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Analyzing non-stationary signals using generalized multiple fundamental frequency model

Swagata Nandi[a], Debasis Kundu[b,*]

[a]Indian Statistical Institute, 7, S.J.S. Sansanwal Marg, New Delhi 110016, India
[b]Department of Mathematics, Indian Institute of Technology Kanpur, Kanpur 208016, India

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Abstract

In this paper, we propose a new generalized multiple frequency model to analyze non-stationary signals. The model under the assumption of additive stationary errors can be used quite effectively to analyze different signals. We propose the usual least-squares estimators to estimate the unknown parameters and it is shown that the estimators are strongly consistent. We obtain the asymptotic distributions also. The performance of the proposed model is compared with the multiple frequency model using Monte Carlo simulations. Finally, several real data are analyzed using both the proposed model and the multiple frequency model.

Keywords: Multiple fundamental frequency; Sinusoidal model; Asymptotic distribution

1. Introduction

Parametric modeling of signals and estimating the parameters of different models are important problems in Statistical Signal Processing. Several models have been proposed in the past 25 years to analyze several stationary signals. Statistical performance of different methods depend very much on the suitability of the selected model and also on the estimation procedure. Several models like Moving Average (MA), Autoregressive (AR) or

* Corresponding author. Tel.: +91 512 597 636; fax: +91 512 590 007.
E-mail addresses: nandi@isid.ac.in (S. Nandi), kundu@iitk.ac.in (D. Kundu).

Autoregressive Moving Average (ARMA) are being used extensively for analyzing stationary signals. In practice, many signals are non-stationary in nature and it is well known that analyzing non-stationary signals is quite difficult in general. Quasi-stationarity is often used in analyzing non-stationary signals. The main idea about quasi-stationarity is to assume stationarity over a short data segment (see for example Isaksson et al., 1981; Kay, 1988; McAulay and Quatieri, 1986; Dahlhaus, 1997; Ombao et al., 2001) and the analysis of the signal is performed over this short data length. Therefore, by this method a compromise is needed between the model validity over the entire data length and the estimation of the unknown parameters. In this paper we introduce a new model to analyze non-stationary signals which is a generalization of the fundamental frequency model as well as the harmonic model with multiple fundamentals and provide an estimation procedure under the assumption of stationary noise random variables.

We consider the following model in stationary noise:

$$y(t) = \sum_{k=1}^{M} f_k(t; \boldsymbol{\theta}_k) + X(t), \quad t = 1, \ldots, N. \tag{1}$$

We assume that there are M fundamental frequencies and the other frequencies appear in the model with a certain relationship associated with each fundamental frequency. Here $f_k(t; \boldsymbol{\theta}_k)$ is the contribution of the kth fundamental frequency and is a sum of q_k sinusoidal components of the following form:

$$f_k(t; \boldsymbol{\theta}_k) = \sum_{j=1}^{q_k} \rho^0_{k_j} \cos\{[\lambda^0_k + (j-1)\omega^0_k]t - \phi^0_{k_j}\}, \tag{2}$$

where λ^0_k is the fundamental frequency and the other frequencies associated with λ^0_k are occurring at $\lambda^0_k, \lambda^0_k + \omega^0_k, \ldots, \lambda^0_k + (q_k - 1)\omega^0_k$. Note that, $\lambda^0_k + \omega^0_k, \ldots, \lambda^0_k + (q_k - 1)\omega^0_k$, need not be harmonics of λ^0_k. If $\lambda^0_k = \omega^0_k$, then they are harmonics of λ^0_k. Corresponding to the frequency $\lambda^0_k + (j-1)\omega^0_k$, $\rho^0_{k_j}$ and $\phi^0_{k_j}$ represent the amplitude and phase components, respectively, and they are also unknown.

We make the following assumptions on the model parameters and noise random variables $X(t)$.

Assumption 1.

$$\rho^0_{k_j} > 0, \quad \phi^0_{k_j} \in (-\pi, \pi), \quad \lambda^0_k, \omega^0_k \in (0, \pi), \quad j = 1, \ldots, q_k, \quad k = 1, \ldots, M. \tag{3}$$

Assumption 2. λ^0_k and ω^0_k, $k = 1, \ldots, M$ are such that

$$\lambda^0_k + (i_1 - 1)\omega^0_k \neq \lambda^0_l + (i_2 - 1)\omega^0_l$$

for $i_1 = 1, \ldots, q_k$; $i_2 = 1, \ldots, q_l$ and $k \neq l = 1, \ldots, M$.

Assumption 3. The number of fundamental frequencies, M and the number of components q_k associated with the kth fundamental frequency, $k = 1, \ldots, M$ are known.

Assumption 4. $X(t)$ has the following representation:

$$X(t) = \sum_{k=-\infty}^{\infty} \alpha(k)e(t-k), \tag{4}$$

where $\{e(t)\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero and finite variance σ^2. The arbitrary constants $\alpha(k)$'s are such that

$$\sum_{k=-\infty}^{\infty} |\alpha(k)| < \infty.$$

Assumptions 1–4 are quite general. We need Assumption 2 for identifiability. It only says that the effective frequencies are distinct. Note that $\lambda_M + (q_M - 1)\omega_M < \pi$. We should mention here that under Assumption 4, the signal $y(t)$'s are non-stationary in mean, not in second or higher order structure. Thus, given an observed signal $\{y(t); t = 1, \ldots, N\}$, the aim is to estimate the unknown parameters, namely ρ's, λ's, ω's and ϕ's under Assumptions 1–4.

We are interested to study the model (1) under Assumptions 1–4. Several authors have considered various forms of the model (1) with $M = 1$ and without any restriction on the frequencies, namely,

$$y(t) = \sum_{k=1}^{p} a_k^0 \cos(\beta_k^0 t - \phi_k^0) + X(t), \quad t = 1, \ldots, N, \tag{5}$$

where a_k^0s are non-negative amplitudes, β_k^0s are frequencies and ϕ_k^0s are phases and they are unknown. Note that the multiple frequency model (MFM) (5) is a more general model and can be used when no relationship exists among the frequencies. But, if the frequencies are related, then this additional information is helpful to reduce the number of non-linear parameters. In Section 4, we shall see how the model (1) is used to analyze different real data.

The model (5) is a well-studied model and several authors considered the model with different assumptions on the noise random variables and proposed different estimation procedures. References may be made to works of Walker (1971), Hannan (1971, 1973), Rice and Rosenblatt (1988), Kundu (1997) and so on. Observe that the proposed model is a generalization of the following fundamental frequency model:

$$y(t) = \sum_{j=1}^{q} \rho_j^0 \cos(j\lambda^0 t - \phi_j^0) + X(t), \quad t = 1, \ldots, N. \tag{6}$$

When $M=1$, $q_1=q$ and $\lambda_1^0=\omega_1^0=\lambda^0$, model (1) coincides with model (6). The model (6) was considered by Hannan (1974), Baldwin and Thomson (1978), Quinn and Thomson (1991), Nandi and Kundu (2003) and Kundu and Nandi (2004) to analyze different real-life data sets. Quinn and Thomson (1991) obtained the theoretical properties of an equivalent estimator of the generalized least-squares estimator (LSEs). Nandi and Kundu (2003) discussed the theoretical properties of the LSEs for the model (6). The model (6) has only one non-linear parameter, and so if all the frequencies are harmonics of a particular frequency, the fundamental frequency model is the best one as maximum reduction of the number of parameters is possible. Recently, Irizarry (2000) considered a similar model to analyze several musical sound (harmonical) data. The model is expressed as follows:

$$y(t) = \sum_{j=1}^{J} \left\{ \sum_{k=1}^{K_j} (A_{j,k} \cos(k\theta_j t) + B_{j,k} \sin(k\theta_j t)) \right\} + X(t) \tag{7}$$

and is a harmonic model with multiple fundamental frequencies $\theta_1, \ldots, \theta_J$. This model is exactly equal to the fundamental frequency model (6) with $J = 1$, $A_{1,k} = \phi_k \cos(\phi_k)$ and $B_{1,k} = \rho_k \sin(\phi_k)$. Irizarry (2000) proposed window-based estimators using the method of weighted least squares to estimate the unknown parameters and established the consistency and asymptotic normality properties of the estimators. The model (1) is a generalization of model (7). If $\lambda_k^0 = \omega_k^0$ in model (1), it coincides with (7) (writing $\rho_{k_j} \cos(\phi_{k_j}) = A_{j,k}$ and $\rho_{k_j} \sin(\phi_{k_j}) = B_{j,k}$). Therefore, the proposed model is a generalization of the harmonic model (7) with multiple fundamentals (so also of the fundamental frequency model (6)) and a particular case of the frequency model (5) which has several applications in different field of science.

The presence of this kind of periodicity is a convenient approximation, but many real-life phenomena can be described quite effectively, using models (6), (7) and similarly by using model (1). We shall see later on in this paper that incidentally several short-duration speech data can be successfully modeled using (1). Baldwin and Thomson (1978) and Quinn and Thomson (1991) used the model (6) to describe the visual observation of S. Carinae, a variable star in the Southern Hemisphere sky. Greenhouse et al. (1987) proposed the use of higher-order harmonic terms of one or more fundamentals and ARMA processes for the errors (so model (6) and (7)) for fitting biological rhythms and illustrated it by analyzing human core body temperature data. The harmonic regression model has also been used to assess the static properties of human circadian systems; see Brown and Czeisler (1992) and Brown and Liuthardt (1999). To analyze the periodic changes in the functional activity of specific groups of neurons in the human SCN (suprachiasmatic nucleus), the annual cycles of peptidergic activity could be described by a multiple harmonic regression model with ARMA errors (Hofman, 2001). Musical sound waves produced by musical instruments can be analyzed using above-mentioned models (Rodet, 1997). Irizarry (2000) studied a segment of sound produced by a pipe organ playing two consecutive notes using model (7). Sircar and Syali (1996) proposed an amplitude modulated model with i.i.d. error by exploiting some special features of some short-length voiced speech signals and analyzed "aaa" and "uuu" sound data. Nandi et al. (2004) also studied this amplitude-modulated model with stationary error and analyzed the same data sets. The analysis of these two data sets are also included in this paper using model (1).

We propose the usual LSEs to estimate the unknown parameters of the model (1) and obtain the theoretical properties of the estimators. We note that the model (1) is highly non-linear in its parameters. Therefore, all the theoretical results of the LSEs are asymptotic and it is not possible to obtain the finite sample behavior theoretically. It also does not satisfy the sufficient conditions given in Jennrich (1969) or Wu (1981) for the LSEs to be consistent. So the consistency and asymptotic normality properties are not automatically followed from standard results already available in the literature. We need to prove them in some different way. We observe that the LSEs are consistent and they are asymptotically normally distributed under the assumption that the error process is a linear process. The asymptotic distribution provides us to approximate the variances of the estimates for finite samples and to construct the error bounds of all the estimators. We compare models (1) and (5) using numerical examples based on simulated as well as real-life data sets.

The rest of the paper is organized as follows. In Section 2, we define the LSEs of the unknown parameters and derive the asymptotic properties of the LSEs. Simulation results are presented in Section 3 to see the small sample performance. Different real data sets are analyzed in Section 4 and finally we conclude the paper in Section 5. The proofs are given in the appendix.

2. Asymptotic properties

In this section, we define the usual LSEs and obtain their theoretical properties. We denote the parameter vector $\mathbf{\Psi}$ as $\mathbf{\Psi} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_M)$ for the model (1), where $\boldsymbol{\theta}_k = (\rho_{k_1}, \ldots, \rho_{k_{q_k}}, \phi_{k_1}, \ldots, \phi_{k_{q_k}}, \lambda_k, \omega_k)$, $k = 1, \ldots, M$. $\mathbf{\Psi}^0$ denotes the true parameter value. Here, M refers to the total number of fundamental frequencies present and q_k, the number of frequencies associated with the kth fundamental frequency. Under Assumption 3, the other parameters of this model are identifiable provided they satisfy Assumption 2.

Least-squares method consists of choosing $\hat{\mathbf{\Psi}}$ by minimizing the criterion

$$Q_N(\mathbf{\Psi}) = \sum_{t=1}^{N} \left(y(t) - \sum_{k=1}^{M} \sum_{j=1}^{q_k} \rho_{k_j} \cos\{(\lambda_k + (j-1)\omega_k)t - \phi_{k_j}\} \right)^2. \tag{8}$$

Note that obtaining the LSEs involves a $2\sum_{k=1}^{M} q_k + 2M$ dimensional minimization search. When M and q_k, $k = 1, \ldots, M$ are large, the LSEs may be very expensive. But ρ_{k_j}s and ϕ_{k_j}s can be expressed as functions of the frequencies λ_ks and ω_ks, so using the separable regression technique of Richards (1961), it involves a $2M$ dimensional search.

We now present results describing the asymptotic properties of the LSEs for the parameter $\mathbf{\Psi}^0$ of the model defined in (1). We prove all the results in the appendix.

