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Inequalities for permanents involving Perron complements

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Dedicated to Professor Peter Lancaster on his 75th birthday

Abstract

Let $A \in \mathbb{R}^{n,n}$ and let α and β be nonempty complementary subsets of $\{1, \ldots, n\}$ of increasing integers. For $\lambda > \rho(A[\beta])$, we define the *generalized Perron complement of* $A[\beta]$ *in* A *at* λ as the matrix $\mathscr{P}_\lambda(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](\lambda I - A[\beta])^{-1}A[\beta, \alpha]$. For the classes of the nonnegative matrices and of the positive semidefinite matrices, we study the relationship between the permanents of the whole matrices and the permanents of their Perron complement. Our conditions, which hold in many cases of interest, are such that the value of the permanent increases as we pass from the whole matrix to its generalized Perron complement.

For nonnegative and irreducible matrices, we also study the relationship between the maximum circuit geometric mean of the entire matrix and the maximum circuit geometric mean of its Perron complements.

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1. Introduction

Let $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ and recall that the *permanent* of A is the quantity given by

$$\mathrm{per}(A) = \sum_{\sigma \in \mathscr{S}_n} \prod_{i=1}^{n} a_{i,\sigma(i)}. \tag{1.1}$$

Permanents of matrices arise in many contexts, but particularly in combinatorial applications, see Minc [12] and Brualdi and Ryser [3, Chapter 7]. Among the classes of matrices to which permanents have been applied are the nonnegative matrices and doubly stochastic matrices. An $n \times n$ nonnegative matrix $A = (a_{i,j})$ is *stochastic* if

$$\sum_{j=1}^{n} a_{i,j} = 1 \quad \forall i = 1, \ldots, n. \tag{1.2}$$

It is *doubly stochastic* if

$$\sum_{i=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j} = 1 \quad \forall i, j = 1, \ldots, n. \tag{1.3}$$

It is well known that in the beginning of the 1980s, two Russian scientists, Egoricev [4] and Falikman [5], independently, settled the *van der Waerden conjecture* showing that

$$\min_{A \in \Omega_n} \mathrm{per}(A) = \frac{n!}{n^n} = \mathrm{per}(J_n), \tag{1.4}$$

where Ω_n is the class of all doubly stochastic matrices of order n and J_n is the $n \times n$ matrix whose entries are all equal to $1/n$. Moreover, they showed that J_n is the unique matrix in Ω_n on which the minimum is attained.

Let $A \in \mathbb{R}^{n,n}$ be the space of all real $n \times n$ matrices and let γ and δ be *nonempty ordered subsets* of $\langle n \rangle := \{1, \ldots, n\}$, *both of strictly increasing integers*. By $A[\gamma, \delta]$ we shall denote the submatrix of A whose rows and columns are determined by γ and δ, respectively. Also, $A(\gamma, \delta)$ will denote the submatrix of A obtained by deleting rows in γ and columns in δ, respectively. Matrices $A[\gamma, \delta)$ and $A(\gamma, \delta]$ are defined similarly. In the special case when $\gamma = \delta$, we shall use $A[\gamma]$ and $A(\gamma)$ to denote $A[\gamma, \gamma]$ and $A(\gamma, \gamma)$, respectively.

In connection with a divide and conquer algorithm for computing the stationary distribution vector for a Markov chain, Meyer [10,11] introduced, for an $n \times n$ nonnegative and irreducible matrix A, the notion of the Perron complement. Again, if $\beta \subset \langle n \rangle$, then the *Perron complement of* $A[\beta]$ in A is given by

$$\mathscr{P}(A/A[\beta]) = A(\beta) + A(\beta, \beta][\rho(A)I - A[\beta]]^{-1} A[\beta, \beta), \tag{1.5}$$

