Sankhyā : The Indian Journal of Statistics
1982, Volume 44, Series B, Pt. 3 pp. 351-354.

AN AXIOMATISATION OF THE ENTROPY MEASURE OF INEQUALITY

By SATYA RANJAN CHAKRAVARTY
Indian Statistical Institute

SUMMARY. This note provides an axiomatisation of a social welfare function, which for a given mean income and a given population size ranks all possible income distributions in exactly the opposite way as the entropy measure of inequality.

While the normative aspects of the Gini coefficient have been discussed extensively (see, for example, Newbery, 1970; Sheshinski, 1972; Kats, 1972; Dasgupta, Sen and Starrett, 1973; Rothschild and Stiglitz, 1973; and Sen, 1973, 1974), Theil's ingenious measure (Theil, 1967) derived from the notion of entropy in information theory is yet to be discussed thoroughly. Here we make a complete axiomatisation of a social welfare function that will rank all possible income distributions with the same number of earners and the same mean income, in the same way as the Theil measure.

Let us assume that s_i represents the income share of individual i in a community S consisting of n individuals; $s_i = \frac{y_i}{n\lambda} \geqslant 0$, where $y_i \geqslant 0$ is the income of the i-th individual and $\lambda > 0$ is the mean income. We assume that λ is finite and is the same for all income distributions under comparison.

Let

$$\underline{s} = (s_1, s_2, \ldots, s_n). \qquad \ldots \quad (1)$$

Assume that $(\underline{s}, i)$ is the state of individual i in the social state $\underline{s}$. We choose the following weighted sum from of the social welfare function :

$$W(\underline{s}) = \sum_{i=1}^{n} s_i \, v(s_i). \qquad \ldots \quad (2)$$

The weight on s_i is given by $v(s_i) > 0$, a function of income share of the i-th individual.

We now propose a set of axioms which seem to be appealing within a specific framework. These axioms correspond to exactly the entropy measure helping us to understand it.

Axiom M (Independent monotonicity): For all $\underline{s}$, all individuals regard $(\underline{s}, i)$ to be at least as good as $(\underline{s}, j)$ if and only if $s_i > s_j$.

Axiom M states that an individual decides his preference on the basis of his income and he prefers more to less.

Axiom E (Relative equity): If everyone prefers $(\underline{s}, i)$ to $(\underline{s}, j)$, then $v(s_i) < v(s_j)$.

This axiom says if in the social state $\underline{s}$, the position of an individual j is regarded to be worse off than another individual i, then individual j deseves greater weightage in the evaluation scheme (2).

Axiom D (Weight differential information): If everyone prefers $(\underline{s}, i)$ to $(\underline{s}, j)$, then the weight differential $v(s_j)-v(s_i)$ will depend only on s_i/s_j, the income share of the i-th individual relative to that of individual j, that is only on their relative income.

This axiom implies that so long as the relative income of two individuals remain constant, the weight differential will remain unchanged.

Theorem: *A social welfare function of the form (2) satisfying axioms M, E and D must rank all income distributions, with a given total income, over a given population, in exactly the same way as the negative of Theil's entropy measure of inequality.*

Proof: Consider any two individuals i and j. By axiom D, $v(s_j)-v(s_i)$ depends only on $\frac{s_i}{s_j}$.

Therefore we can write

$$v(s_j)-v(s_i)=f\left(\frac{s_i}{s_j}\right) \quad \ldots \quad (3)$$

where f is some real valued function whose domain of definition is the interval $(0, \infty)$.

Choose $c \in (0, 1)$

$$\begin{aligned} f\left(s_i.\frac{1}{s_j}\right) &= f\left(\frac{cs_i}{cs_j}\right) \\ &= v(cs_j)-v(cs_i) \quad [\text{from } (3)] \\ &= [v(c)-v(cs_i)]+[v(cs_j)-v(c)] \\ &= f\left(\frac{cs_i}{c}\right)+f\left(\frac{c}{cs_j}\right) \end{aligned}$$

$$= f(s_i)+f\left(\frac{1}{s_j}\right). \qquad \ldots \quad (4)$$

Equation (4) is the wellknown Caucny equation (see Eichhorn, 1978, pp. 12–13). Equation (4) along with axioms M and E implies that

$$v(s_i) = A \log\left(\frac{1}{s_i}\right) \qquad \ldots \quad (5)$$

where $A > 0$ is independent of s_i.

This yields,

$$W(\underline{s}) = \sum_{i=1}^{n} s_i \left(A \log \frac{1}{s_i}\right)$$

$$= A \sum_{i=1}^{n} s_i \log\left(\frac{1}{s_i}\right). \qquad \ldots \quad (6)$$

The Theil entropy measure of inequality based on $\underline{s}$ is

$$T(\underline{s}) = \log n - \sum_{i=1}^{n} s_i \log \frac{1}{s_i}. \qquad \ldots \quad (7)$$

It is clear that given n and λ, the odrering of S yielded by $W(\underline{s})$ is exactly the opposite that given by $T(\underline{s})$. □

The theorem provides a complete axiomatisation of the Theil measure of inequality. Given the weight differential information axiom, the other axioms are necessary and sufficient for the social welfare function to be a negative transformation of T. Therefore the axiom system specifies a set of necessary and sufficient conditions for the welfare interpretation of the entropy measure.

Acknowledgment. The author is grateful to Professor N. Bhattacharya for his guidance in preparing the note. Comments of Professors A. B. Atkinson and Bhaskar Dutta are also acknowledged with thanks.

REFERENCES

DASGUPTA, P., SEN, A. and STARRETT, P. (1973): Notes on the measurement of inequality. *Journal of Economic Theory*, 6, 180-187.

EICHHORN, W. (1978): *Functional Equations in Economics*, Addison-Wesley: Reading (Mass). London, Amsterdam, Don Mills (Ontario), Sydney, Tokyo.

KATS, A. (1971): On the social welfare function and the parameters of income distribution. *Journal of Economic Theory*, 5, 377-382.

NEWBERY, D. (1970): A theorem on the measurement of inequality. *Journal of Economic Theory*, 2, 264-266.

ROTHSCHILD, M. and STIGLITZ, J. E. (1973): Some further results on the measurement of inequality. *Journal of Economic Theory*, 6, 188-204.

SEN, A. (1973): *On Economic Inequality*, Oxford University Press, London.

——— (1974): Information bases of alternative welfare approaches. *Journal of Public Economics*, 3, 387-403.

SHESHINSKI, E. (1971): Relation between a social welfare function and the Gini index of income inequality. *Journal of Economic Theory*, 4, 98-100.

THEIL, H. (1967): *Economics and Information Theory*, North-Holland, Amsterdam,

Paper received: November, 1981.