Theorem 2.1. *Under Assumption* 1, 3 *and* 4, *the LSE* $\hat{\mathbf{\Psi}}$ *of* $\mathbf{\Psi}^0$ *is a strongly consistent estimator of* $\mathbf{\Psi}^0$.

Now for asymptotic distribution, let us define a diagonal matrix $\mathbf{V}$ as follows:

$$\mathbf{V} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 & \cdots & \mathbf{0} \\ \vdots & \cdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_M \end{bmatrix},$$

where for each $k = 1, \ldots, M$, $\mathbf{D}_k$ is a diagonal matrix of the following form:

$$\mathbf{D}_k = \begin{bmatrix} \frac{1}{N^{1/2}}\mathbf{I}_{q_k} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{N^{1/2}}\mathbf{I}_{q_k} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{N^{3/2}} & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \frac{1}{N^{3/2}} \end{bmatrix}.$$

Here $\mathbf{I}_{q_k}$ denotes the identity matrix of order q_k. Let $R = 2\sum_{k=1}^{M} q_k + 2M$. The following theorem states the asymptotic distribution of the LSE $\hat{\mathbf{\Psi}}$ of $\mathbf{\Psi}^0$.

Theorem 2.2. *Under the same assumption as Theorem* 2.1 *and Assumption* 2,

$$(\hat{\mathbf{\Psi}} - \mathbf{\Psi}^0)\mathbf{V}^{-1} \to \mathcal{N}_R(\mathbf{0}, 2\sigma^2\mathbf{\Sigma}^{-1}\mathbf{G}\mathbf{\Sigma}^{-1}),$$

as $N \to \infty$. *The matrices* $\mathbf{\Sigma}$ *and* $\mathbf{G}$ *are as follows:*

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2 & \cdots & \mathbf{0} \\ \vdots & \cdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{\Sigma}_M \end{bmatrix} \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 & \cdots & \mathbf{0} \\ \vdots & \cdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_M \end{bmatrix},$$

where

$$\mathbf{\Sigma}_k = \begin{pmatrix} \mathbf{I}_{q_k} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_k & -\frac{1}{2}\mathbf{P}_k\mathbf{J}_k & -\frac{1}{2}\mathbf{P}_k\mathbf{L}_k \\ \mathbf{0} & -\frac{1}{2}\mathbf{J}_k^{\mathrm{T}}\mathbf{P}_k & \frac{1}{3}\mathbf{J}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{J}_k & \frac{1}{3}\mathbf{J}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{L}_k \\ \mathbf{0} & -\frac{1}{2}\mathbf{L}_k^{\mathrm{T}}\mathbf{P}_k & \frac{1}{3}\mathbf{L}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{J}_k & \frac{1}{3}\mathbf{L}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{L}_k \end{pmatrix}, \quad k = 1, \ldots, M$$

and

$$\mathbf{G}_k = \begin{pmatrix} \mathbf{C}_k & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_k\mathbf{C}_k & -\frac{1}{2}\mathbf{P}_k\mathbf{C}_k\mathbf{J}_k & -\frac{1}{2}\mathbf{P}_k\mathbf{C}_k\mathbf{L}_k \\ \mathbf{0} & -\frac{1}{2}\mathbf{J}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{C}_k & \frac{1}{3}\mathbf{J}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{C}_k\mathbf{J}_k & \frac{1}{3}\mathbf{J}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{C}_k\mathbf{L}_k \\ \mathbf{0} & -\frac{1}{2}\mathbf{L}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{C}_k & \frac{1}{3}\mathbf{L}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{C}_k\mathbf{J}_k & \frac{1}{3}\mathbf{L}_k^{\mathrm{T}}\mathbf{P}_k\mathbf{C}_k\mathbf{L}_k \end{pmatrix}, \quad k = 1, \ldots, M.$$

Here

$$\mathbf{P}_k = \mathrm{diag}\{{\rho_{k_1}^0}^2, \ldots, {\rho_{k_{q_k}}^0}^2\}, \quad \mathbf{J}_k = (1, 1, \ldots, 1)_{q_k \times 1}^{\mathrm{T}},$$
$$\mathbf{L}_k = (0, 1, \ldots, q_k - 1)_{q_k \times 1}^{\mathrm{T}}, \quad \mathbf{C}_k = \mathrm{diag}\{c_k(1), \ldots, c_k(q_k)\}, \quad k = 1, \ldots, M$$

and

$$
\begin{aligned}
c_k(j) &= \left(\sum_{l=-\infty}^{\infty} \alpha(l) \cos\{(\lambda_k^0 + (j-1)\omega_k^0)l\} \right)^2 \\
&\quad + \left(\sum_{l=-\infty}^{\infty} \alpha(l) \sin\{(\lambda_k^0 + (j-1)\omega_k^0)l\} \right)^2 \\
&= \left| \sum_{l=-\infty}^{\infty} \alpha(l) e^{-i\{(\lambda_k^0+(j-1)\omega_k^0)l\}} \right|^2, \quad j = 1, \ldots, q_k; \;\; k = 1, \ldots, M.
\end{aligned}
$$

Remark 2.1. The asymptotic distribution of the LSEs indicates that $\hat{\boldsymbol{\theta}}_j$ and $\hat{\boldsymbol{\theta}}_k$ are asymptotically independent if $j \neq k$, that is, the estimators of the unknown parameters corresponding to different fundamental frequencies are independent.

Remark 2.2. As $\boldsymbol{\Sigma}$ and $\mathbf{G}$ are block-diagonal matrices, $\boldsymbol{\Sigma}^{-1}\mathbf{G}\boldsymbol{\Sigma}^{-1}$ is also a block-diagonal matrix with diagonal blocks as $\boldsymbol{\Sigma}_k^{-1}\mathbf{G}_k\boldsymbol{\Sigma}_k^{-1}$, $k = 1, \ldots, M$. The off-diagonal blocks are $\mathbf{0}$ matrices. This implies that for each $k = 1, \ldots, M$,

$$
(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k^0)\mathbf{D}_k^{-1} \rightarrow \mathcal{N}_{2q_k+2}(\mathbf{0}, 2\sigma^2\boldsymbol{\Sigma}_k^{-1}\mathbf{G}_k\boldsymbol{\Sigma}_k^{-1}),
$$

as $N \rightarrow \infty$.

Remark 2.3. It can be seen from the definition of the matrix $\mathbf{D}_k$ and Remark 2.2 that the normalization factor associated with $\hat{\lambda}_k$ and $\hat{\omega}_k$ is $N^{3/2}$ whereas with $\hat{\rho}_{k_j}$ and $\hat{\phi}_{k_j}$, it is $N^{1/2}$. This indicates that for a given sample size N, the frequencies can be estimated more accurately than the other parameters and the rate of convergence is much higher in case of estimators of λ_k^0 and ω_k^0.

Remark 2.4. Note that the matrices $\boldsymbol{\Sigma}_k$ and $\mathbf{G}_k$ are of the form

$$
\boldsymbol{\Sigma}_k = \begin{pmatrix} \mathbf{I}_{q_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_k \end{pmatrix}
$$

and

$$
\mathbf{G}_k = \begin{pmatrix} \mathbf{C}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_k \end{pmatrix}
$$

So

$$
\boldsymbol{\Sigma}_k^{-1}\mathbf{G}_k\boldsymbol{\Sigma}_k^{-1} = \begin{pmatrix} \mathbf{C}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_k^{-1}\mathbf{H}_k\mathbf{F}_k^{-1} \end{pmatrix}
$$

which implies that, the amplitude estimators are independent of the corresponding phase and frequency parameter estimators.

Remark 2.5. From Theorem 2.2, it can be seen that asymptotic distribution of the LSEs of the unknown parameters is independent of the true values of the phases.

Remark 2.6. For the linear processes, i.e. processes satisfying Assumption 4, $\sigma^2 c_k(j)$ is exactly equal to the spectral density function of the process. Also it can be shown that

$$\sigma^2 c_k(j) = E\left(\frac{1}{N}\left|\sum_{t=1}^{N} X(t)e^{-i(\lambda_k^0+(j-1)\omega_k^0)t}\right|^2\right), \quad j=1,\ldots,q_k; \; k=1,\ldots,M,$$

which is the expected value of the periodogram function

$$I(\lambda) = \frac{1}{N}\left|\sum_{t=1}^{N} X(t)e^{-i\lambda t}\right|^2 \tag{9}$$

of the error random variables $X(t)$. Thus, for simulation study, $\sigma^2 c_k(j)$ can be estimated by local averaging of the periodogram function of the error process across the point estimate of the effective frequencies $\hat{\lambda}_k, \hat{\lambda}_k + \hat{\omega}_k, \ldots, \hat{\lambda}_k + (q_k - 1)\hat{\omega}_k$. Another approach to estimate $\sigma^2 c_k(j)$ is to model $\{X(t)\}$ as an autoregressive process and then using the estimated autoregressive parameters, $c_k(j)$'s can be estimated. The later approach is suitable for analyzing real data sets but cannot be implemented in experiments based on simulations.

3. Simulation results

In this section, we present results of numerical experiments based on simulations. We compare the performance of the LSEs of the proposed model defined in (1) with the LSEs of the unknown parameters of the corresponding MFM as defined in (5). In case of MFM, $\beta_1 = \lambda_1, \beta_2 = \lambda_1 + \omega_1, \ldots, \beta_{q_1} = \lambda_1 + (q_1 - 1)\omega_1, \ldots, \beta_{\sum_{k=1}^{M} q_k} = \lambda_{q_M} + (q_M - 1)\omega_{q_M}$. We consider the following model for simulation studies with $M = 2$, $q_1 = 3$, $q_2 = 3$:

$$\begin{aligned} y(t) = &\sum_{j=1}^{3} \rho_{1_j}^0 \cos\{(\lambda_1^0 + (j-1)\omega_1^0)t - \phi_{1_j}^0\} \\ &+ \sum_{j=1}^{3} \rho_{2_j}^0 \cos\{(\lambda_2^0 + (j-1)\omega_2^0)t - \phi_{2_j}^0\} + X(t) \end{aligned} \tag{10}$$

with $X(t) = 0.5\, e(t-1) + e(t)$ and

$\rho_{1_1}^0 = 0.10$, $\rho_{1_2}^0 = 0.45$, $\rho_{1_3}^0 = 0.40$, $\phi_{1_1}^0 = 0.55$, $\phi_{1_2}^0 = 0.60$, $\phi_{1_3}^0 = 0.15$,

$\rho_{2_1}^0 = 0.20$, $\rho_{2_2}^0 = 0.60$, $\rho_{2_3}^0 = 0.40$, $\phi_{2_1}^0 = 0.10$, $\phi_{2_2}^0 = 0.50$, $\phi_{2_3}^0 = 0.20$,

$\lambda_1^0 = 0.439822978$, $\omega_1^0 = 0.157079635$,

$\lambda_2^0 = 1.130973372$, $\omega_2^0 = 0.188495562$.

Here $e(t)$s are i.i.d. Gaussian random variables with mean zero and finite variance σ^2. We report the results for $N = 200$ and for error variances $\sigma^2 = 0.2$ and 0.4. We generate a data set from the model (10) and compute LSEs of different parameters by minimizing the residual sum of squares given in (8) and 95% confidence intervals for each parameter using Theorem 2.2. For minimization we use routines "amoeba" and "amotry" (based on downhill simplex method) given in Press et al. (1992). As already mentioned in Section 2, for interval estimation we need to estimate $\sigma^2 c_k(j)$, $j = 1, \ldots, q_k$, $k = 1, \ldots, M$. We use smoothed periodogram (averaging the periodogram function over a window $(-L, L)$ across the point estimate of the frequency) of the estimated error process. We replicate the process 5000 times and report average estimates, mean-squared errors (MSEs), average confidence lengths and coverage percentages. The results are reported in Tables 1 and 2.