where $\rho(\cdot)$ denotes the *spectral radius* of a matrix. Recall that as A is irreducible, $\rho(A) > \rho(A[\beta])$, so that the expression on the right hand side of (1.5) is well defined. Meyer has derived several interesting and useful properties of $\mathscr{P}(A/A[\beta])$. The first is that $\rho(\mathscr{P}(A/A[\beta])) = \rho(A)$. The second is that if A is stochastic, then so is $\mathscr{P}(A/A[\beta])$. In the latter case, Meyer has shown how, if $\beta_1, \ldots, \beta_s$ are disjoint subsets whose union is $\langle n \rangle$, the stationary distribution vector for the (entire) Markov process can be aggregated from the stationary distribution vectors of its Perron complements $\mathscr{P}(A/A[\beta_1]), \ldots, \mathscr{P}(A/A[\beta_s])$.

Actually in this paper we shall work with a generalized form of the Perron complement: Let $A \in \mathbb{R}^{n,n}$, let $\beta \subset \langle n \rangle$, and let $\lambda \in \mathbb{R}$ be such that $\lambda I - A[\beta]$ is invertible. Then the *generalized Perron complement of* $A[\beta]$ *in* A *at* λ is given by the matrix

$$\mathscr{P}_\lambda(A/A[\beta]) = A(\beta) + A(\beta, \beta][\lambda I - A[\beta]]^{-1}A[\beta, \beta). \tag{1.6}$$

We mention that generalized Perron complements were already used in [13], [6, Theorem 2.4], and in Lu [9]. It is immediate that if $A \in \mathbb{R}^{n,n}$ is a nonnegative matrix or a positive semidefinite matrix, then, in particular, the generalized Perron complements in A exist for all $\beta \subset \langle n \rangle$ and for all $\lambda > \rho(A)$.

In this paper we shall derive several inequalities on the permanents of the generalized Perron complements of irreducible nonnegative matrices and of positive semidefinite matrices. For example in Theorems 2.4 and 4.4, of Sections 2 and 4, respectively, we shall show that if $A \in \mathbb{R}^{n,n}$ is any one of the two types of matrices just mentioned and $\beta \subset \langle n \rangle$, then

$$\operatorname{per}(\mathscr{P}_\lambda(A/A[\beta]))\det(\lambda I - A[\beta]) \geqslant \operatorname{per}(A), \quad \forall \lambda \geqslant 2\rho(A).$$

In Lemma 2.1 we shall show that if $A \in \mathbb{R}^{n,n}$ is a nonnegative matrix and $\beta \in \langle n \rangle$, then

$$\operatorname{per}(\mathscr{P}_\lambda(A/A[\beta])) \geqslant \operatorname{per}(A), \quad \forall \lambda > \rho(A[\beta]).$$

In Section 3 we shall turn our attention from permanents to maximal circuit geometric means in irreducible stochastic nonnegative matrices. We shall show, for example, in Lemma 3.1, that if $\beta \subset \langle n \rangle$, with $|\beta| = 1$, then $\mu(\mathscr{P}_\lambda(A/A[\beta])) \geqslant (\mu(A))^2$, where, for an $n \times n$ nonnegative matrix, $\mu(\cdot)$ denotes the maximum circuit geometric mean.

For background material on nonnegative matrices, M-matrices, directed graphs, permanents, etc., we refer the reader to the books by Bapat and Raghavan [1] and Berman and Plemmons [2]. For background material on matrix theory, linear algebra, and matrix computations see the books by Horn and Johnson [7] and Golub and van Loan [8].

2. Permanent of Perron complement

In this section we develop inequalities between the permanent of a nonnegative matrix and its generalized Perron complements.

