easy to see that for any $0 < \alpha \leqslant 1$, the series

$\sum_{n=1}^{\infty} \frac{1}{n^{1-\alpha}\phi(n^\alpha)}$ is also convergent [see Remark (ii) below]. Now for any $0 < \alpha \leqslant 1$, we have, as before,

$$\sum_{n=1}^{\infty} \frac{1}{n} E[|X_n| I_{\{|X_n| > n^\alpha\}}] \leqslant \sum_{n=1}^{\infty} \frac{1}{n^{1-\alpha}\phi(n^\alpha)} E[\phi(|X_n|)].$$

By applying Remark 3(ii) of Landers–Rogge[7] and using (2) and the above-mentioned consequence of $\phi \in \Phi_s$, it follows that the series on the right-hand-side of the above inequality converges, thus proving that $\{X_n, n \geqslant 1\}$ is SCI(α) for all $0 < \alpha \leqslant 1$, and hence for all $\alpha > 0$. □

Remark. (i) If $\{a_n\}$ and $\{b_n\}$ are two sequences of non-negative reals such that $\lim_{n\to\infty} a_n = 0$ and $\sup_n \frac{1}{n}\sum_{i=1}^{n} b_i < \infty$, then it is easy to see that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i b_i = 0$.

(ii) If $\phi \in \Phi_s$, then $\sum_{n=1}^{\infty} \frac{1}{n^{1-\alpha}\phi(n^\alpha)} < \infty$, for any $0 < \alpha \leqslant 1$. For $\alpha = 1$, there is nothing to prove. For $0 < \alpha < 1$, it just follows using the integral test:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1-\alpha}\phi(n^\alpha)} \leqslant \frac{1}{\phi(1)} + \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{1}{x^{1-\alpha}\phi(x^\alpha)}\,dx = \frac{1}{\phi(1)} + \int_1^\infty \frac{1}{x^{1-\alpha}\phi(x^\alpha)}\,dx$$
$$= \frac{1}{\phi(1)} + \alpha^{-1}\int_1^\infty \frac{1}{\phi(y)}\,dy \leqslant \frac{1}{\phi(1)} + \alpha^{-1}\sum_{n=1}^{\infty} \frac{1}{\phi(n)} < \infty.$$

To get a feel for the conditions CI(α) and SCI(α), here is a simple example.

Example 2.1. Let $\{X_n, n \geqslant 1\}$ be a sequence of exponentially distributed random variables with expectations λ_n^{-1}. Then for any $\alpha > 0$, easy computation shows that for every $n \geqslant 1$,

$$E[|X_n| I_{\{|X_n| > n^\alpha\}}] = e^{-\lambda_n n^\alpha}\left(n^\alpha + \frac{1}{\lambda_n}\right).$$

It can be easily verified that if, for example, $\lambda_n = \beta n^{-\alpha}\log n$, then the sequence $\{X_n, n \geqslant 1\}$ is CI(α) if and only if $\beta > \alpha$ and in that case it is actually SCI(α).

Lemma 2.1 should already indicate that a sequence $\{X_n, n \geqslant 1\}$ may be CI(α), respectively SCI(α), for *some* $\alpha > 0$, and yet fail to be CUI, respectively SCUI. Here is an example that confirms this.

Example 2.2. Fix any $0 < \alpha \leqslant 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables with X_n taking the values 0, n^α and n with probabilities $1 - n^{-\alpha}$, $n^{-\alpha} - n^{-2}$ and n^{-2} respectively.

We first show that this sequence is CI(α); in fact, it is SCI(α). First of all, $E[|X_n|] = 1 + \frac{1}{n^2}(n - n^\alpha)$. Since $\frac{1}{n}\sum_{i=1}^n \frac{1}{i^2}(i - i^\alpha) \leqslant \frac{1}{n}\sum_{i=1}^n \frac{1}{i} \sim \frac{\log n}{n} \to 0$ as $n \to \infty$, it follows that $\sup_{n\geqslant 1} \frac{1}{n}\sum_{i=1}^n E[|X_i|] < \infty$. Also, $\sum_{n=1}^\infty \frac{1}{n} E[|X_n| \, I_{\{|X_n| > n^\alpha\}}] \leqslant \sum_{n=1}^\infty \frac{1}{n^2} < \infty$, thus completing the proof that $\{X_n, n \geqslant 1\}$ is SCI(α).