Table 1

The average LSEs, the MSEs, the average confidence lengths and the coverage probabilities of the different parameters using the proposed model for $\sigma^2 = 0.2$

Parameter	Average LSE (true value)	MSE (asym. var.)	Av. conf. length (ex. conf. length)	95% cov. prob.
ρ_{1_1}	0.125963062	3.95831512e − 03	0.337497294	0.95
	(0.10)	(4.3096542e − 03)	(0.2573400)	
ρ_{1_2}	0.455777884	4.15113848e − 03	0.326178163	0.94
	(0.45)	(4.1541611e − 03)	(0.2526549)	
ρ_{1_3}	0.404897749	3.96256289e − 03	0.315576911	0.95
	(0.40)	(3.9579375e − 03)	(0.2466156)	
ϕ_{1_1}	0.494047403	0.688315213	4.24205685	0.91
	(0.55)	(0.4414515)	(2.604519)	
ϕ_{1_2}	0.596532404	2.38425396e − 02	0.774129033	0.94
	(0.60)	(2.2643777e − 02)	(0.5898757)	
ϕ_{1_3}	0.151415542	2.99340468e − 02	0.855129421	0.95
	(0.15)	(2.7628275e − 02)	(0.6515728)	
λ_1	0.439792693	1.55844623e − 06	5.46646677e − 03	0.95
	(0.4398230)	(1.0486136e − 06)	(4.0141521e − 03)	
ω_1	0.157105446	7.16287389e − 07	3.64642008e − 03	0.95
	(0.1570796)	(4.5592520e − 07)	(2.6468716e − 03)	
ρ_{2_1}	0.210262716	3.31078214e − 03	0.198635176	0.81
	(0.20)	(3.3515587e − 03)	(0.2269392)	
ρ_{2_2}	0.600659609	2.89968797e − 03	0.189876512	0.82
	(0.60)	(2.9973800e − 03)	(0.2146135)	
ρ_{2_3}	0.404682457	2.62100901e − 03	0.175994471	0.81
	(0.40)	(2.6255806e − 03)	(0.2008625)	
ϕ_{2_1}	0.0978257582	9.84198451e − 02	1.11533058	0.83
	(0.10)	(9.2979610e − 02)	(1.195308)	
ϕ_{2_2}	0.501318812	9.60094389e − 03	0.359809935	0.85
	(0.50)	(1.0157371e − 02)	(0.3950724)	
ϕ_{2_3}	0.203538164	1.92437172e − 02	0.506877661	0.84
	(0.20)	(2.0647796e − 02)	(0.5632781)	
λ_2	1.13098848	5.99011685e − 07	3.59018217e − 03	0.95
	(1.1309734)	(9.1906406e − 07)	(3.7580191e − 03)	
ω_2	0.188493758	3.12252752e − 07	2.60419305e − 03	0.95
	(0.1884956)	(4.8829617e − 07)	(2.7392253e − 03)	

Table 2
The average LSEs, the MSEs, the average confidence lengths and the coverage probabilities of the different parameters using the proposed model for $\sigma^2 = 0.4$

Parameter	Average LSE (true value)	MSE (asym. var.)	Av. conf. length (ex. conf. length)	95% cov. prob.
ρ_{1_1}	0.151604086	8.27635452e − 03	0.476105064	0.95
	(0.10)	(8.6193085e − 03)	(0.3639337)	
ρ_{1_2}	0.46168986	8.26609321e − 03	0.461868048	0.94
	(0.45)	(8.3083222e − 03)	(0.3573080)	
ρ_{1_3}	0.411106139	7.85315502e − 03	0.446626604	0.95
	(0.40)	(7.9158749e − 03)	(0.3487671)	
ϕ_{1_1}	0.411025107	1.22210026	5.07852125	0.88
	(0.55)	(0.8829030)	(3.683347)	
ϕ_{1_2}	0.594926059	4.85680364e − 02	1.11277223	0.94
	(0.60)	(4.5287553e − 02)	(0.8342102)	
ϕ_{1_3}	0.152366251	6.14459403e − 02	1.23698533	0.95
	(0.15)	(5.5256549e − 02)	(0.9214631)	
λ_1	0.439781189	3.03807201e − 06	7.8823017e − 03	0.95
	(0.4398230)	(2.0972273e − 06)	(5.6768684e − 03)	
ω_1	0.157114819	1.42341173e − 06	5.31302486e − 03	0.95
	(0.1570796)	(9.1185041e − 07)	(3.7432418e − 03)	
ρ_{2_1}	0.220800266	6.48095924e − 03	0.281173646	0.81
	(0.20)	(6.7031174e − 03)	(0.3209405)	
ρ_{2_2}	0.602861106	5.78639749e − 03	0.268543154	0.82
	(0.60)	(5.9947600e − 03)	(0.3035093)	
ρ_{2_3}	0.409267187	5.22927288e − 03	0.249112844	0.81
	(0.40)	(5.2511613e − 03)	(0.2840624)	
ϕ_{2_1}	0.0954661742	0.232197076	1.66008222	0.82
	(0.10)	(0.1859592)	(1.690421)	
ϕ_{2_2}	0.501951993	1.93329826e − 02	0.514545619	0.85
	(0.50)	(2.0314742e − 02)	(0.5587168)	
ϕ_{2_3}	0.20508866	3.9053704e − 02	0.72747165	0.85
	(0.20)	(4.1295592e − 02)	(0.7965956)	
λ_2	1.13099003	1.19849585e − 06	5.12786489e − 03	0.94
	(1.1309734)	(1.8381281e − 06)	(5.3146416e − 03)	
ω_2	0.188494474	6.31346211e − 07	3.73536511e − 03	0.94
	(0.1884956)	(9.7659233e − 07)	(3.8738493e − 03)	

For comparison, we have also reported asymptotic variances and expected confidence lengths computed using the true values of the parameters. We perform same experiments on model (10), but using MFM, instead of model (1). In this case total number of non-linear parameters is $q_1 + q_2 = 6$. The results for MFM are reported in Tables 3 and 4.

Some of the points are quite clear from Tables 1 and 2. It is observed that for all the parameter estimators as the variance increases, average biases and MSEs increase. It verifies the consistency property of the LSEs. The non-linear frequency estimators are more accurate than the amplitude and phase estimators as the theory suggests. The MSEs and the corresponding asymptotic variances of all the estimators are quite close to each other. The coverage percentages of the parameters associated with first fundamental frequency

Table 3
The average LSEs, the MSEs, the average confidence lengths and the coverage probabilities of different parameters using MFM for $\sigma^2 = 0.2$

Parameter	Average LSE (true value)	MSE (asym. var.)	Av. conf. length (ex. conf. length)	95% cov. prob.
a_1	0.100683041	6.62101229e − 05 (4.30965424e − 03)	0.220231146 (0.257339984)	1.0
a_2	0.451257676	3.44792061e − 04 (4.15416108e − 03)	0.216402516 (0.25265491)	1.0
a_3	0.400900543	2.59834371e − 04 (3.95793747e − 03)	0.212928489 (0.246615589)	1.0
a_4	0.200601995	9.68176464e − 05 (3.35155893e − 03)	0.152528748 (0.226939201)	1.0
a_5	0.601447225	5.48608368e − 04 (2.99737975e − 03)	0.138830096 (0.214613453)	0.95
a_6	0.400943607	2.53332662e − 04 (2.62558088e − 03)	0.135277435 (0.200862452)	0.98
ϕ_1	0.551747084	4.09032963e − 03 (1.72386169)	4.39787626 (5.14680004)	1.0
ϕ_2	0.60472846	6.82766503e − 03 (8.20575058e − 02)	0.960425317 (1.12291074)	0.99
ϕ_3	0.151506901	3.14068445e − 03 (9.8948434e − 02)	1.06390452 (1.233078)	1.0
ϕ_4	0.0995254889	1.77992496e − 03 (0.335155904)	1.52432573 (2.26939178)	1.0
ϕ_5	0.501218498	5.90951648e − 03 (3.33042182e − 02)	0.462355971 (0.715378225)	0.95
ϕ_6	0.200857386	2.44277902e − 03 (6.56395182e − 02)	0.675922155 (1.00431228)	0.99
β_1	0.439534605	5.60617264e − 05 (1.29289634e − 04)	3.80868129e − 02 (4.45725955e − 02)	0.95
β_2	0.596982002	2.0185114e − 06 (6.15431281e − 06)	8.31752364e − 03 (9.72469151e − 03)	0.97
β_3	0.754041851	2.13030876e − 06 (7.42113252e − 06)	9.21367202e − 03 (1.06787682e − 02)	0.98
β_4	1.1309551	9.4032639e − 06 (2.51366928e − 05)	1.3201056e − 02 (1.96535103e − 02)	0.93
β_5	1.31950402	9.81566586e − 07 (2.49781647e − 06)	4.00412921e − 03 (6.1953566e − 03)	0.89
β_6	1.50798428	1.40446798e − 06 (4.92296385e − 06)	5.85365482e − 03 (8.69759917e − 03)	0.94

attain the nominal level for all the parameters except ϕ_{1_1} when $\sigma^2 = 0.4$, but in case of the second one, the amplitude and phase estimators do not attain the nominal level and they are quite poor, whereas the performance of the frequency estimators are satisfactory in all the cases considered. Since the expected confidence lengths are quite close to the average confidence lengths for all the parameters, the estimation of $\sigma^2 c_k(j)$ is quite reasonable and the asymptotic results can be used in making finite sample inference.

Table 4
The average LSEs, the MSEs, the average confidence lengths and the coverage probabilities of different parameters using MFM for $\sigma^2 = 0.4$

Parameter	Average LSE (true value)	MSE (asym. var.)	Av. conf. length (ex. conf. length)	95% cov. prob.
a_1	0.103271469	3.12825432e − 04 (8.61930847e − 03)	0.310845464 (0.363933712)	1.0
a_2	0.453495115	8.75412254e − 04 (8.30832217e − 03)	0.303890526 (0.357308)	0.99
a_3	0.402786911	6.71417103e − 04 (7.91587494e − 03)	0.299821109 (0.348767132)	1.0
a_4	0.20187901	3.19670828e − 04 (6.70311786e − 03)	0.213473439 (0.320940495)	0.99
a_5	0.603555858	1.3390953e − 03 (5.9947595e − 03)	0.195176035 (0.303509265)	0.95
a_6	0.402185053	6.61559461e − 04 (5.25116175e − 03)	0.190110669 (0.284062415)	0.97
ϕ_1	0.553592145	8.26491881e − 03 (3.44772339)	6.60993528 (7.27867413)	1.0
ϕ_2	0.610250294	1.67299565e − 02 (0.164115012)	1.345227 (1.5880357)	0.99
ϕ_3	0.152599573	7.37958541e − 03 (0.197896868)	1.49412251 (1.74383569)	1.0
ϕ_4	.0999217778	3.84876621e − 03 (0.670311809)	2.12972307 (3.20940495)	1.0
ϕ_5	0.50403589	1.38942935e − 02 (6.660084364e − 02)	0.648625791 (1.01169753)	0.95
ϕ_6	0.202204913	6.29422953e − 03 (0.131279036)	0.948941886 (1.42031193)	0.99
β_1	0.439267337	9.31811592e − 05 (2.58579268e − 04)	5.72436824e − 02 (6.30351603e − 02)	0.95
β_2	0.597042143	4.41594557e − 06 (1.23086256e − 05)	1.16500212e − 02 (1.37527911e − 02)	0.97
β_3	0.754063606	4.70306395e − 06 (1.4842265e − 05)	1.2939482e − 02 (1.51020577e − 02)	0.97
β_4	1.13074279	2.5412297e − 05 (5.02733856e − 05)	1.84439197e − 02 (2.77942587e − 02)	0.90
β_5	1.31954181	2.17460683e − 06 (4.99563293e − 06)	5.61727071e − 03 (8.76155775e − 03)	0.88
β_6	1.50802243	3.02026615e − 06 (9.84592771e − 06)	8.2180649e − 03 (1.23002622e − 02)	0.93

Now comparing the estimators obtained using the proposed model and MFM, we observe that the amplitude and phase parameter estimators are estimated more accurately in terms of biases and MSEs if MFM is used. On the other hand the non-linear frequency estimators are more accurate if model (1) is used. The average confidence lengths for amplitudes are larger in case of model (1), whereas for phase estimators, they are larger in case of MFM. In case of MFM, 95% coverage percentages cover all the time for all the phase and amplitude estimators except a_5 and ϕ_5, for the model considered. For the frequencies also, they do not attain the nominal level in general.