Let A be an $n \times n$ matrix and let $\beta \subset \langle n \rangle$. Recall that if $\lambda I - A[\beta]$ is nonsingular, then the Perron complement of $A[\beta]$ in A at λ is given by:

$$\mathscr{P}_\lambda(A/A[\beta]) = A(\beta) + A(\beta, \beta](\lambda I - A[\beta])^{-1}A[\beta, \beta).$$

Lemma 2.1. *Let A be an $n \times n$ nonnegative matrix and let $\lambda > a_{n,n}$. Then*

$$\operatorname{per}(\mathscr{P}_\lambda(A/a_{n,n})) \geqslant \frac{1}{\lambda - a_{n,n}}\operatorname{per}(A) + \frac{\lambda - 2a_{n,n}}{\lambda - a_{n,n}}\operatorname{per}(A[\langle n-1 \rangle]). \tag{2.1}$$

Furthermore, if $\lambda \geqslant 2a_{n,n}$*, then*

$$\operatorname{per}(\mathscr{P}_\lambda(A/a_{n,n}))(\lambda - a_{n,n}) \geqslant \operatorname{per}(A). \tag{2.2}$$

Proof. Let $B = A[\langle n-1\rangle]$, $x = A[\langle n-1\rangle, n]$, and $y = A[n, \langle n-1\rangle]$. Then

$$\mathscr{P}_\lambda(A/a_{n,n}) = B + \frac{1}{\lambda - a_{n,n}} xy. \tag{2.3}$$

Denote by $B(i, j)$ the $(n-2) \times (n-2)$ submatrix of B obtained by deleting row i and column j. Since A is nonnegative, $\lambda > a_{nn}$ and the permanent is a multilinear function of the columns, it follows from (2.3) that

$$\operatorname{per}(\mathscr{P}_\lambda(A/a_{n,n})) \geqslant \operatorname{per}(B) + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \frac{x_i y_j}{\lambda - a_{n,n}} \operatorname{per}(B(i,j)). \tag{2.4}$$

Clearly,

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} x_i y_j (\operatorname{per} B(i,j)) = \operatorname{per}\left(\begin{bmatrix} B & x \\ y & 0 \end{bmatrix}\right) = \operatorname{per}(A) - a_{n,n}\operatorname{per}(B). \tag{2.5}$$

Now (2.1) follows easily after substituting (2.5) in (2.4). If $\lambda \geqslant 2a_{nn}$, then (2.2) is a simple consequence of (2.1). □

As an example of this lemma consider the case when $A \in \Omega_n$. First, it is an immediate outcome of Meyer's results on the Perron complement mentioned in the introduction, that all the Perron complements of A are now doubly stochastic matrices of a smaller size. Thus for any subsets $\beta \subseteq \gamma \subset \langle n\rangle$, with γ of cardinality $|\gamma| = n - 1$, we have that

$$\operatorname{per}(A) \leqslant \operatorname{per}(\mathscr{P}_1(A/A[\beta])) \leqslant \operatorname{per}(\mathscr{P}_1(A/A[\gamma])) = 1.$$

In this connection we also mention that when $A = J_n$, then for any $\beta \subset \langle n\rangle$ with $|\beta| = k$, $\mathscr{P}_1(A/A[\beta]) = J_{n-k}$. It should be noted though that even when $A \neq J_n$ with $A \in \Omega_n$, it can be that for some $\beta \subset \langle n\rangle$, with $|\beta| = k$, $\mathscr{P}_1(A/A[\beta]) = J_{n-k}$ as the following example shows: Let

$$A = \left[\begin{array}{cc|c} \frac{169}{440} & \frac{188}{495} & \frac{17}{72} \\ \frac{1}{8} & \frac{1}{9} & \frac{55}{72} \\ \hline \frac{27}{55} & \frac{28}{55} & 0 \end{array}\right] \in \Omega_3.$$

Then for $\beta = \{3\}$, we find that:

$$\mathscr{P}_1(A/a_{3,3})) = \begin{bmatrix} \frac{169}{440} & \frac{188}{495} \\ \frac{1}{8} & \frac{1}{9} \end{bmatrix} + \frac{1}{1-0}\begin{bmatrix} \frac{17}{72} \\ \frac{55}{72} \end{bmatrix}\begin{bmatrix} \frac{27}{55} & \frac{28}{55} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = J_2.$$

In order to prove the main result of this section (Theorem 2.4) we require the following two lemmas. The first of these lemmas was observed implicitly in [13] and explicitly in Lu [9, Lemma 3].

