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Unsharp spin-$\frac{1}{2}$ observables and CHSH inequalities

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Abstract

A quantitative study of the unsharpness needed for unsharp spin-$\frac{1}{2}$ observables to satisfy the Bell/CHSH inequalities shows that the result obtained by Busch has to be corrected due to unexpected correlations. It is also demonstrated that the original Bell inequality is inappropriate for unsharp spin observables.

Bell inequalities have been derived as necessary and sufficient conditions for the existence of a joint probability distribution for three observables with pair distributions as marginals [1,2]. In another version [3,4] it can be stated that Bell inequalities and Clauser–Horne–Shimony–Holt (CHSH) inequalities are the necessary and sufficient condition for the classical representation of the probability sequence obtained from three and four observables (here spin-$\frac{1}{2}$ observables) respectively in a quantum state (here singlet state). Now a joint probability in standard quantum mechanics is forbidden for incompatible observables. So it has been suggested that violation of the Bell inequalities is connected with the existence of incompatible observables rather than with some non-local influence exerted by the measurement of one local observable on the measurement result of another local observable [5]. This idea has been supported by the fact that in generalized quantum mechanics there exists a joint measurement, hence a joint distribution of two incompatible observables if the measurement is sufficiently unsharp or imprecise [6,7]. Here we are not going to discuss the interpretational problem. We shall find the amount of unsharpness needed to make quantitatively the observables satisfy the Clauser–Horne–Shimony–Holt (CHSH) inequalities in the case of two spin-$\frac{1}{2}$ particles in a singlet state. Then we shall show that a previous result on the amount of unsharpness in the case of Bell inequalities obtained by Busch [2,6,8] should be corrected. This result also demonstrates that a direct application of the Bell inequality for unsharp spin observables is problematic due to unexpected correlation. To show this let us consider unsharp spin-$\frac{1}{2}$ observables in the two-dimensional complex Hilbert space,

$$E(n_i)=\tfrac{1}{2}(I+\lambda\hat{n}_i\cdot\sigma_i),\quad i=1,2,3,4,$$

where λ is a real number with $0\leqslant\lambda\leqslant 1$, the $\hat{n}_i$ are unit vectors and the $\boldsymbol{\sigma}$ are Pauli matrices. When $\lambda=1$, $E(n_i)$ is a projection operator representing a sharp observable.

In Bohm's version of the EPR experiment let us consider a system consisting of two spin-$\frac{1}{2}$ particles in the singlet state,

$$\psi=\phi\otimes\frac{1}{\sqrt{2}}[\varphi_+(n)\otimes\varphi_-(n) - \varphi_-(n)\otimes\varphi_+(n)]\,. \tag{1}$$

$\varphi_{\pm}(n)$ are the eigenvectors of $n\cdot\sigma$, and the spatial wave function ϕ represents a superposition of two one-particle waves propagating into opposite directions.

Let us now designate the probability and joint probability distributions of the $E(n_i)$ in the state ψ by P_i and P_{ij}.

Then

$$P_i = \langle\psi| I\otimes E(n_i)\psi\rangle = \langle\psi| E(n_i)\otimes I\psi\rangle = \tfrac{1}{2}\,, \tag{2}$$

where I is a unit operator and $i=1, 2, 3, 4$,

$$P_{ij} = \langle\psi| E(n_i)\otimes E(n_j)\psi\rangle = \tfrac{1}{4}(1-\lambda^2\hat{n}_i\cdot\hat{n}_j) = (1-2\epsilon)\times\tfrac{1}{2}\sin^2(\tfrac{1}{2}\theta_{ij}) + \tfrac{1}{2}\epsilon\,, \tag{3}$$

where $\epsilon=\frac{1}{2}(1-\lambda^2)$ is the measure of the degree of unsharpness and θ_{ij} is the angle between $\hat{n}_i$ and $\hat{n}_j$.