4. Data analysis

In this section we use the proposed model (1) for analyzing several real datasets. We would like to mention here that several short-duration voiced speech signal, namely "eee", "aaa", "aww", "uuu" and "ahh", can be analyzed using model (1) and we present the analysis in this section. But as already mentioned in introduction that there are many other applications where this model can be satisfactorily used. The plots of the observed data sets and corresponding periodogram functions are provided in Figs. 1–10. The data set "ahh" contains 340 signal values whereas each of all the other data sets contains 512 signal values, all sampled at 10 kHz frequency. We have estimated M and q_k, $k=1,\ldots,M$ from the periodogram plots. The periodogram is a powerful tool for locating the frequencies visually. If the observed data are periodic, the plot of the periodogram function exhibits large positive values (the squares of the amplitudes associated with the frequencies) at the true values of the underlying frequencies present in the data and at all the other points it is close to zero. Note that, we have considered the simple periodogram function, not the smoothed periodogram, which is commonly known as spectrogram in the time series literature. Also, we have calculated $I(\lambda)$, for each point of a grid (fine enough) of $(0,\pi)$. We have not calculated $I(\lambda)$ only at the so-called Fourier frequencies $\{2\pi j/N, j=0,1,\ldots,N-1\}$. So the number of peaks in the plot of the periodogram function gives an estimate of the number of effective frequencies present in the underlying model, or it roughly estimates the number of components p if one uses model (5). It may be quite subjective sometimes, depending on the error variance and magnitude of the amplitudes. The periodogram may show only the more dominant frequencies. In such cases, when the effects of these frequencies are removed from the observed series and the periodogram function of the residual series is plotted, then it may show some peaks corresponding to other frequencies. If the error variance is too

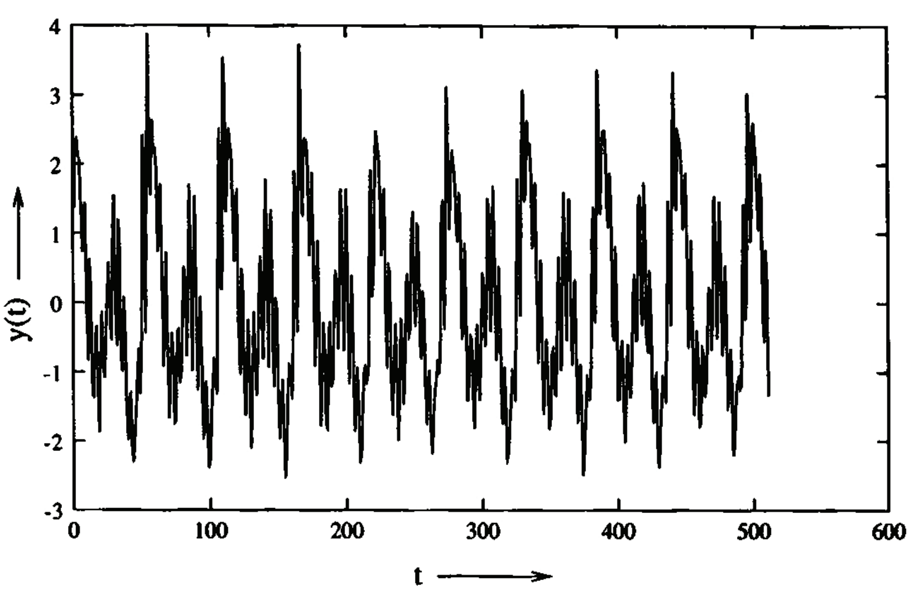

Fig. 1. The plot of the observed "eee" vowel sound.

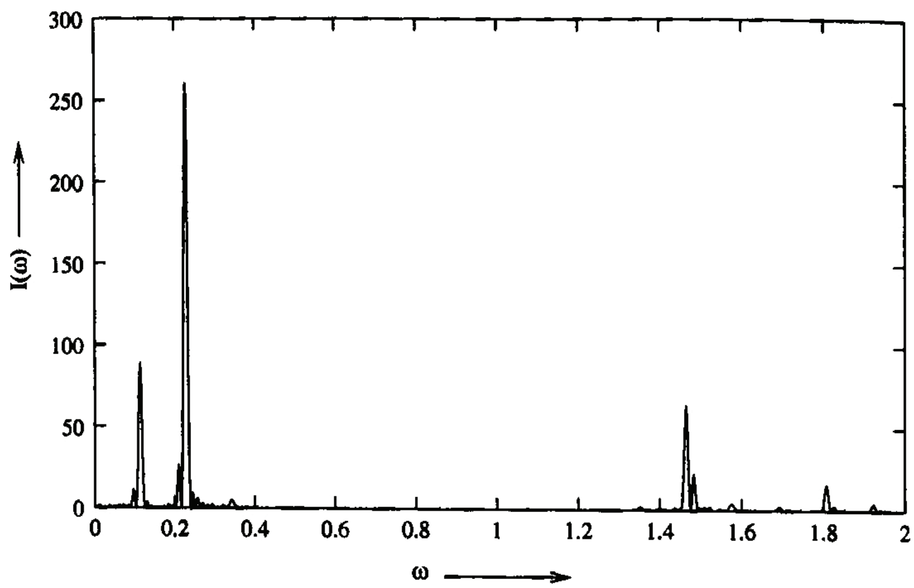

Fig. 2. The plot of the periodogram function of "eee" sound.

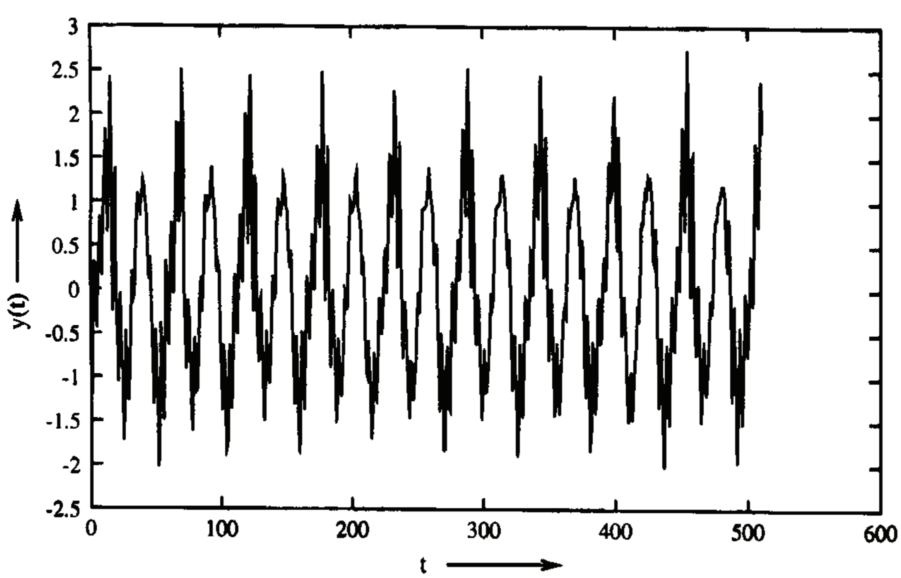

Fig. 3. The plot of the observed "aaa" vowel sound.

high, periodogram plot may not exhibit a significant distinct peak at λ^*, even if this λ^* has a significant contribution to the data. Also, in case, two frequencies are "close enough" then periodogram may show only one peak. In such cases it is recommended to use of larger sample size, if it is possible and use of a finer grid may provide some more information about the presence of another frequency.

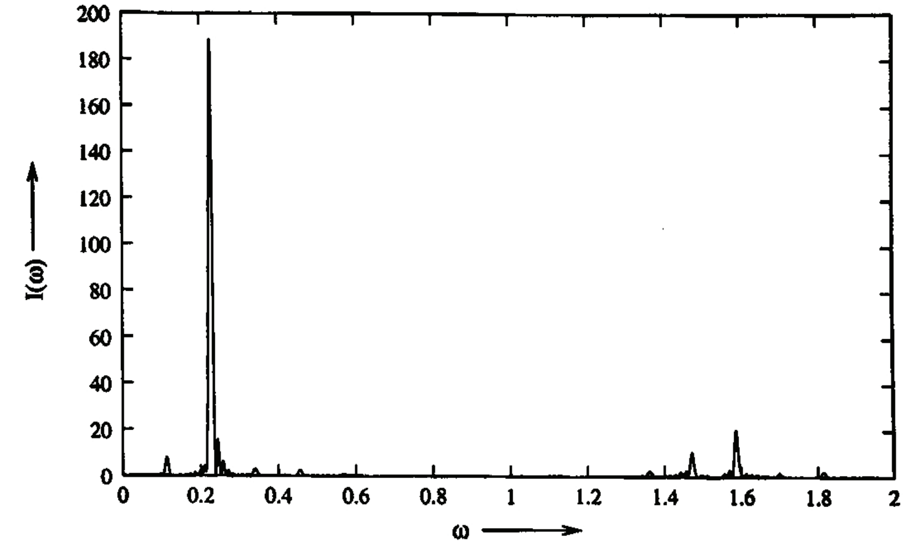

Fig. 4. The plot of the periodogram function of "aaa" vowel sound.

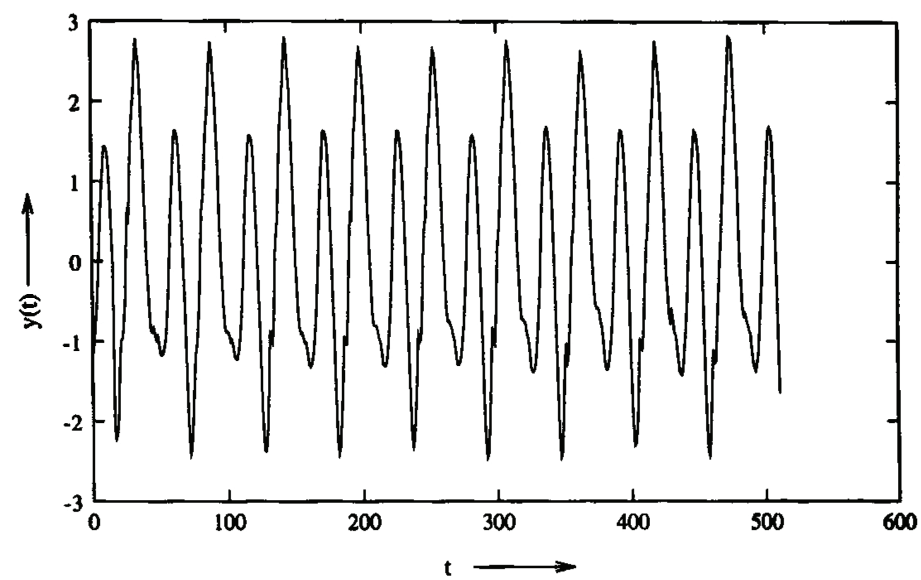

Fig. 5. The plot of the observed "uuu" vowel sound.

The initial estimates of the frequencies are obtained from the plot of the periodogram function. Using these initial estimates as starting values, the LSEs of the unknown parameters are obtained for all the data sets. Using Theorem 2.2, we also calculate 95% confidence intervals of the LSEs. To see how the proposed model (1) performs as compared to the general MFM (5), we estimate the LSEs of the unknown parameters of MFM. We also obtain 95% confidence intervals using the asymptotic distribution (Kundu, 1997) of the LSEs of the parameters of MFM in case of each data set. In analyzing these data sets we

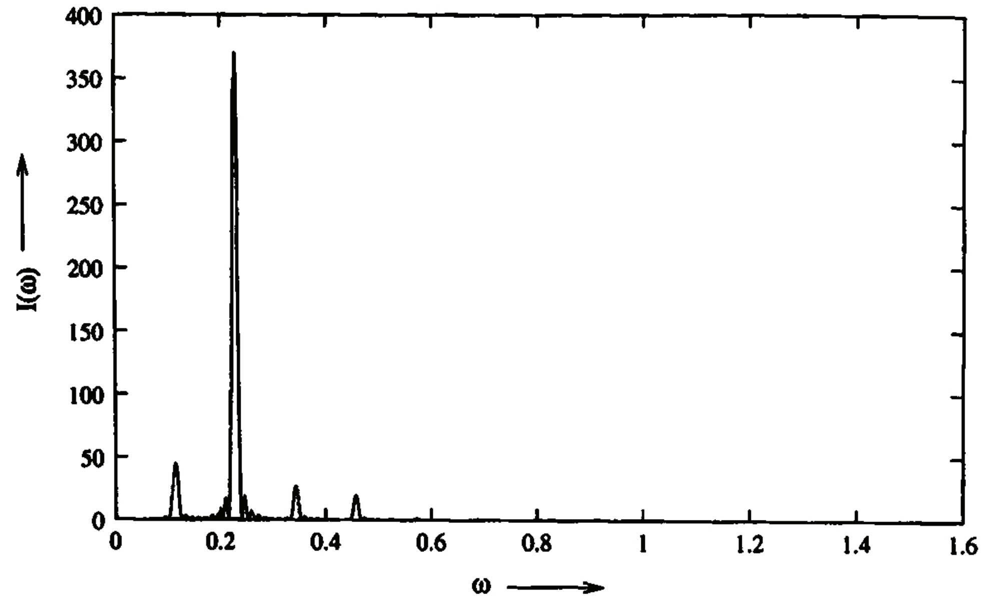

Fig. 6. The plot of the periodogram function of "uuu" vowel sound.