Lemma 2.2. *Let A be an $n \times n$ nonnegative matrix and let $\lambda_1 \geqslant \lambda_2 > \rho(A)$. Then for any $\beta \subset \langle n \rangle$,*

$$\rho(\mathscr{P}_{\lambda_1}(A/A[\beta])) \leqslant \rho(\mathscr{P}_{\lambda_2}(A/A[\beta])).$$

The second lemma is as follows.

Lemma 2.3 [10]. *Let A be an $n \times n$ nonnegative, irreducible matrix with $\rho = \rho(A)$ and let $\beta \subset \langle n \rangle$, $\beta \neq \langle n \rangle$. Then $\rho(\mathscr{P}_\rho(A/[\beta])) = \rho$.*

The main result of this section can be stated as follows:

Theorem 2.4. *Let A be an $n \times n$ nonnegative, irreducible matrix with $\rho = \rho(A)$ and let $\lambda \geqslant 2\rho$. Then for any $\beta \subset \langle n \rangle$,*

$$\operatorname{per}(\mathscr{P}_\lambda(A/A[\beta])) \det(\lambda I - A[\beta]) \geqslant \operatorname{per}(A).$$

Proof. We use induction on the cardinality of β, namely on $|\beta|$. Without loss of generality, let $\beta = \{k, \ldots, n\}$. If $\beta = \{n\}$, then the result follows from (2.2) of Lemma 2.1. So let $|\beta| > 1$, assume the result to be true for $\gamma = \{k+1, \ldots, n\}$ and proceed by induction. Then

$$\operatorname{per}(\mathscr{P}_\lambda(A/A[\gamma])) \det(\lambda I - A[\gamma]) \geqslant \operatorname{per}(A). \tag{2.6}$$

It follows from Lemmas 2.2 and 2.3 that the spectral radius of $\mathscr{P}_\lambda(A/A[\gamma])$ is less than ρ. Thus any diagonal entry of $\mathscr{P}_\lambda(A/A[\gamma])$ is less than $\lambda/2$ and it follows from Lemma 2.1 that

$$\operatorname{per}(\mathscr{P}_\lambda(\mathscr{P}_\lambda(A/A[\gamma])/\tilde{a}_{k,k}))(\lambda - \tilde{a}_{k,k}) \geqslant \operatorname{per}(\mathscr{P}_\lambda(A/A[\gamma])), \tag{2.7}$$

where $\tilde{a}_{k,k}$ is the (k,k)-element of $\mathscr{P}_\lambda(A/A[\gamma])$. By the quotient formula for the Perron complement, see [6], we have

$$\mathscr{P}_\lambda((\mathscr{P}_\lambda A/A[\gamma])/\tilde{a}_{k,k}) = \mathscr{P}_\lambda(A/A[\beta])$$

and hence (2.7) implies that

$$\operatorname{per}(\mathscr{P}_\lambda(A/A[\beta]))(\lambda - \tilde{a}_{k,k}) \geqslant \operatorname{per}(\mathscr{P}_\lambda(A/A[\gamma])). \tag{2.8}$$

The result follows from (2.6) and (2.8) in view of the identity (which is the familiar Schur-complement formula for the determinant) that

$$\det(\lambda I - A[\beta]) = (\lambda - \tilde{a}_{k,k}) \det(\lambda I - A[\gamma]). \qquad \square$$

3. Circuit geometric means

If A is an $n \times n$ matrix and if $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$, then the entries of A: $a_{i_1,i_2}, a_{i_2,i_3}, \ldots, a_{i_k,i_1}$ are said to constitute a *circuit* in A and $(a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_k i_1})^{1/k}$ is the corresponding *circuit geometric mean*. The *maximum circuit geometric mean* of A is then the maximum geometric mean over all circuits in A and we shall denote it by $\mu(A)$. In this section we shall obtain certain inequalities between the maximum circuit geometric means of an $n \times n$ nonnegative and irreducible matrix A and its Perron complements.