We now show that $\{X_n, n \geqslant 1\}$ is not even CUI. Indeed, we show that for any $\lambda > 0$, $\sup_{n\geqslant 1} \frac{1}{n}\sum_{i=1}^n E[|X_i| \, I_{\{|X_i| > \lambda\}}] \geqslant 1$. Given any $\lambda > 0$, $\exists n_0 = n_0(\lambda)$ such that $n_0^\alpha \leqslant \lambda < (n_0+1)^\alpha$. But then, for any $i \geqslant n_0$, $E[|X_i| \, I_{\{|X_i| > \lambda\}}] = E[X_i] = 1 + \frac{1}{i^2}(i - i^\alpha) \geqslant 1$. It follows that for $n \geqslant n_0$, $\frac{1}{n}\sum_{i=1}^n E[|X_i| \, I_{\{|X_i| > \lambda\}}] \geqslant \frac{n - n_0}{n}$, whence one obtains that $\sup_{n\geqslant 1} \frac{1}{n}\sum_{i=1}^n E[|X_i| \, I_{\{|X_i| > \lambda\}}] \geqslant 1$.

Remark. If we slightly modify the random variables in Example 2.2 as follows: $X_1 \equiv 1$ and, for $n \geqslant 2$, X_n takes the values 0, $\log n$ and n with probabilities $1 - (\log n)^{-1}$, $(\log n)^{-1} - n^{-2}$ and n^{-2} respectively, we get an example of a sequence $\{X_n, n \geqslant 1\}$ which is SCI(α) for *all* $\alpha > 0$ but not CUI.

We are now ready to proceed to our main results showing sufficiency of CI(α), respectively SCI(α), for some $\alpha \in (0, \frac{1}{2})$ (instead of CUI, respectively SCUI) for weak law of large numbers, respectively strong law of large numbers, to hold. The first theorem concerns non-negative random variables with non-positive correlation and, in view of Lemma 2.1 and Example 2.2, is an improvement over Theorem 2 of Landers–Rogge.[7]

Theorem 2.1. Let $\{X_n, n \geqslant 1\}$ be a sequence of non-negative random variables satisfying $E[X_i X_j] \leqslant E[X_i]\,E[X_j]$ $\forall\, i \neq j$. Let $S_n = \sum_{i=1}^n X_i$.

(a) If $\{X_n, n \geqslant 1\}$ is CI(α) for some $\alpha \in (0, \frac{1}{2})$, then

$$\frac{S_n - E[S_n]}{n} \to 0 \quad \text{as } n \to \infty \quad \text{in } L_1 \text{ and hence in probability.}$$

(b) If $\{X_n, n \geqslant 1\}$ is SCI(α) for some $\alpha \in (0, \frac{1}{2})$, then

$$\frac{S_n - E[S_n]}{n} \to 0 \qquad \text{as} \quad n \to \infty \text{ almost surely.}$$

Proof. (a) For each $n \geqslant 1$, let $Y_n = X_n I_{\{X_n \leqslant n^\alpha\}}$ and $T_n = \sum_{i=1}^n Y_i$. It suffices to prove that

$$\frac{1}{n}(S_n - T_n) \to 0 \qquad \text{in } L_1, \qquad \frac{1}{n}(E[S_n] - E[T_n]) \to 0 \tag{7}$$

$$\frac{1}{n}(T_n - E[T_n]) \to 0 \qquad \text{in } L_1. \tag{8}$$

We first show (7). Since the X_n, $n \geqslant 1$ are non-negative, we have

$$E\left[\left|\frac{1}{n}(S_n - T_n)\right|\right] = E\left[\frac{1}{n}(S_n - T_n)\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i I_{\{X_i > i^\alpha\}}],$$

and the last expression above goes to 0 as $n \to \infty$, since the sequence $\{X_n, n \geqslant 1\}$ is CI(α).