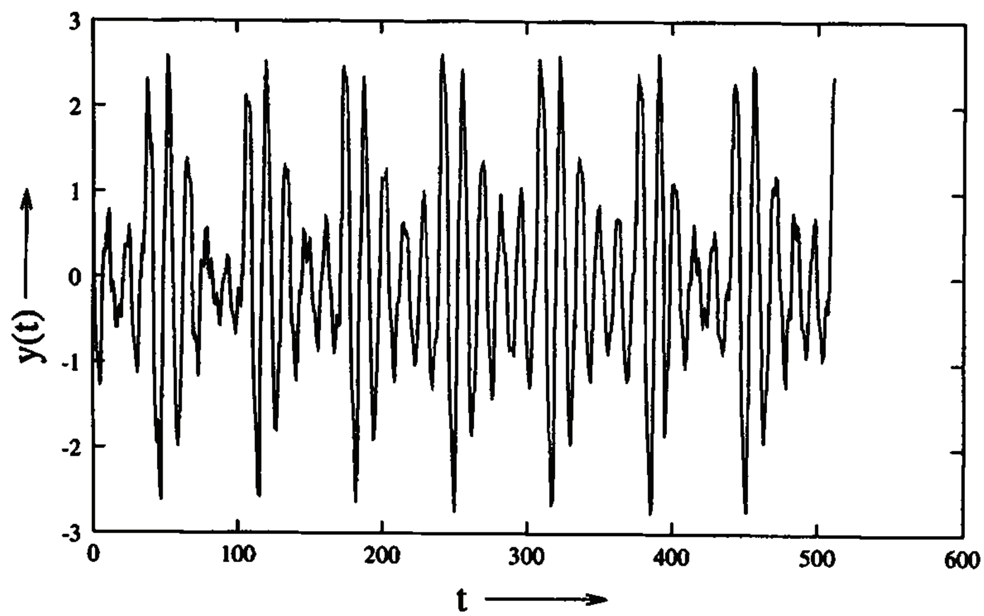

Fig. 7. The plot of the observed "aww" sound.

use estimated error random variables to estimate $\sigma^2 c_k(j)$, $j = 1, \ldots, q_k$; $k = 1, \ldots, M$. We use run test (Draper and Smith, 1981) to test whether the estimated error is independent or not. For "eee" data set the estimated errors are independent for both models, whereas for all the other data sets, the test statistic value confirms that the errors are correlated. Using autocorrelation and partial autocorrelation function we model the error processes as different autoregressive (AR) processes in all such cases. In case of "aaa" with model (1)

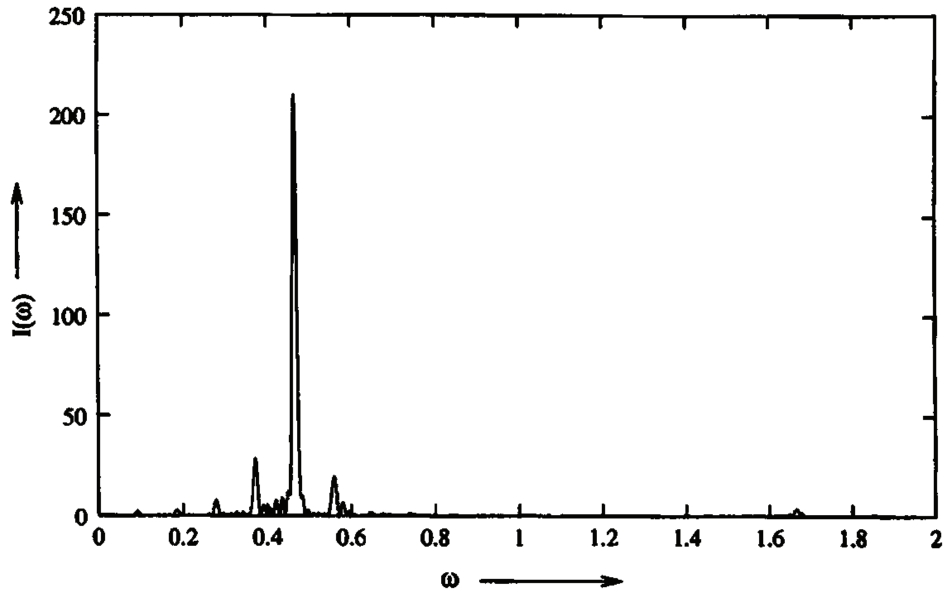

Fig. 8. The plot of the periodogram function of "aww" sound.

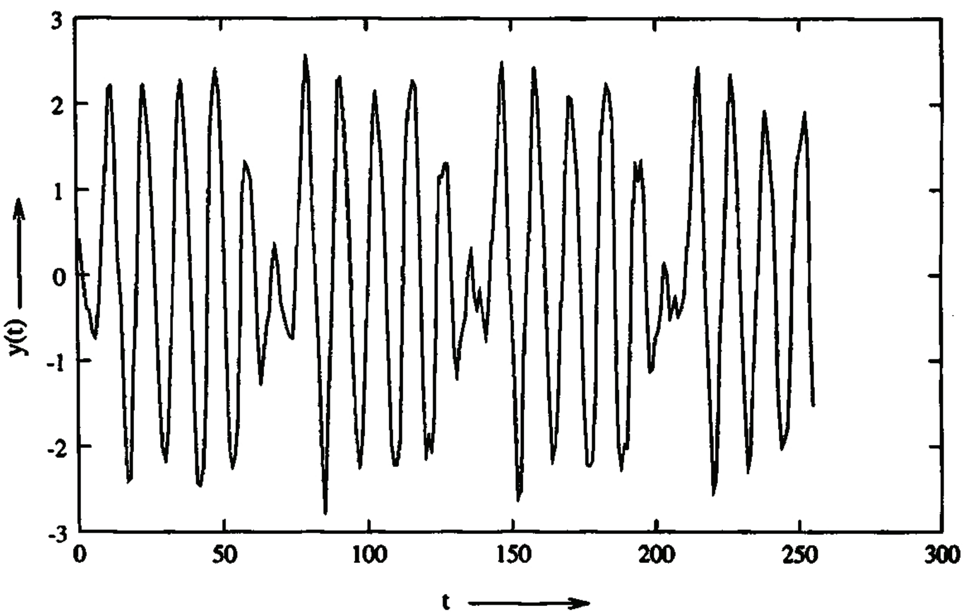

Fig. 9. The plot of the observed "ahh" sound.

the error is modeled as an AR(3) process whereas with MFM, it is modeled as an AR(1) process. For "ahh", "aww" and "uuu" the residuals are modeled as different AR(3) processes. We estimate the AR parameters using Yule–Walker equation. Finally, we again use the run test to verify whether the independence assumption on $\hat{e}(t)$ is satisfied at 95% level of significance or not. We see that in all the cases $\hat{e}(t)$ satisfies the independence assumption except "aww" data set when MFM is used for estimation. As the estimated error is used

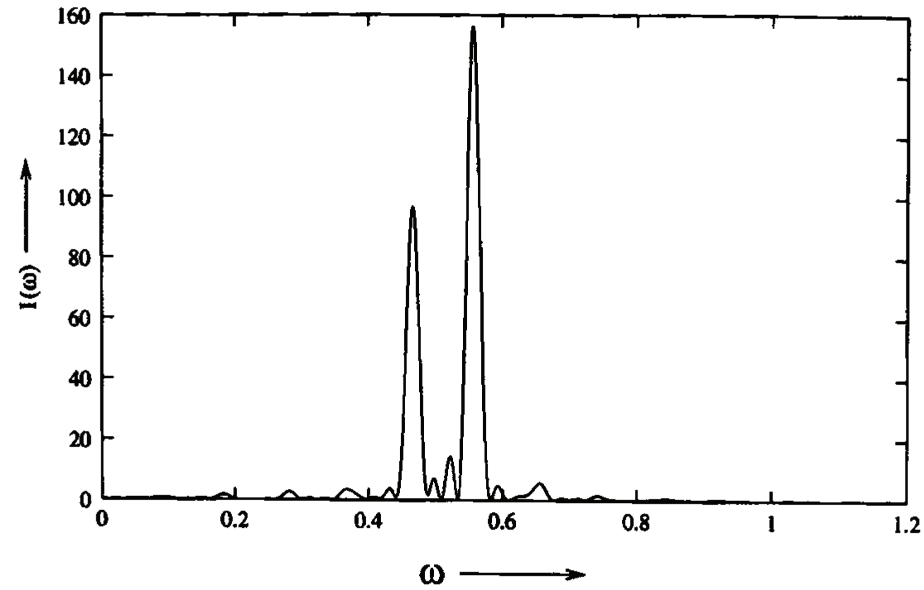

Fig. 10. The plot of the periodogram function of "ahh" sound.

to estimate $\sigma^2 c_k(j)$'s for these data sets, in case of "aww" data, we apply run test to the $\hat{e}(1), \ldots, \hat{e}(256)$ and then independence is satisfied. For "aww" data, we have used the fact that frequencies appear as harmonics of the first fundamental frequency. In each case the roots of the characteristic equation of the estimated AR process are less than one in absolute value, so the estimated error is stationary and can be expressed as linear process given in (4). The results for model (1) are provided in Tables 5, 7, 9, 11 and 13 and for MFM in Tables 6, 8, 10, 12 and 14.

The predicted signals say $\hat{y}(t)$ for all data sets are provided in Figs. 11–15 for "eee", "aaa", "uuu", "aww" and "ahh", respectively. For comparison, we have plotted the predicted values using LSEs of model (1), predicted values using LSEs of MFM and the original signal in the same figure. The fitted values match quite well with the original signal in all the cases.

We observe that in case of "eee" and "aaa" data sets the confidence intervals of all the parameters corresponding to first fundamental frequency λ_1^0 and amplitudes corresponding to second fundamental frequency are slightly larger in case they are estimated using model (1) than those obtained with MFM. But the confidence lengths of λ_2 and ω_2 and corresponding phases are much smaller in case of model (1). In case of "ahh" data set there is only one frequency and the confidence lengths of the phases are larger in case of MFM. For "aww" data set confidence intervals of amplitudes associated with λ_1 is much higher in model (1), whereas for phases, it is the other way. The confidence interval for frequency λ_1 (here it is used that $\lambda_1 = \omega_1$) is much lower as compared to β_1 of MFM. For second frequency, they are almost identical (but in this case there is only one frequency, so theoretically asymptotic variances are equal) for both the models. For "uuu" data set the confidence intervals of all the parameters are smaller in case of MFM. But in "uuu" data set like "ahh", $M = 1$ and the total number of parameters is 10. If we use the information that $\lambda_1 = \omega_1$, i.e. if the

Table 5
Results for "eee" data set using the proposed model

Parameter	Estimate	Lower bound	Upper bound
ρ_{1_1}	0.792912781	0.745130777	0.840694785
ρ_{1_2}	1.41441321	1.36663115	1.46219528
ρ_{1_3}	0.203153163	0.15537113	0.250935197
ϕ_{1_1}	0.681846261	0.434020698	0.929671824
ϕ_{1_2}	0.913759112	0.747117102	1.08040118
ϕ_{1_3}	−0.0415502712	−0.298853695	0.215753138
λ_1	0.1140192	0.113080189	0.114958212
ω_1	0.113874406	0.11353147	0.114217341
ρ_{2_1}	0.110554226	0.0627721995	0.158336252
ρ_{2_2}	0.708103955	0.660321951	0.755885959
ρ_{2_3}	0.170492932	0.122710906	0.218274966
ρ_{2_4}	0.141166449	0.0933844224	0.188948482
ρ_{2_5}	0.344861776	0.297079742	0.392643809
ρ_{2_6}	0.16529128	0.117509253	0.213073313
ϕ_{2_1}	0.427151084	−0.0344740637	0.888776243
ϕ_{2_2}	0.479345709	0.34814921	0.610542178
ϕ_{2_3}	−2.71241069	−3.01004577	−2.41477561
ϕ_{2_4}	0.815808356	0.451096743	1.18051994
ϕ_{2_5}	−0.773955345	−1.01284397	−0.535066724
ϕ_{2_6}	−0.577183068	−0.966903508	−0.187462628
λ_2	1.3514533	1.35081983	1.35208678
ω_2	0.11346291	0.113172941	0.113752879

Data set: "eee". $M = 2, q_1 = 3, q_2 = 6$.
$\hat{X}(t) = e(t)$.
Run test: z for $\hat{e}(t) = -1.10903132$.
Residual sum of squares: 0.154122174.

fundamental frequency model given in (6) is used, the number of non-linear parameters reduces to one from two. Using the asymptotic distribution of $(\hat{\lambda}_1 - \hat{\omega}_1)$, it is observed that the confidence interval of $(\lambda_1 - \omega_1)$ is $(-0.004721, 0.002557)$ which includes zero. Thus, we accept the hypothesis $H_0 : \lambda_1 - \omega_1 = 0$. So for this particular data set it is reasonable to use the fundamental frequency model rather than the proposed model (1). If we use the fundamental frequency model (6), then we see that the confidence intervals for phases and frequency λ_1 is much lower as compared to MFM (not reported here). Note that the asymptotic variances of a particular frequency of MFM is inversely proportional to the square of the associated amplitude and they are independent of the other frequencies, which is not true in case of the proposed model. In this case the asymptotic variances of λ_k and ω_k depend on all the amplitudes ρ_{k_j}, $j = 1, \ldots, q_k$. Thus, we have seen that several short-duration voiced speech data can be analyzed using the model (1). In analyzing these data sets, neither of the two models outperforms the other in all respects. But the advantage of

Table 6
LSE and confidence intervals for "eee" data with multiple frequency model

Parameters	LSE	Lower limit	Upper limit
a_1	0.792754471	0.745822906	0.839686036
a_2	1.41381097	1.36687934	1.46074259
a_3	0.203779265	0.1568477	0.250710845
a_4	0.111959115	0.0650275424	0.158890679
a_5	0.7039783	0.657046735	0.750909865
a_6	0.172563389	0.125631824	0.219494954
a_7	0.145034194	0.0981026217	0.191965759
a_8	0.355013132	0.308081567	0.401944697
a_9	0.176291764	0.129360199	0.223223329
ϕ_1	0.687658191	0.569256902	0.80605948
ϕ_2	0.912467241	0.846077085	0.978857398
ϕ_3	0.0317988843	−0.428812951	0.492410719
ϕ_4	0.628177047	−0.21019274	1.46654689
ϕ_5	0.495831549	0.362499118	0.62916398
ϕ_6	2.82846522	2.28453088	3.37239957
ϕ_7	0.847214043	0.200034678	1.49439347
ϕ_8	−0.360235542	−0.624628961	−0.0958421007
ϕ_9	0.00214044028	−0.530290186	0.534571111
β_1	0.114038095	0.113637552	0.114438638
β_2	0.227885813	0.227661222	0.228110403
β_3	0.342097998	0.340539783	0.343656212
β_4	1.35217214	1.34933603	1.35500824
β_5	1.46495438	1.46450329	1.46540546
β_6	1.5753814	1.57354128	1.57722151
β_7	1.69190764	1.68971825	1.69409704
β_8	1.80700564	1.80611122	1.80790007
β_9	1.92089081	1.91908967	1.92269194

$\hat{X}(t) = e(t)$.
Run test: z for $\hat{e}(t) = -1.03238821$.
Residual sum of squares: 0.148744926.

using the proposed model is that the total number of non-linear parameters to be estimated, reduces as compared to the number of the effective frequencies. Several non-stationary data follow a particular relationship among the frequencies and it is captured by the proposed model.