We begin with the following lemma.

Lemma 3.1. *Let A be an $n \times n$ irreducible, stochastic matrix. Then*

$$\mu(\mathscr{P}(A/a_{n,n})) \geqslant \mu(A)^2.$$

Proof. Let $P = \mathscr{P}(A/a_{n,n})$. First suppose that $\mu(A)$ is the circuit geometric mean of a circuit which does not pass through n. Without loss of generality, let $\mu(A) = (a_{1,2}a_{2,3}\cdots a_{k-1,k}a_{k,1})^{1/k}$, where $k < n$. Since $p_{i,j} \geqslant a_{i,j}$, for $1 \leqslant i, j \leqslant n-1$, we have that

$$p_{1,2}p_{2,3}\cdots p_{k-1,k}p_{k,1} \geqslant a_{1,2}a_{2,3}\cdots a_{k-1,k}a_{k,1},$$

and it follows that $\mu(P) \geqslant \mu(A)$.

Now suppose that $\mu(A)$ is the circuit geometric mean of a circuit which passes through n. Without loss of generality, let $\mu(A) = (a_{1,2}a_{2,3}\cdots a_{k-1,k}a_{k,n}a_{n,1})^{1/(k+1)}$, where $k < n$. Then

$$\begin{aligned} p_{1,2}p_{2,3}\cdots p_{k-1,k}p_{k,1} &\geqslant a_{1,2}a_{2,3}\cdots a_{k-1,k}\left(a_{k,1} + \frac{a_{k,n}a_{n,1}}{1-a_{n,n}}\right) \\ &\geqslant a_{1,2}a_{2,3}\cdots a_{k-1,k}a_{k,n}a_{n,1}, \end{aligned}$$

since $\frac{1}{1-a_{n,n}} > 1$. Thus

$$\begin{aligned} (p_{1,2}p_{2,3}\cdots p_{k-1,k}p_{k,1})^{1/k} &\geqslant (a_{1,2}a_{2,3}\cdots a_{k-1,k}a_{k,n}a_{n,1})^{1/k} \\ &= \mu(A)^{(k+1)/k} \\ &\geqslant \mu(A)^2. \end{aligned}$$

Hence $\mu(P) \geqslant \mu(A)^2$ and the proof is complete. □

The following example shows that the inequality in Lemma 3.1 cannot be improved. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \epsilon & 0 & 1-\epsilon \\ 0 & 1 & 0 \end{bmatrix},$$

where $0 < \epsilon \leqslant 1/2$. Then

$$P = \mathscr{P}(A/a_{3,3}) = \begin{bmatrix} 0 & 1 \\ \epsilon & 1-\epsilon \end{bmatrix}$$

and we see that $\mu(A) = \sqrt{1-\epsilon} = \sqrt{\mu(P)}$ and so equality holds in the inequality of Lemma 3.1.

A repeated application of Lemma 3.1 gives the following result.

Theorem 3.2. *Let A be an irreducible, stochastic matrix partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is $k \times k$. Then

$$\mu(\mathscr{P}(A/A_{22})) \geqslant \mu(A)^{2(n-k)}.$$

4. Permanent inequalities for positive semidefinite matrices

In this final section of the paper we develop inequalities between the permanent of a positive semidefinite matrix and its generalized Perron complements. We shall employ the following here: if A and B are $n \times n$ positive semidefinite matrices, then $A \succ B$ will mean that $A - B$ is positive semidefinite.

We begin with a preliminary result.

Lemma 4.1. *Let B and C be $n \times n$ positive semidefinite matrices. Then*

$$\operatorname{per}(B + C) \geqslant \operatorname{per}(B) + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \operatorname{per} B(i, j). \tag{4.1}$$

Proof. The proof involves familiar ideas from multilinear algebra. We include a proof since it is not readily available in the literature.