Turning now to (8), we actually show convergence to 0 in L_2. Since

$$0 \leqslant E\left[\frac{1}{n^2}(T_n - E[T_n])^2\right] \leqslant \frac{1}{n^2}\sum_{i=1}^{n} E[Y_i^2] + \frac{1}{n^2}\sum_{i \neq j}^{n} (E[Y_i Y_j - E[Y_i]\, E[Y_j]]),$$

it suffices to show that

$$\frac{1}{n^2}\sum_{i=1}^{n} E[Y_i^2] \to 0, \tag{9}$$

and,

$$\limsup_{n\to\infty} \frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} (E[Y_i Y_j - E[Y_i]\, E[Y_j]]) \leqslant 0. \tag{10}$$

To obtain (9), we note that

$$\frac{1}{n^2}\sum_{i=1}^{n} E[Y_i^2] = \frac{1}{n^2}\sum_{i=1}^{n} E[X_i^2 I_{\{|X_i| \leqslant i^\alpha\}}] \leqslant \frac{1}{n^2}\sum_{i=1}^{n} i^{2\alpha},$$

which is of the order of $\frac{n^{2\alpha+1}}{n^2}$ and hence goes to 0 (as $n \to \infty$) since $\alpha < \frac{1}{2}$.

To prove (10) now, we use the facts that the X_n, $n \geqslant 1$ are non-negative and $E[X_i X_j] \leqslant E[X_i]\, E[X_j]$ for $i \neq j$ to obtain

$$\begin{aligned}
\frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} (E[Y_i Y_j - E[Y_i]\, E[Y_j]]) &\leqslant \frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} (E[X_i X_j - E[Y_i]\, E[Y_j]]) \\
&\leqslant \frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} (E[X_i]\, E[X_j] - E[Y_i]\, E[Y_j]) \\
&\leqslant \frac{1}{n^2}\sum_{i,j=1}^{n} (E[X_i]\, E[X_j] - E[Y_i]\, E[Y_j]) \\
&= \frac{1}{n^2}\sum_{i,j=1}^{n} (E[X_i] - E[Y_i])\, E[X_j] + \frac{1}{n^2}\sum_{i,j=1}^{n} E[Y_i](E[X_j] - E[Y_j])
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{n}\sum_{i=1}^{n} E[X_i I_{\{X_i > i^\alpha\}}]\right)\left(\frac{1}{n}\sum_{j=1}^{n} E[X_j]\right) \\
&\quad + \left(\frac{1}{n}\sum_{i=1}^{n} E[Y_i]\right)\left(\frac{1}{n}\sum_{j=1}^{n} E[X_j I_{\{X_j > j^\alpha\}}]\right) \\
&\leqslant 2\left(\frac{1}{n}\sum_{i=1}^{n} E[X_i I_{\{X_i > i^\alpha\}}]\right)\left(\frac{1}{n}\sum_{j=1}^{n} E[X_j]\right),
\end{aligned}
$$

and this last expression goes to 0 as $n \to \infty$ because $\{X_n, n \geqslant 1\}$ is CI(α).

(b) We define $Y_n = X_n I_{\{X_n \leqslant n\}}$ and $T_n = \sum_{i=1}^{n} Y_i$, for each $n \geqslant 1$. The argument proceeds essentially along the same steps as in the proof of Landers–Rogge,[7] Theorem 2(ii). We prove

$$
\frac{1}{n}(E[S_n] - E[T_n]) \to 0 \tag{11}
$$

$$
\sum_{n=1}^{\infty} P\{X_n \neq Y_n\} < \infty, \tag{12}
$$

and

$$
\frac{T_n - E[T_n]}{n} \to 0 \quad \text{almost surely.} \tag{13}
$$

These will imply that

$$
\frac{S_n - E[S_n]}{n} \to 0 \quad \text{almost surely.}
$$

For proving (11), we note that, since the X_n are non-negative,

$$
\left|\frac{E[S_n] - E[T_n]}{n}\right| = \frac{E[S_n] - E[T_n]}{n} = \frac{1}{n}\sum_{i=1}^{n} E[X_i I_{\{X_i > i\}}],
$$

which goes to 0 as $n \to \infty$, since $\{X_n\}$ is SCI(α) (and hence CI(α)) with $\alpha < \frac{1}{2}$.