5. Conclusions

In this paper, we propose a new model with multiple fundamental frequencies in stationary noise. The model is a particular model of the multiple frequency model (5) and a

Table 7
Results for "aaa" data set using the proposed model

Parameter	Estimate	Lower bound	Upper bound
ρ_{1_1}	0.220683411	0.16870077	0.272666067
ρ_{1_2}	1.21162212	1.16022503	1.2630192
ρ_{1_3}	0.160044715	0.109573394	0.210516036
ρ_{1_4}	0.138260633	0.088989988	0.187531278
ϕ_{1_1}	2.11266017	1.35893762	2.8663826
ϕ_{1_2}	2.74804449	2.53007388	2.9660151
ϕ_{1_3}	−0.129586205	−0.89653182	0.637359381
ϕ_{1_4}	0.377879053	−1.02523708	1.78099513
λ_1	0.113897234	0.11110048	0.116693988
ω_1	0.113964394	0.111329563	0.116599225
ρ_{2_1}	0.146804646	0.109372251	0.184237033
ρ_{2_2}	0.245368242	0.209165379	0.28157112
ρ_{2_3}	0.377789408	0.342714161	0.412864655
ρ_{2_4}	0.0969588906	0.0629100502	0.131007731
ρ_{2_5}	0.133374184	0.100253083	0.166495293
ϕ_{2_1}	2.94918633	2.42440987	3.47396278
ϕ_{2_2}	0.0518640578	−0.357301384	0.4610295
ϕ_{2_3}	1.36409092	0.98188448	1.74629736
ϕ_{2_4}	2.10673928	1.55073655	2.66274214
ϕ_{2_5}	−0.706613243	−1.30023837	−0.112988077
λ_2	1.36052275	1.35873103	1.36231446
ω_2	0.11399442	0.113307588	0.1146212

Data set: "aaa". $M=2, q_1=4, q_2=5$.
$\hat{X}(t)=0.696457803\ \hat{X}(t-1)-0.701408327\ \hat{X}(t-2)+0.618664384\ \hat{X}(t-3)+e(t)$.
Run test: z for $\hat{X}(t)=-5.87313509$, z for $\hat{e}(t)=0.628699183$.
Residual sum of squares: 0.101041436.

generalization of the fundamental frequency model (6) as well as the harmonic model (7). It is observed that several non-stationary signals can be analyzed using this model. To analyze the data sets, the number of fundamental frequencies, M and the number of frequencies associated with kth fundamental frequency q_k, $k=1,\ldots,M$ are estimated using the periodogram function. We have proposed the usual LSEs to estimate the unknown parameters. The estimators are strongly consistent and asymptotically normal. The asymptotic distribution indicates that the estimators of the unknown parameters corresponding to different fundamental frequencies are independent and amplitude estimators are independent of the corresponding phase and frequency estimators. The experimental results indicate that the asymptotic results can be used in making finite sample inferences. Several real data are analyzed and the estimated signals match quite well with the observed signals in each case. The asymptotic distribution is used to construct the confidence bounds of each parameter at 95% level of significance. In this paper, we have not considered the problem of estimating M and q_k. Some information theoretic criteria combined with the special structure

Table 8
LSE and confidence intervals for "aaa" data with multiple frequency model

Parameters	LSE	Lower limit	Upper limit
a_1	0.226719305	0.190961957	0.262476653
a_2	1.21300876	1.17725146	1.24876606
a_3	0.163902849	0.128145501	0.199660197
a_4	0.137947291	0.102189943	0.173704639
a_5	0.143389478	0.10763213	0.179146826
a_6	0.266157418	0.230400071	0.301914752
a_7	0.38930583	0.353548497	0.425063163
a_8	0.101906128	0.0661487803	0.137663469
a_9	0.134322852	0.098565504	0.1700802
ϕ_1	2.1619103	1.84647751	2.47734308
ϕ_2	2.54826617	2.48930979	2.60722256
ϕ_3	−0.0690812841	−0.505404949	0.367242366
ϕ_4	0.371035159	−0.147385299	0.889455616
ϕ_5	3.04547048	2.54672599	3.54421496
ϕ_6	−0.602789819	−0.871483028	−0.334096611
ϕ_7	0.90565449	0.721956491	1.08935249
ϕ_8	1.58608973	0.884319425	2.28785992
ϕ_9	−1.12101579	−1.65342474	−0.588606775
β_1	0.114094153	0.113027073	0.115161233
β_2	0.227092206	0.226892769	0.227291644
β_3	0.34200263	0.340526581	0.34347868
β_4	0.455799341	0.454045564	0.457553118
β_5	1.36083913	1.35915196	1.3625263
β_6	1.47200274	1.47109377	1.47291172
β_7	1.58692384	1.5863024	1.58754528
β_8	1.70071459	1.69834054	1.70308864
β_9	1.81501627	1.81321514	1.8168174

$\hat{X}(t) = 0.31577149\ \hat{X}(t-1) + e(t)$.
Run test: z for $\hat{X}(t) = -3.13001537$, z for $\hat{e}(t) = 0.26275149$.
Residual sum of squares: 0.0874886289.

of the proposed model may be used to estimate them. Further research is needed in this direction.

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Table 9
Results for "uuu" data set using the proposed model

Parameter	Estimate	Lower bound	Upper bound
ρ_{1_1}	0.633288026	0.544679284	0.721896768
ρ_{1_2}	1.70542562	1.61684871	1.79400253
ρ_{1_3}	0.419593066	0.331111431	0.508074701
ρ_{1_4}	0.343196869	0.254873455	0.431520283
ϕ_{1_1}	−2.61283183	−3.14323187	−2.08243179
ϕ_{1_2}	1.33654237	1.08859777	1.58448696
ϕ_{1_3}	−2.88876963	−3.43958592	−2.33795333
ϕ_{1_4}	−2.69739866	−3.66095066	−1.73384655
λ_1	0.113104612	0.111106128	0.115103096
ω_1	0.114186443	0.112432718	0.115940168

Data set: "uuu". $M = 1, q_1 = 4$.
$\hat{X}(t) = 1.2119236\ \hat{X}(t-1) - 0.518095672\ \hat{X}(t-2) + 0.0920835361\ \hat{X}(t-2) + e(t)$.
Run test: z for $\hat{X}(t) = -14.5708122$, z for $\hat{e}(t) = -1.03364444$.
Residual sum of squares: 0.0981521457.

Table 10
LSE and confidence intervals for "uuu" data with multiple frequency model

Parameters	LSE	Lower limit	Upper limit
a_1	0.63175	0.57488	0.68861
a_2	1.71710	1.66195	1.77225
a_3	0.43196	0.37941	0.48452
a_4	0.35917	0.30972	0.40861
ϕ_1	−2.41835	−2.59838	−2.23832
ϕ_2	1.52894	1.46471	1.59318
ϕ_3	−2.30741	−2.55075	−2.06407
ϕ_4	−2.08317	−2.35851	−1.80782
β_1	0.11390	0.11329	0.11451
β_2	0.22804	0.22782	0.22825
β_3	0.34376	0.34294	0.34458
β_4	0.45793	0.45700	0.45886

$\hat{X}(t) = 1.09189606\ \hat{X}(t-1) - 0.502943218\ \hat{X}(t-2) + 0.102927223\ \hat{X}(t-2) + e(t)$.
Run test: z for $\hat{X}(t) = -12.1590595$, z for $\hat{e}(t) = -1.48203266$.
Residual sum of squares: 0.0620818324.

Appendix A

For notational convenience, first we prove the results for $M = 1$, i.e. the model has only one fundamental frequency and then we sketch the outline of the proof for general M.

Table 11
Results for "aww" data set using the proposed model

Parameter	Estimate	Lower bound	Upper bound
ρ_{1_1}	0.127831191	0.0166739132	0.238988474
ρ_{1_2}	0.164348915	0.0448353849	0.283862442
ρ_{1_3}	0.242797658	0.107381746	0.378213555
ρ_{1_4}	0.490236014	0.328855962	0.651616037
ρ_{1_5}	1.11415863	0.920606256	1.30771101
ρ_{1_6}	0.362586141	0.164035648	0.561136603
ϕ_{1_1}	−2.36564922	−3.23676491	−1.49453342
ϕ_{1_2}	0.397528142	−0.337061763	1.13211799
ϕ_{1_3}	−2.53827643	−3.11740494	−1.95914781
ϕ_{1_4}	0.833990335	0.444618791	1.22336185
ϕ_{1_5}	−1.08417308	−1.39682209	−0.771524072
ϕ_{1_6}	−2.51970673	−3.14991474	−1.88949871
λ_1	0.0922982097	0.0920951292	0.0925012901
ρ_{2_1}	0.168908238	0.14887704	0.188939437
ϕ_{2_1}	0.61500001	0.377815574	0.852184474
λ_2	1.66371846	1.66291606	1.66452086

Data set: "aww". $M=2, q_1=6, q_2=1$.
$\hat{X}(t)=1.31350672\ \hat{X}(t-1)-0.511514068\ \hat{X}(t-2)-0.103843123\ \hat{X}(t-2)+e(t)$.
Run test: z for $\hat{X}(t)=-15.0958452$, z for $\hat{e}(t)=-0.89579612$.
Residual sum of squares: 0.484496683.

For $M=1$, $\Psi=\boldsymbol{\theta}_1=\boldsymbol{\theta}$, $q_1=q$, say and let us write $\boldsymbol{\theta}=(\rho_1,\ldots,\rho_q,\phi_1,\ldots,\phi_q,\lambda,\omega)$. $\boldsymbol{\theta}^0=(\rho_1^0,\ldots,\rho_q^0,\phi_1^0,\ldots,\phi_q^0,\lambda^0,\omega^0)$ and $\hat{\boldsymbol{\theta}}=(\hat{\rho}_1,\ldots,\hat{\rho}_q,\hat{\phi}_1,\ldots,\hat{\phi}_q,\hat{\lambda},\hat{\omega})$ denote the true value of $\boldsymbol{\theta}$ and the LSE of $\boldsymbol{\theta}^0$, respectively. We need the following lemmas to prove the theorems.

Lemma 1. *If* $X(t)$ *satisfies Assumption* 4, *then*

$$\lim_{N\to\infty}\sup_{0\leqslant\gamma\leqslant\pi}\left|\frac{1}{N^{L+1}}\sum_{t=1}^{N}U(t)t^L\cos(\gamma t)\right|=0\quad a.s.,$$

$$\lim_{N\to\infty}\sup_{0\leqslant\gamma\leqslant\pi}\left|\frac{1}{N^{L+1}}\sum_{t=1}^{N}U(t)t^L\sin(\gamma t)\right|=0\quad a.s.$$

for $L=0,1,2,\ldots$.

Table 12
LSE and confidence intervals for "aww" data with multiple frequency model

Parameters	LSE	Lower limit	Upper limit
a_1	0.120224684	0.0445712507	0.195878118
a_2	0.152249157	0.0754743591	0.229023963
a_3	0.238393679	0.160011902	0.316775471
a_4	0.514508545	0.434411645	0.594605446
a_5	1.3179431	1.23688722	1.39899898
a_6	0.456269532	0.376150042	0.536388993
a_7	0.172196656	0.154326186	0.190067127
ϕ_1	−2.42837739	−3.68691158	−1.16984332
ϕ_2	0.851819396	−0.156722158	1.86036098
ϕ_3	−2.27399015	−2.93157291	−1.61640739
ϕ_4	1.53354931	1.22219622	1.8449024
ϕ_5	0.00387152098	−0.119132072	0.126875117
ϕ_6	−1.31130302	−1.66249669	−0.960109353
ϕ_7	0.664068937	0.456509978	0.871627867
β_1	0.0920180827	0.0877605751	0.0962755904
β_2	0.186266482	0.182854667	0.189678296
β_3	0.278118253	0.275893718	0.280342788
β_4	0.372019708	0.370966434	0.373072982
β_5	0.46590662	0.46549052	0.46632272
β_6	0.558983684	0.557795644	0.560171723
β_7	1.66409254	1.6633904	1.66479468

$\hat{X}(t) = 1.21117878\ \hat{X}(t-1) - 0.603570342\ \hat{X}(t-2) + 0.0734062716\ \hat{X}(t-2) + e(t)$.
Run test: z for $\hat{X}(t) = -13.1944799$ (using whole data set),
z for $\hat{X}(t) = -8.97054672$ (using first 256 data points).
z for $\hat{e}(t) = -1.72880948$ (using first 256 data points).
Residual sum of squares: 0.193716243.