If A is a square matrix, then $\otimes^n(A)$ will denote the Kronecker product of A with itself, taken n times. Let z be the column vector of order n^n, with its coefficients indexed by all sequences $i_1, i_2, \ldots, i_n$ of integers from $1, 2, \ldots, n$, and with its entries defined as follows. The entry of z indexed by $i_1, i_2, \ldots, i_n$ is 1 if and only if $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$, and is zero otherwise. We have the following basic identity: If A is an $n \times n$ matrix then

$$\operatorname{per}(A) = \frac{1}{n!}\langle \otimes^n(A)z, z\rangle. \tag{4.2}$$

Now if B and C are $n \times n$ positive semidefinite matrices, then

$$\otimes^n(B + C) \succ \otimes^n(B) + \sum_{i=1}^{n} B \otimes \cdots \otimes B \otimes C \otimes B \cdots \otimes B, \tag{4.3}$$

where C appears at the ith position in the summation. It follows from (4.3) that

$$\frac{1}{n!}\langle \otimes^n(B + C)z, z\rangle \geqslant \frac{1}{n!}\langle \otimes^n(B)z, z\rangle + \frac{1}{n!}\sum_{i=1}^{n}\langle B \otimes \cdots \otimes B \otimes C \otimes B \cdots \otimes Bz, z\rangle. \tag{4.4}$$

Observe now that

$$\langle B \otimes \cdots \otimes B \otimes C \otimes B \cdots \otimes Bz, z\rangle = (n-1)! \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \mathrm{per}(B(i,j)). \tag{4.5}$$

The result follows from (4.3)–(4.5). □

Lemma 4.2. *Let A be an $n \times n$ positive semidefinite matrix and let $\lambda > a_{n,n}$. Then*

$$\mathrm{per}(\mathscr{P}_\lambda((A/a_{n,n})) \geqslant \frac{1}{\lambda - a_{nn}} \mathrm{per}(A) + \frac{\lambda - 2a_{n,n}}{\lambda - a_{n,n}} \mathrm{per}(A[\langle n\rangle]). \tag{4.6}$$

Furthermore, if $\lambda \geqslant 2a_{n,n}$, then

$$\mathrm{per}(\mathscr{P}_\lambda(A/a_{n,n}))(\lambda - a_{n,n}) \geqslant \mathrm{per}(A). \tag{4.7}$$

Proof. As in the proof of Lemma 2.1, let $B = A[\langle n-1\rangle]$, $x = A[\langle n-1\rangle, n]$, and $y = A[n, \langle n-1\rangle]$. Then

$$\mathscr{P}_\lambda(A/a_{n,n}) = B + \frac{1}{\lambda - a_{n,n}} xy. \tag{4.8}$$

Denote by $B(i,j)$ the $(n-2) \times (n-2)$ submatrix of B obtained by deleting row i and column j. Since $\lambda > a_{nn}$, by Lemma 4.1 and (4.8) we have

$$\mathrm{per}(\mathscr{P}_\lambda(A/a_{n,n})) \geqslant \mathrm{per}(B) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \frac{x_i y_j}{\lambda - a_{n,n}} \mathrm{per}(B(i,j)). \tag{4.9}$$

The rest of the proof is similar to that of Lemma 2.1. □

If A is a positive semidefinite matrix, we continue to denote its spectral radius by $\rho(A)$. Observe that then $\rho(A)$ is just the largest eigenvalue of A. The next result is analogous to Lemma 2.2.

Lemma 4.3. *Let A be an $n \times n$ positive semidefinite matrix and let $\lambda_1 \geqslant \lambda_2 > \rho(A)$. Then for any $\beta \subset \langle n\rangle$,*

$$\rho(\mathscr{P}_{\lambda_1}(A/A[\beta])) \leqslant \rho(\mathscr{P}_{\lambda_2}(A/A[\beta])) \leqslant \lambda_2. \tag{4.10}$$