Condition (12) follows because

$$
\sum_{n=1}^{\infty} P\{X_n \neq Y_n\} = \sum_{n=1}^{\infty} P\{X_n > n\} \leqslant \sum_{n=1}^{\infty} \frac{E[X_n I_{\{X_n > n\}}]}{n} \leqslant \sum_{n=1}^{\infty} \frac{E[X_n I_{\{X_n > n^\alpha\}}]}{n}
$$

and the last sum converges because $\{X_n, n \geqslant 1\}$ is SCI(α).

To verify (13), which is just SLLN for the sequence $\{Y_n\}$, we observe that, since $\sup_n \frac{1}{n}\sum_{i=1}^{n} E[Y_i] \leqslant \sup_n \frac{1}{n}\sum_{i=1}^{n} E[X_i] < \infty$, we have only to prove that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{i} \frac{1}{i^2} \rho_{ij} < \infty,$$

where $\rho_{ij} = [\operatorname{cov}(Y_i, Y_j)]^+$ [see, for example, Chandra–Goswami,[3] Theorem 1, and Chandra–Goswami[4] for its corrected proof].

First note that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} E[Y_n^2] &= \sum_{n=1}^{\infty} \frac{1}{n^2} E[X_n^2 I_{\{X_n \leqslant n^\alpha\}}] + \sum_{n=1}^{\infty} \frac{1}{n^2} E[X_n^2 I_{\{n^\alpha < X_n \leqslant n\}}] \\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} n^{2\alpha} + \sum_{n=1}^{\infty} \frac{1}{n} E[X_n I_{\{n^\alpha < X_n \leqslant n\}}] \\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{n^{2-2\alpha}} + \sum_{n=1}^{\infty} \frac{1}{n} E[X_n I_{\{X_n > n^\alpha\}}],\end{aligned}$$

and both the sums above converge—the first sum converges because $\alpha < \frac{1}{2}$, while the second converges because $\{X_n, n \geqslant 1\}$ is SCI(α).

This means that $\sum_{i=1}^{\infty} \frac{1}{i^2} \rho_{ii} < \infty$ and therefore it suffices for us to show that

$$\sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{1}{i^2} \rho_{ij} < \infty. \tag{14}$$

Since the X_i are non-negative with $E[X_i X_j] \leqslant E[X_i]\, E[X_j]\ \forall i \neq j$, we have, for $i \neq j$,

$$\begin{aligned}\rho_{ij} &\leqslant (E[X_i X_j] - E[Y_i]\, E[Y_j])^+ \leqslant E[X_i]\, E[X_j] - E[Y_i]\, E[Y_j] \\ &= (E[X_i] - E[Y_i])\, E[X_j] + E[Y_i](E[X_j] - E[Y_j]) \\ &\leqslant E[X_j]\, E[X_i I_{\{X_i > i\}}] + E[X_i]\, E[X_j I_{\{X_j > j\}}].\end{aligned}$$

To prove (14) therefore, it suffices to prove that

$$\sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{1}{i^2} E[X_j]\, E[X_i I_{\{X_i > i\}}] < \infty \quad \text{and} \quad \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \frac{1}{i^2} E[X_i]\, E[X_j I_{\{X_j > j\}}] < \infty.$$