Let us now consider one of the four CHSH inequalities given in Eq. (14) of Ref. [4], and let the quantum state be a singlet state of two spin-$\frac{1}{2}$ particles and let E_1 and E_2 refer to the unsharp polarization measurement along directions $\hat{n}_1$ and $\hat{n}_2$ respectively for one particle, say the left one, and similarly E_3 and E_4 for the right particle,

$$0\leqslant P_1+P_4-P_{13}-P_{14}-P_{24}+P_{23}\leqslant 1\,. \tag{4}$$

Writing the expressions of P_i and P_{ij} from Eqs. (2) and (3) we get from (4)

$$f=|\cos\theta_{13}+\cos\theta_{14}+\cos\theta_{24}-\cos\theta_{23}| \leqslant \frac{2}{1-2\epsilon} = F\,. \tag{5}$$

The left hand side attains its maximum when the n_i are coplanar and

$$\theta_{24}=\theta_{14}=\theta_{13}=\theta,\quad \theta_{23}=3\theta,\quad \theta=45°\,.$$

So $f=2\sqrt{2}=f_{\max}$. Equating $f_{\max}$ with F yields a minimal value for ϵ,

$$\epsilon\geqslant\epsilon_{\rm CHSH}=\tfrac{1}{2}(1-1/\sqrt{2})\,,$$

i.e. $\lambda\leqslant\lambda_{\rm CHSH}=1/2^{1/4}$.

Busch has shown [2,6] that any three unsharp spin-$\frac{1}{2}$ observables can satisfy Bell inequalities if $\epsilon\geqslant\epsilon_{\rm B}=\frac{1}{6}$, i.e. $\lambda\leqslant\lambda_{\rm B}=(2/9)^{1/2}$. Now we see that $\epsilon_{\rm B}>\epsilon_{\rm CHSH}$ which is obviously wrong because violation in the case of CHSH inequalities is larger than that for Bell inequalities. This happens because, when we write down the Bell inequalities directly in terms of unsharp spin observables as done in Refs. [2,6], we ignore one important fact. The point is that the original Bell inequalities refer to three observables whereas the underlying experimental situation necessarily deals with four quantities, two of which may or may not be correlated. This was the reason for CHSH to derive their inequality. While for sharp observables it is possible to have strict (anti-)correlation between two observables, each belonging to one component of a singlet state system, this no longer happens in the case of unsharp observables. In this latter case the reduction of the CHSH inequalities to a situation where two spin observables have coincident directions does no longer yield the original Bell inequalities but instead a set of inequalities containing an additional term accounting for the imperfect correlation.

When $E(n_i)$ and $E(n_j)$, each belonging to one component of a singlet state system, are in the same direction, then in the case of sharp spin observables $P_{ij}=0$. Hence for this case the CHSH inequalities reduce to the Bell inequalities in the usual form. But for unsharp spin observables

$$P_{ij}(\hat{n}_i=\hat{n}_j)=\tfrac{1}{4}(1-\lambda^2)=\tfrac{1}{2}\epsilon\neq 0\,. \tag{6}$$

Let us now find the restriction on ϵ and λ for the non-violation of the modified Bell inequality with $P_{ii}\neq 0$ as in (6).

For this purpose let us take the second inequality of (14) of Ref. [4],

$$0\leqslant P_2+P_4-P_{23}-P_{24}-P_{14}+P_{13}\leqslant 1\,. \tag{7}$$

For $\hat{n}_2=\hat{n}_3$ it reduces to

$$P_{12}\leqslant P_{14}+P_{24}+P_{22}\leqslant 1+P_{12}\,. \tag{8}$$

Calculating all the probabilities for unsharp spin observables in a singlet state we get from inequality (8),

$$-\frac{3-2\epsilon}{1-2\epsilon}\leqslant f=\cos\theta_{14}+\cos\theta_{24}-\cos\theta_{12} \leqslant\frac{1+2\epsilon}{1-2\epsilon}=F\quad(\text{say})\,, \tag{9}$$

For first part of the inequality there is no violation even in the case of sharp observables ($\epsilon=0$). Now f obtains its maximum when n_1, n_2 and n_4 are coplanar and $\theta_{14}=\theta_{24}=60°=\frac{1}{2}\theta_{12}$. This gives $f=f_{\max}=\frac{3}{2}$. So the minimal value of ϵ will be found by equating F to f_{max}. Then we get $\epsilon \geqslant \epsilon_{min} = \frac{1}{10}$ and $\lambda \leqslant \lambda_{max} = 2/\sqrt{5}$.

Let us denote these constants by the subscript MB (modified Bell). So finally we get $\epsilon_{CHSH} > \epsilon_{MB}$ and $\lambda_{CHSH} < \lambda_{MB}$, which are consistent.

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