Proof of Lemma 1. For $L = 0$, the result is available in Kundu (1997). For general L, the result follows using the fact that $t/N < 1$. The lemma also follows from Theorem 4.5.1 in Brillinger (1981, p. 98). □

Comment: Lemma 1 is a very strong result and has been proved under different conditions. Walker (1971) proved for i.i.d. errors. Hannan (1973) proved it under ergodic and purely non-deterministic conditions. Kundu (1997) provided the proof for stationary linear processes, Brillinger (1986) and Nandi et al. (2002) proved a version of this lemma for spatial point processes and for i.i.d. stable processes, respectively.

Lemma 2. *Let us define*

$$S_{\delta,K} = \{\boldsymbol{\theta} : |\lambda - \lambda^0| > \delta \text{ or } |\omega - \omega^0| > \delta \text{ or } |\rho_j - \rho_j^0| > \delta \text{ or } |\phi_j - \phi_j^0| > \delta$$
$$\text{for any } j = 1, \ldots, q, \text{ and } \rho_k \leqslant K \text{ for all } k = 1, \ldots, q\}.$$

Table 13
Results for "ahh" data set using the proposed model

Parameter	Estimate	Lower bound	Upper bound
ρ_{1_1}	0.154038683	0.00933268107	0.298744678
ρ_{1_2}	0.110609308	−0.038386818	0.259605438
ρ_{1_3}	0.20130381	0.044975698	0.357631922
ρ_{1_4}	0.219141886	0.0524176359	0.385866135
ρ_{1_5}	0.973635674	0.794148386	1.1531229
ρ_{1_6}	1.39281392	1.20093036	1.58469748
ϕ_{1_1}	1.51953971	0.227566779	2.81151271
ϕ_{1_2}	1.95636904	0.578811526	3.33392644
ϕ_{1_3}	2.49770141	1.72301531	3.2723875
ϕ_{1_4}	−2.94550371	−3.68023229	−2.21077514
ϕ_{1_5}	−2.23193026	−2.43851805	−2.02534246
ϕ_{1_6}	0.776610613	−0.572394848	2.12561607
λ_1	0.0916645825	0.079089947	0.104239218
ω_1	0.0923735052	0.0873730332	0.0973739773

Data set: "ahh". $M=1, q_1=6$.
$\hat{X}(t) = 1.2019335\ \hat{X}(t-1) - 0.639286041\ \hat{X}(t-2) + 0.049728144\ \hat{X}(t-2) + e(t)$.
Run test: z for $\hat{X}(t) = -10.5448523$, z for $\hat{e}(t) = -0.152799755$.
Residual sum of squares: 0.460617959.

If for any $\delta > 0$ and for some $0 < K < \infty$,

$$\liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in S_{\delta,K}} \frac{1}{N}[Q_N(\boldsymbol{\theta}) - Q_N(\boldsymbol{\theta}^0)] > 0 \quad a.s. \tag{11}$$

then $\hat{\boldsymbol{\theta}}$ which minimizes (8) (*when $M=1$, $\boldsymbol{\Psi}=\boldsymbol{\theta}$*), *is a strongly consistent estimator of $\boldsymbol{\theta}^0$.*

Proof of Lemma 2. In this proof we denote $\hat{\boldsymbol{\theta}}$ by $\hat{\boldsymbol{\theta}}_N = (\hat{\rho}_{1N}, \ldots, \hat{\rho}_{qN}, \hat{\phi}_{1N}, \ldots, \hat{\phi}_{qN}, \hat{\lambda}_N, \hat{\omega}_N)$ just to emphasize that it depends on N. Suppose $\hat{\boldsymbol{\theta}}_N$ is not consistent, then we can have one of the following two cases.

Case 1: For all subsequences $\{N_k\}$ of $\{N\}$, at least one $|\hat{\rho}_{jN_k}|$ tends to ∞.

Case 2: There exists a $\delta > 0$, a $0 < K < \infty$ and a subsequence $\{N_k\}$ of $\{N\}$ such that $\hat{\boldsymbol{\theta}}_{N_k} \in S_{\delta,K}$ for all $k = 1, 2, \ldots$.

Now for both the cases, under the definition of $Q_N(\boldsymbol{\theta})$ (see (8)) and because of (11), there exits a K^0, such that for all $k > K^0$,

$$Q_{N_k}(\hat{\boldsymbol{\theta}}_{N_k}) - Q_{N_k}(\boldsymbol{\theta}^0) > 0 \quad \text{a.s.}$$

This contradicts the fact that $\hat{\boldsymbol{\theta}}_{N_k}$ minimizes $Q_{N_k}(\boldsymbol{\theta})$. □

Proof of Theorem 2.1. Let us write

$$S_{\delta,K} = P_1 \cup P_2 \cup \cdots \cup P_q \cup \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_q \cup \Lambda \cup \Omega,$$

Table 14
LSE and confidence intervals for "ahh" data with multiple frequency model

Parameters	LSE	Lower limit	Upper limit
a_1	0.15321	0.04255	0.26386
a_2	0.12292	0.01293	0.23291
a_3	0.19762	0.08844	0.30680
a_4	0.26646	0.15817	0.37476
a_5	1.05137	0.94365	1.15908
a_6	1.47706	1.36958	1.58455
ϕ_1	1.57744	0.13292	3.02196
ϕ_2	2.56365	0.77409	4.35321
ϕ_3	2.45601	1.35103	3.56099
ϕ_4	−1.93182	−2.74466	−1.11898
ϕ_5	−1.58160	−1.78651	−1.37670
ϕ_6	1.38753	1.24199	1.53307
β_1	0.09222	0.08486	0.09958
β_2	0.18906	0.17995	0.19818
β_3	0.27671	0.27108	0.28234
β_4	0.37503	0.37089	0.37917
β_5	0.46543	0.46439	0.46647
β_6	0.55728	0.55654	0.55802

$\hat{X}(t) = 1.01746476\ \hat{X}(t-1) - 0.636902988\ \hat{X}(t-2) + 0.176133722\ \hat{X}(t-2) + e(t)$.
Run test: z for $\hat{X}(t) = -7.69534779$, z for $\hat{e}(t) = -0.594967306$.
Residual sum of squares: 0.239770621.

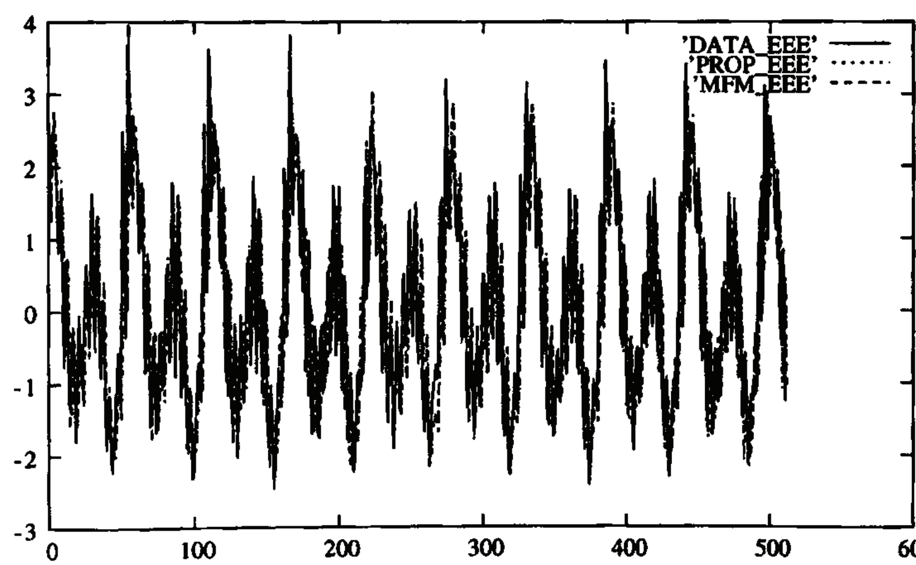

Fig. 11. The plot of the observed (DATA_EEE) and the fitted "eee" sound using LSEs of model (1) (PROP_EEE) and MFM (MFM_EEE).

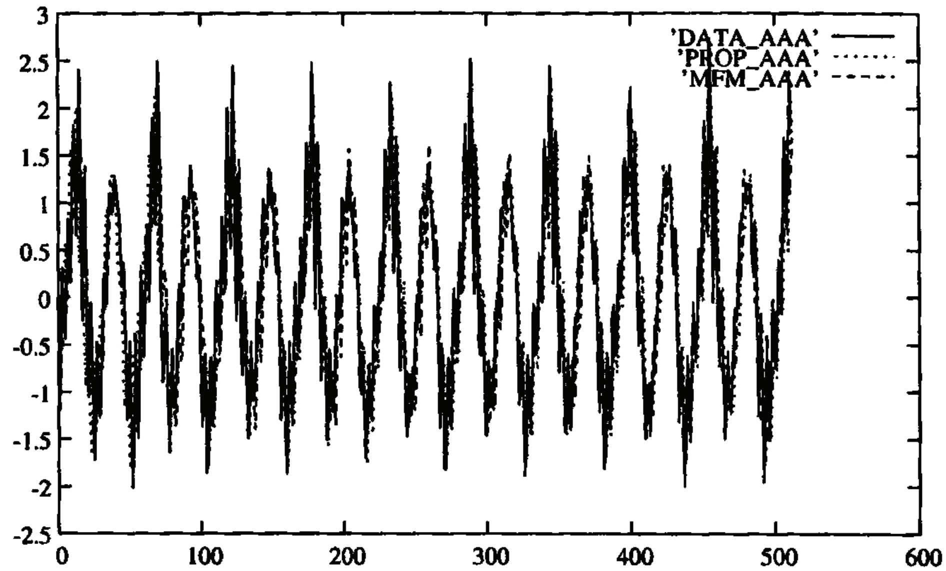

Fig. 12. The plot of the observed (DATA_AAA) and the fitted "aaa" sound using LSEs of model (1) (PROP_AAA) and MFM (MFM_AAA).

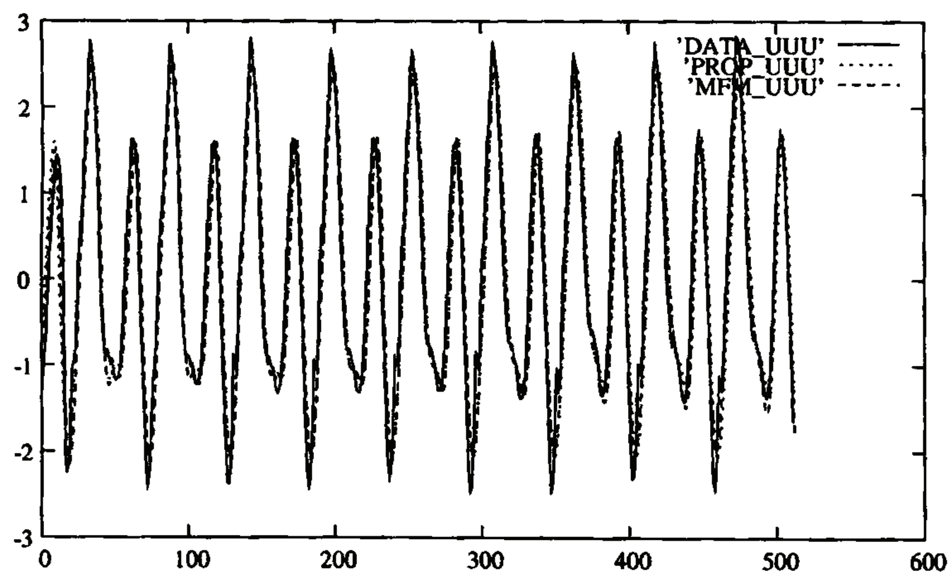

Fig. 13. The plot of the observed (DATA_UUU) and the fitted "uuu" sound using LSEs of model (1) (PROP_UUU) and MFM (MFM_UUU).

where for $j = 1, \ldots, q$,

$$P_j = \{\boldsymbol{\theta} : |\rho_j - \rho_j^0| > \delta, \rho_k \leqslant K \text{ for all } k = 1, \ldots, q\},$$
$$\Phi_j = \{\boldsymbol{\theta} : |\phi_j - \phi_j^0| > \delta, \rho_k \leqslant K \text{ for all } k = 1, \ldots, q\},$$
$$\Lambda = \{\boldsymbol{\theta} : |\lambda - \lambda^0| > \delta, \rho_k \leqslant K \text{ for all } k = 1, \ldots, q\},$$