The convergence of the first series follows from the fact that

$$\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{1}{i^2}E[X_j]\,E[X_iI_{\{X_i>i\}}]\leqslant\sum_{i=1}^{\infty}\frac{1}{i}E[X_iI_{\{X_i>i\}}]\cdot\frac{1}{i}\sum_{j=1}^{i}E[X_j]$$
$$\leqslant\sup_n\frac{1}{n}\sum_{j=1}^{n}E[X_j]\cdot\sum_{i=1}^{\infty}\frac{1}{i}E[X_iI_{\{X_i>i\}}]$$
$$\leqslant\sup_n\frac{1}{n}\sum_{j=1}^{n}E[X_j]\cdot\sum_{i=1}^{\infty}\frac{1}{i}E[X_iI_{\{X_i>i^{\alpha}\}}],$$

while for the convergence of the second, we note that

$$\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{1}{i^2}E[X_i]\,E[X_jI_{\{X_j>j\}}]=\sum_{j=1}^{\infty}E[X_jI_{\{X_j>j\}}]\sum_{i>j}\frac{1}{i^2}E[X_i]$$
$$\leqslant\sum_{j=1}^{\infty}E[X_jI_{\{X_j>j\}}]\cdot\frac{2}{j}\cdot\sup_n\frac{1}{n}\sum_{i=1}^{n}E[X_i]$$

[see Remark 3 (iii) of Landers–Rogge[7]]

$$\leqslant 2\cdot\sup_n\frac{1}{n}\sum_{i=1}^{n}E[X_i]\cdot\sum_{j=1}^{\infty}\frac{1}{j}E[X_jI_{\{X_j>j^{\alpha}\}}].$$

This completes the proof of the theorem. □

Remark. It may be worthwhile to note that our proof of Theorem 2.1(a) shows that WLLN holds for a sequence $\{X_n, n\geqslant 1\}$ of non-negative random variables with property CI(α) even if we replace the condition of non-positive correlations by the weaker condition that

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n}(E[X_iX_j]-E[X_i]\,E[X_j])\leqslant 0\qquad\text{for all } n.$$

For a general sequence $\{X_n, n\geqslant 1\}$ of random variables, noting that if $\{X_n, n\geqslant 1\}$ is CI(α), respectively SCI(α), then both $\{X_n^+, n\geqslant 1\}$ and $\{X_n^-, n\geqslant 1\}$ are CI(α), respectively SCI(α), and applying Theorem 2.1 to the sequences $\{X_n^+, n\geqslant 1\}$ and $\{X_n^-, n\geqslant 1\}$ separately, we get the following result, which is an improvement over Chandra[2] and Chandra–Goswami.[3]

Theorem 2.2. Let $\{X_n, n\geqslant 1\}$ be a sequence of pairwise independent random variables and let $S_n=\sum_{i=1}^{n}X_i$.

(a) If $\{X_n, n \geqslant 1\}$ is CI(α) for some $\alpha \in (0, \frac{1}{2})$, then

$$\frac{S_n - E[S_n]}{n} \to 0 \quad \text{as} \quad n \to \infty \text{ in } L_1 \text{ and hence in probability.}$$

(b) If $\{X_n, n \geqslant 1\}$ is SCI(α) for some $\alpha \in (0, \frac{1}{2})$, then

$$\frac{S_n - E[S_n]}{n} \to 0 \quad \text{as} \quad n \to \infty \text{ almost surely.}$$

3. SOME DEPENDENT SEQUENCES

In this section, we show that even for certain kinds of sequences of dependent random variables with suitable dependence structures, similar truncation technique, as used in the previous section for pairwise independent sequences, can be applied to derive WLLN under the condition CI(α) for appropriate α. In view of the fact that CUI is strictly stronger than CI(α), these results improve Theorems 4 and 5 of Chandra.[2]

Theorem 3.1. Let $\{X_n, n \geqslant 1\}$ be a martingale difference sequence of random variables with respect to a non-decreasing sequence $\{\mathscr{B}_n, n \geqslant 0\}$ of σ-fields. If $\{X_n, n \geqslant 1\}$ is CI(α) for some $\alpha \in (0, \frac{1}{2})$, then $\frac{S_n}{n} \to 0$ as $n \to \infty$ in L_1 and hence in probability, where $S_n = \sum_{i=1}^{n} X_i$.