Proof. First observe that since $\lambda_1 \geqslant \lambda_2 > \rho(A) \geqslant \rho(A[\beta])$, both the Perron complements in the result are well defined. Note that $\lambda_1 I - A[\beta]$ and $\lambda_2 I - A[\beta]$ are positive semidefinite and that

$$\lambda_1 I - A[\beta] \succ \lambda_2 I - A[\beta].$$

It follows that

$$(\lambda_2 I - A[\beta])^{-1} \succ (\lambda_1 I - A[\beta])^{-1}.$$

Thus

$$A_{12}(\lambda_2 I - A[\beta])^{-1}A_{21} \succ A_{12}(\lambda_2 I - A[\beta])^{-1}A_{21}$$

and hence

$$\mathscr{P}_{\lambda_2}(A/A[\beta]) \succ \mathscr{P}_{\lambda_1}(A/A[\beta]).$$

The first inequality (4.10) follows in view of the well-known monotonicity property of the largest eigenvalue. Since $\lambda_2 I - A$ is positive semidefinite, any Schur complement in the matrix is positive semidefinite as well. Thus

$$\lambda_2 I - A(\beta) - A(\beta,\beta][\lambda_2 I - A[\beta]]^{-1}A[\beta,\beta)$$

is positive semidefinite and therefore $\lambda_2 I - \mathscr{P}_\lambda(A/A[\beta])$ is positive semidefinite. The second inequality in (4.10) now follows. □

We now state the main result of this section.

Theorem 4.4. *Let A be an $n \times n$ positive semidefinite matrix with $\rho = \rho(A)$ and let $\lambda \geqslant 2\rho$. Then for any $\beta \subset \langle n \rangle$,*

$$\operatorname{per}(\mathscr{P}_\lambda(A/A[\beta]))\det(\lambda I - A[\beta]) \geqslant \operatorname{per}(A).$$

Proof. We use induction on $|\beta|$. Without loss of generality, let $\beta = \{k,\dots,n\}$. If $\beta = \{n\}$, then the result follows from (4.7) of Lemma 4.1. So let $|\beta| > 1$, assume the result to be true for $\gamma = \{k+1,\dots,n\}$, and proceed by induction. Then

$$\operatorname{per}(\mathscr{P}_\lambda(A/A[\gamma]))\det(\lambda I - A[\gamma]) \geqslant \operatorname{per}(A). \tag{4.11}$$

Setting $\lambda_1 = \lambda$ and taking the limit as λ_2 approaches $\rho(A)$ in (4.10) it follows that the largest eigenvalue of $\mathscr{P}_\lambda(A/A[\gamma])$ is less than ρ. Then, since $\mathscr{P}_\lambda(A/A[\gamma])$ is positive semidefinite for any $\lambda \geqslant 2\rho$, any diagonal entry of $\mathscr{P}_\lambda(A/A[\gamma])$ is less than ρ and hence is less than $\lambda/2$. Now using Lemma 4.1 we have that

$$\operatorname{per}(\mathscr{P}_\lambda(\mathscr{P}_\lambda(A/A[\gamma])/\tilde{a}_{k,k}))(\lambda - \tilde{a}_{k,k}) \geqslant \operatorname{per}(\mathscr{P}_\lambda(A/A[\gamma])), \tag{4.12}$$

where, as before, $\tilde{a}_{k,k}$ is the (k,k)-element of $\mathscr{P}_\lambda(A/A[\gamma])$. Now, by the quotient formula for the Perron complement, see [6], we have that:

$$\mathscr{P}_\lambda(\mathscr{P}_\lambda(A/A[\gamma])/\tilde{a}_{k,k}) = \mathscr{P}_\lambda(A/A[\beta]),$$

and hence (4.12) implies that

$$\operatorname{per}(\mathscr{P}_\lambda(A/A[\beta])(\lambda - \tilde{a}_{kk})) \geqslant \operatorname{per}(\mathscr{P}_\lambda(A/A[\gamma])). \tag{4.13}$$

The result follows from (4.12) and (4.13) in view of the identity

$$\det(\lambda I - A[\beta]) = (\lambda - \tilde{a}_{kk})\det(\lambda I - A[\gamma]). \quad \square$$

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