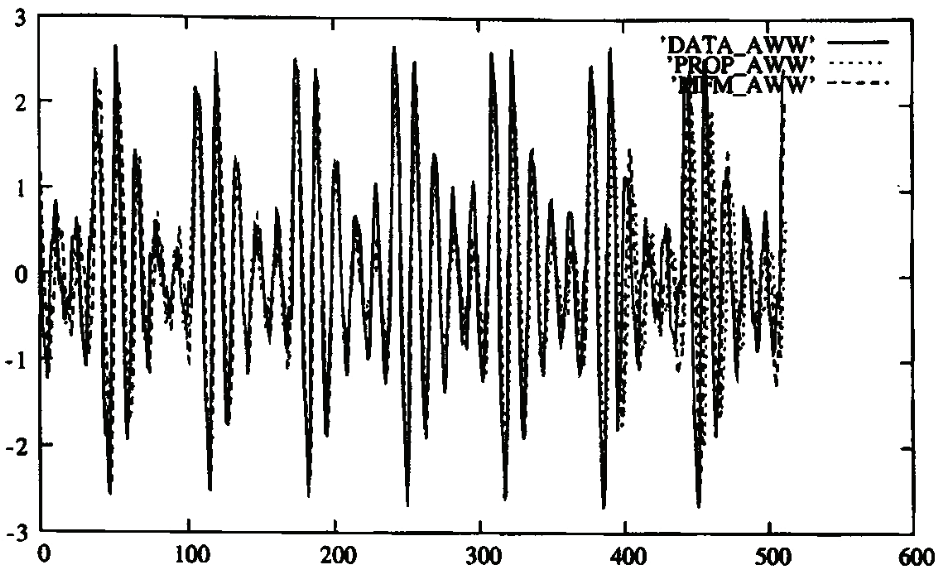

Fig. 14. The plot of the observed (DATA_AWW) and the fitted "aww" sound using LSEs of model (1) (PROP_AWW) and MFM (MFM_AWW).

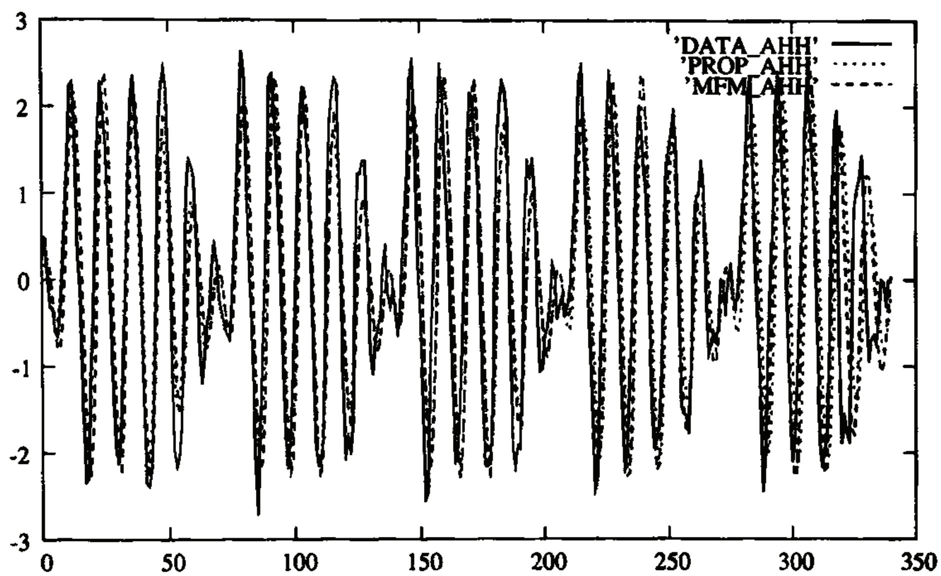

Fig. 15. The plot of the observed (DATA_AHH) and the fitted "ahh" sound using LSE of model (1) (PROP_AHH) and MFM (MFM_AHH).

and

$$\Omega = \{\boldsymbol{\theta} : |\omega - \omega^0| > \delta,\ \rho_k \leqslant K \text{ for all } k = 1, \ldots, q\}.$$

Now observe that

$$
\begin{aligned}
&\frac{1}{N}[Q_N(\boldsymbol{\theta}) - Q_N(\boldsymbol{\theta}^0)] \\
&= \frac{1}{N}\sum_{t=1}^{N}\left\{\sum_{j=1}^{q}\rho_j \cos\{(\lambda + (j-1)\omega)t - \phi_j\}\right. \\
&\quad \left. - \sum_{j=1}^{q}\rho_j^0 \cos\{(\lambda^0 + (j-1)\omega^0)t - \phi_j^0\}\right\}^2 \\
&\quad + \frac{2}{N}\sum_{t=1}^{N} X(t)\left\{\sum_{j=1}^{q}\rho_j \cos\{(\lambda + (j-1)\omega)t - \phi_j\}\right. \\
&\quad \left. - \sum_{j=1}^{q}\rho_j^0 \cos\{(\lambda^0 + (j-1)\omega^0)t - \phi_j^0\}\right\} \\
&= f_N(\boldsymbol{\theta}) + g_N(\boldsymbol{\theta}) \quad \text{(say)}.
\end{aligned}
$$

For any $\delta > 0$ and a fixed $0 < K < \infty$,

$$
\begin{aligned}
&\liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in P_i} f_N(\boldsymbol{\theta}) \\
&= \liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in P_i} \frac{1}{N}\sum_{t=1}^{N}\left\{\sum_{j=1}^{q}\rho_j^0 \cos\{(\lambda^0 + (j-1)\omega^0)t - \phi_j^0\}\right. \\
&\quad \left. - \sum_{j=1}^{q}\rho_j \cos\{(\lambda + (j-1)\omega)t - \phi_j\}\right\}^2 \\
&= \liminf_{N\to\infty} \inf_{|\rho_i - \rho_i^0| > \delta} \frac{1}{N}\sum_{t=1}^{N}[(\rho_i^0 - \rho_i)\cos\{(\lambda^0 + (i-1)\omega^0)t - \phi_i^0\}]^2 \\
&= \inf_{|\rho_i - \rho_i^0| > \delta} \frac{1}{2}(\rho_i - \rho_i^0)^2 > \frac{1}{2}\delta^2 > 0 \quad \text{a.s.}, \quad i = 1, \ldots, q.
\end{aligned}
$$

Similarly it can be proved that

$$
\liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in \Phi_i} f_N(\boldsymbol{\theta}) > 0 \quad \text{a.s.}, \quad i = 1, \ldots, q
$$

and

$$
\liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in \Lambda} f_N(\boldsymbol{\theta}) > 0 \quad \text{a.s.}, \quad \liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in \Omega} f_N(\boldsymbol{\theta}) > 0 \quad \text{a.s.}
$$

This proves that

$$
\liminf_{N\to\infty} \inf_{\boldsymbol{\theta}\in S_{\delta,K}} f_N(\boldsymbol{\theta}) > 0 \quad \text{a.s.} \tag{12}
$$

Using Lemma 1, it follows that

$$\lim_{N\to\infty} \sup_{\boldsymbol{\theta}\in S_{\delta,K}} g_N(\boldsymbol{\theta}) = 0 \quad \text{a.s.} \tag{13}$$

Now using (12) and (13) in Lemma 2, the theorem follows. □

Proof of Theorem 2.2. Let $Q'_N(\boldsymbol{\theta})$, the first derivative vector of $Q_N(\boldsymbol{\theta})$ be defined as follows:

$$Q'_N(\boldsymbol{\theta}) = \left(\frac{\partial Q_N(\boldsymbol{\theta})}{\partial \rho_1}, \ldots, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \rho_q}, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \phi_1}, \ldots, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \phi_q}, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \lambda}, \frac{\partial Q_N(\boldsymbol{\theta})}{\partial \omega}\right).$$

Similarly $Q''_N(\boldsymbol{\theta})$, the $(2q+2)\times(2q+2)$ matrix of second derivatives of $Q_N(\boldsymbol{\theta})$ is also defined. Expanding $Q'_N(\boldsymbol{\theta})$ at $\hat{\boldsymbol{\theta}}$ around $\boldsymbol{\theta}^0$ using Taylor Series expansion, we have

$$Q'_N(\hat{\boldsymbol{\theta}}) - Q'_N(\boldsymbol{\theta}^0) = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)^{\mathrm{T}} Q''_N(\bar{\boldsymbol{\theta}}), \tag{14}$$

where $\bar{\boldsymbol{\theta}} = \alpha\hat{\boldsymbol{\theta}} + (1-\alpha)\boldsymbol{\theta}^0$ for some $0<\alpha<1$. Consider a $(2q+2)\times(2q+2)$ diagonal matrix $\mathbf{D}_1$ (=$\mathbf{D}$ with $M=1$ and $q_1=q$) as follows:

$$\mathbf{D}_1 = \begin{pmatrix} N^{-1/2}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & N^{-1/2}\mathbf{I}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & N^{-3/2} & 0 \\ \mathbf{0} & \mathbf{0} & 0 & N^{-3/2} \end{pmatrix}.$$

Therefore, (14) can be written as

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)^{\mathrm{T}}\mathbf{D}_1^{-1} = -[Q'_N(\boldsymbol{\theta}^0)\mathbf{D}_1][\mathbf{D}_1 Q''_N(\bar{\boldsymbol{\theta}})\mathbf{D}_1]^{-1}. \tag{15}$$

As $\hat{\boldsymbol{\theta}}$ is a strongly consistent estimator of $\boldsymbol{\theta}^0$ and $\bar{\boldsymbol{\theta}}$ lies between $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}^0$, it can be shown that

$$\lim_{N\to\infty} [\mathbf{D}_1 Q''_N(\bar{\boldsymbol{\theta}})\mathbf{D}_1] = \lim_{N\to\infty} [\mathbf{D}_1 Q''_N(\boldsymbol{\theta}^0)\mathbf{D}_1] = \boldsymbol{\Sigma}, \tag{16}$$

where $\boldsymbol{\Sigma}$ is same as defined in the statement of Theorem 2.2. Now, using the Central Limit Theorem of a linear process (see Fuller, 1976, pp. 251–256) it can be proved that

$$Q'_N(\boldsymbol{\theta}^0)\mathbf{D}_1 \to \mathcal{N}_{(2q+2)}(\mathbf{0}, 2\sigma^2\mathbf{G}), \tag{17}$$

and $\mathbf{G}(=\mathbf{G}_1)$ is same as defined earlier. Now Theorem 2.2 follows immediately using (16) and (17) in (14). □

A.1. Outline of the proofs of the results when more than one fundamental frequency are present in the model ($M>1$)

The consistency of $\hat{\boldsymbol{\Psi}}$, the LSE of $\boldsymbol{\Psi}^0$ (when $M>1$ in model (1)) follows exactly the same way as the proof of Theorem 2.1, considering the entire set of parameters, i.e. considering $\hat{\boldsymbol{\Psi}}$, $\boldsymbol{\Psi}^0$ and $\boldsymbol{\Psi}$ instead of $\hat{\boldsymbol{\theta}}$, $\boldsymbol{\theta}^0$ and $\boldsymbol{\theta}$, respectively.

The asymptotic normality of the LSEs of the model (1) ($M > 1$) can be obtained along the same line as the proof of Theorem 2.2. Expanding $Q'_N(\hat{\mathbf{\Psi}})$ by Taylor series similarly as (14), an equivalent expression to (15)

$$(\hat{\mathbf{\Psi}} - \mathbf{\Psi}^0)^{\mathrm{T}}\mathbf{D}^{-1} = -[Q'_N(\mathbf{\Psi}^0)\mathbf{D}][\mathbf{D}Q''_N(\bar{\mathbf{\Psi}})\mathbf{D}]^{-1} \tag{18}$$

can be obtained for the general model having more than one fundamental frequency. The left-hand side of (18) is a $1 \times R$ ($R = 2\sum_{k=1}^{M} q_k + 2M$) random vector whereas the right-hand side is a product of $1 \times R$ ($Q'_N(\mathbf{\Psi}^0)\mathbf{D}$) random vector and a $R \times R$ ($[\mathbf{D}Q''_N(\hat{\mathbf{\Psi}})\mathbf{D}]^{-1}$) random matrix. Using similar techniques, the $R \times R$ matrix converges to a block diagonal matrix $\mathbf{\Sigma}$ of M blocks with kth block as $\mathbf{\Sigma}_k$ of order $2q_k + 2$. The $1 \times R$ random vector $Q'_N(\mathbf{\Psi}^0)\mathbf{D}$ converges to a R-variate normal distribution with mean vector zero and the dispersion matrix $2\sigma^2\mathbf{G}$, having a block-diagonal form with kth diagonal block as $2\sigma^2\mathbf{G}_k$. Therefore, asymptotic distribution of $\hat{\mathbf{\Psi}}$ is the same as given in Theorem 2.2. □

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