Proof. For each $n \geqslant 1$, let $Y_n = X_n I_{\{|X_n| \leqslant n^\alpha\}}$ and $Z_n = Y_n - E[Y_n \mid \mathscr{B}_{n-1}]$. Then, using the martingale difference property of $\{X_n, n \geqslant 1\}$, it is easy to see that

$$\frac{S_n}{n} = \frac{1}{n} \sum_{i=1}^{n} Z_i + \frac{1}{n} \sum_{i=1}^{n} X_i I_{\{|X_i| > i^\alpha\}} - \frac{1}{n} \sum_{i=1}^{n} E[X_i I_{\{|X_i| > i^\alpha\}} \mid \mathscr{B}_{i-1}].$$

Since $\{X_n, n \geqslant 1\}$ is CI(α), it follows that both $\frac{1}{n} \sum_{i=1}^{n} X_i I_{\{|X_i| > i^\alpha\}}$ and $\frac{1}{n} \sum_{i=1}^{n} E[X_i I_{\{|X_i| > i^\alpha\}} \mid \mathscr{B}_{i-1}]$ converge to 0 in L_1 as $n \to \infty$, the latter using Jensen's inequality for conditional expectations.

To complete the proof, we now show that $\frac{1}{n} \sum_{i=1}^{n} Z_i \to 0$ in L_1 as $n \to \infty$.

We actually have convergence to 0 in L_2. This is because

$$E\left[\left(\frac{1}{n} \sum_{i=1}^{n} Z_i\right)^2\right] = \frac{1}{n^2} \sum_{i=1}^{n} E[Z_i^2] \leqslant \frac{1}{n^2} \sum_{i=1}^{n} E[Y_i^2],$$

the equality in the first step being a consequence of the fact that $\{Z_n, n \geqslant 1\}$ is a martingale difference sequence and the inequality in the second step following from $E[(Y_i - E[Y_i \mid \mathscr{B}_{i-1}])^2] = E[Y_i^2] - E[(E[Y_i \mid \mathscr{B}_{i-1}])^2]$. But the fact that $\frac{1}{n^2}\sum_{i=1}^n E[Y_i^2] \to 0$ is just (9) in the proof of Theorem 2.1(a). □

Theorem 3.2. Let $\{X_n, n \geqslant 1\}$ be a "pairwise" m-dependent sequence of random variables, that is, X_n and $X_{n'}$ are independent whenever $|n-n'| > m$. If $\{X_n, n \geqslant 1\}$ is CI(α) for some $\alpha \in (0, \frac{1}{2})$, then $\frac{S_n - E[S_n]}{n} \to 0$ as $n \to \infty$ in L_1 and hence in probability, where $S_n = \sum_{i=1}^n X_i$.

Proof. For each $n \geqslant 1$, let $Y_n = X_n I_{\{|X_n| \leqslant n^\alpha\}}$ and $T_n = \sum_{i=1}^n Y_i$. Then, as in the proof of Theorem 2.1(a), we need to prove that

$$\frac{1}{n}(S_n - T_n) \to 0 \qquad \text{in } L_1, \qquad \frac{1}{n}(E[S_n] - E[T_n]) \to 0 \tag{15}$$

$$\frac{1}{n}(T_n - E[T_n]) \to 0 \qquad \text{in } L_1. \tag{16}$$

Since $\{X_n, n \geqslant 1\}$ is CI(α), (15) follows because

$$\frac{1}{n}\sum_{i=1}^n E[|X_i - Y_i|] = \frac{1}{n}\sum_{i=1}^n E[|X_i| \, I_{\{|X_i| > i^\alpha\}}] \to 0, \qquad \text{as} \quad n \to \infty.$$

For (16), we actually prove convergence to 0 in L_2 as in the proof of Theorem 2.1(a), by showing that

$$\frac{1}{n^2}\sum_{i=1}^n E[Y_i^2] \to 0 \qquad \text{as} \quad n \to \infty, \tag{17}$$

and,

$$\limsup_{n\to\infty} \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i<j}}^n \operatorname{Cov}(Y_i Y_j) \leqslant 0. \tag{18}$$

The proof of (17) is exactly that of (9) in the proof of Theorem 2.1(a).

As for (18), we use the hypothesis of pairwise m-dependence and the inequality $\operatorname{Cov}(Y_i, Y_j) \leqslant 4 i^\alpha j^\alpha \ \forall i \neq j$ to note that

$$\frac{1}{n^2}\sum_{\substack{i,j=1 \\ i<j}}^n \operatorname{Cov}(Y_i, Y_j) = \frac{1}{n^2}\sum_{k=1}^m \sum_{i=1}^{n-k} \operatorname{Cov}(Y_i, Y_{i+k})$$

$$\leqslant \frac{4}{n^2}\sum_{k=1}^m \sum_{i=1}^{n-k} i^\alpha (i+k)^\alpha \leqslant \frac{4m}{n^2}\sum_{i=1}^n i^{2\alpha},$$

which is of the order of $\frac{n^{2\alpha+1}}{n^2}$ and therefore $\to 0$, since $\alpha < \frac{1}{2}$. □

Theorem 3.3. Let $\{X_n, n \geqslant 1\}$ be a ϕ-mixing sequence of random variables, such that for some $0 < \theta < 1$ and $r \geqslant 1$,

$$n^{-\theta} \sum_{k=1}^{n-1} (\phi(k))^r \to 0 \qquad \text{as} \quad n \to \infty. \tag{19}$$

If $\{X_n, n \geqslant 1\}$ is CI(α) for some $\alpha \in (0, \beta]$ where $\beta = \frac{1-\theta}{2r}$, then, as $n \to \infty$, $\frac{S_n - E[S_n]}{n} \to 0$ in L_1 and hence in probability, where $S_n = \sum_{i=1}^n X_i$.

Proof. Let Y_n and T_n be as in the proof of previous theorem. Noting that $0 < \alpha \leqslant \beta < \frac{1}{2}$, the proof proceeds exactly as in the previous theorem until it comes to proving (18). For proving (18), we use the fact that $\text{Cov}(Y_i, Y_j) \leqslant 2i^\alpha j^\alpha \phi(j-i)\ \forall i < j$, [see Lemma 2, p. 171 of Billingsley[1]] to obtain

$$\frac{1}{n^2} \sum_{\substack{i,j=1 \\ i<j}}^{n} \text{Cov}(Y_i, Y_j) = \frac{1}{n^2} \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \text{Cov}(Y_i, Y_{i+k}) \leqslant \frac{2}{n^2} \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} i^\alpha (i+k)^\alpha \phi(k)$$

$$\leqslant 2n^{\alpha-2} \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} i^\alpha \phi(k) \leqslant 2n^{\alpha-2} \sum_{k=1}^{n-1} (n-k)^{\alpha+1} \phi(k).$$

Now, if the condition (19) holds for $r = 1$, then from the above inequality, we get that

$$\frac{1}{n^2} \sum_{\substack{i,j=1 \\ i<j}}^{n} \text{Cov}(Y_i, Y_j) \leqslant 2n^{\alpha-2} \cdot n^{\alpha+1} \sum_{k=1}^{n-1} \phi(k)$$

$$\leqslant 2n^{2\beta-1} \sum_{k=1}^{n-1} \phi(k) = 2n^{-\theta} \sum_{k=1}^{n-1} \phi(k) \to 0 \qquad \text{as} \quad n \to \infty.$$

On the other hand, if the hypothesis (19) holds for some $r > 1$ and if $s > 1$ is such that $r^{-1} + s^{-1} = 1$, we then have, by Hölder's inequality,

$$\left| \sum_{k=1}^{n-1} (n-k)^{\alpha+1} \phi(k) \right| \leqslant \left(\sum_{k=1}^{n-1} k^{(\alpha+1)s} \right)^{\frac{1}{s}} \left(\sum_{k=1}^{n-1} (\phi(k))^r \right)^{\frac{1}{r}}$$

$$\leqslant (n^{(\alpha+1)s+1})^{\frac{1}{s}} \left(\sum_{k=1}^{n-1} (\phi(k))^r \right)^{\frac{1}{r}}$$

$$= n^{\alpha+1+\frac{1}{s}} \left(\sum_{k=1}^{n-1} (\phi(k))^r \right)^{\frac{1}{r}},$$

so that,

$$\frac{1}{n^2}\sum_{\substack{i,j=1\\i<j}}^{n} \mathrm{Cov}(Y_i, Y_j) \leqslant 2n^{\alpha-2}\cdot n^{\alpha+1+\frac{1}{s}}\left(\sum_{k=1}^{n-1}(\phi(k))^r\right)^{\frac{1}{r}}$$

$$= 2n^{2\alpha-\frac{1}{r}}\left(\sum_{k=1}^{n-1}(\phi(k))^r\right)^{\frac{1}{r}}$$

$$\leqslant 2n^{2\beta-\frac{1}{r}}\left(\sum_{k=1}^{n-1}(\phi(k))^r\right)^{\frac{1}{r}}$$

$$= 2\left(n^{-\theta}\sum_{k=1}^{n-1}(\phi(k))^r\right)^{\frac{1}{r}} \to 0 \qquad \text{as} \quad n\to\infty,$$

by the hypothesis (19) and the proof is complete. □

The proof of the above theorem actually allows us to have the following slightly modified version of the theorem.

Theorem 3.4. Let $\{X_n, n\geqslant 1\}$ be a ϕ-mixing sequence of random variables and assume that there exist $0<\theta<1$ and $r\geqslant 1$ such that

$$n^{-\theta}\sum_{k=1}^{n-1}(\phi(k))^r \to 0 \qquad \text{as} \quad n\to\infty.$$

Let $\beta_0 = \sup\{\frac{1-\theta}{2r} : 0<\theta<1, r\geqslant 1, \lim_{n\to\infty} n^{-\theta}\sum_{k=1}^{n-1}(\phi(k))^r = 0\}$. If $\{X_n, n\geqslant 1\}$ is CI(α) for some $\alpha\in(0,\beta_0)$, then $\frac{S_n-E[S_n]}{n}\to 0$ as $n\to\infty$ in L_1 and hence in probability, where $S_n=\sum_{i=1}^n X_i$.

Proof. Since $\alpha<\beta_0$, there exist $0<\theta<1$ and $r\geqslant 1$ with $\frac{1-\theta}{2r}>\alpha$, such that $\lim_{n\to\infty} n^{-\theta}\sum_{k=1}^{n-1}(\phi(k))^r=0$. Now the proof can be completed as above with $\beta=\frac{1-\theta}{2r}$. □

This last theorem has the following corollary that says that for ϕ-mixing sequences with exponentially decaying ϕ, the condition CI(α) with $\alpha<\frac{1}{2}$ suffices for WLLN to hold.

Corollary 3.5. Let $\{X_n, n\geqslant 1\}$ be a ϕ-mixing sequence of random variables with $\phi(k)\leqslant a\rho^k\,\forall k$ where a and $\rho<1$ are constants. If $\{X_n, n\geqslant 1\}$ is CI(α) for some $\alpha\in(0,\frac{1}{2})$, then $\frac{S_n-E[S_n]}{n}\to 0$ as $n\to\infty$ in L_1 and hence in probability, where $S_n=\sum_{i=1}^n X_i$.

Proof. One needs simply to observe that for any $\theta>0$ and any $r\geqslant 1$, $n^{-\theta}\sum_{k=1}^{n-1}(\phi(k))^r\leqslant a^r\rho^r n^{-\theta}(1-\rho^{r(n-1)})(1-\rho^r)^{-1}\to 0$ as $n\to\infty$, so that, in this case, we have $\beta_0 = \sup\{\frac{1-\theta}{2r} : \theta > 0, r \geqslant 1\} = \frac{1}{2}$. Hence the result follows from Theorem 3.4. □

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