The Euler class group of a polynomial algebra

Mrinal Kanti Das
Harish-Chandra Research Institute, Allahabad.
Chhatnag Road, Jhusi, Allahabad - 211 019, India.
e-mail : mrinal@mri.ernet.in

1 Introduction

Let A be a smooth affine domain of dimension n over a field k and I be a prime ideal of $A[T]$ of height r such that $A[T]/I$ is smooth and $2r \geq n+3$. Let $f_1(T), \cdots, f_r(T) \in I$ such that $I = (f_1(T), \cdots, f_r(T)) + (I^2T)$. Furthermore, assume that $A/(f_1(0), \cdots, f_r(0))$ is also smooth. In this set up Nori asked the following question (for motivation, see ([M 2], Introduction):

Question: Do there exist $g_1, \cdots, g_r$ such that $I = (g_1, \cdots, g_r)$ with $g_i - f_i \in (I^2T)$?

This question has been answered affirmatively by Mandal ([M 2]) when I contains a monic polynomial, even without any smoothness assumptions.

When I does not contain a monic polynomial, Nori's question has been answered in the affirmative in the following cases:

1) A is a local ring of a smooth affine algebra over an infinite field ([M-V], Theorem 4).

2) A is a smooth affine algebra over an infinite field and $r = n$ (i.e dim $A[T]/I = 1$) ([B-RS 1], Theorem 3.8).

Moreover, an example is given in ([B-RS 1], Example 6.4) in the case when dim $(A[T]/I) = 1$, which shows that the question of Nori does not have an affirmative answer in general without the smoothness assumption.

So, in view of this example of ([B-RS 1]) one wonders where the obstruction for I to have a set of generators satisfying the required properties lies. In this paper we investigate this question when dim $A[T]/I = 1$. Taking a cue from the above result of Mandal, we prove

Theorem. *Let A be a Noetherian ring of dimension $n \geq 3$, containing the field of rationals. Let $I \subset A[T]$ be an ideal of height n. Suppose that $I = (f_1, \cdots, f_n) + (I^2T)$ and there exist $F_1, \cdots, F_n \in IA(T)$ such that $IA(T) = (F_1, \cdots, F_n)$ and $F_i = f_i$ mod $I^2A(T)$. Then, there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ and $g_i = f_i$ mod (I^2T).*

Let A be a Noetherian ring of dimension $n \geq 2$ and let $J \subset A$ be an ideal of height r such that J/J^2 is generated by r elements. It is of interest to know when a set of r generators of J/J^2 can be lifted to a set of r generators of J. This question was investigated in ([B-RS 3]) for ideals of height n, where, an abelian group $E^n(A)$, called the Euler class group of A is defined and corresponding to a set of generators of J/J^2 an element of this group is attached and it is shown that this set of generators of J/J^2 can be lifted to a set of generators of J if the corresponding element of $E^n(A)$ is zero.

Now let $R = A[T]$ where A is a Noetherian ring of dimension n. Since every ideal $I \subset A[T]$ of height $n+1$ contains a monic polynomial, it can be shown, using a result of Mandal ([M 1]), that $E^{n+1}(R) = 0$.

As one of the interesting consequences of our theorem, we can define a notion of the n^{th} Euler class group of $A[T]$ (denoted by $E^n(A[T])$), where A is a Noetherian ring of dimension n. Further, to any set of n generators of I/I^2 where $I \subset A[T]$ is an ideal of height n we attach an element of this group and show that if this element is zero, then the set of generators of I/I^2 can be lifted to a set of generators of I (Theorem 4.7). Moreover, there is a canonical injective homomorphism from $E^n(A)$ to $E^n(A[T])$ which is an isomorphism when A is a smooth affine domain over an infinite field. (This is an algebraic analogue of a well known result in algebraic topology as if A is a smooth affine domain over reals, then the set X of real points of Spec A is a manifold of dimension n and the groups $E^n(A)$ and $E^n(A[T])$ are algebraic analogues of the n-th cohomology groups $H^n(X)$ and $H^n(X \times I)$.)

The layout of this paper is as follows: In Section 3, we prove our main theorem (3.10). In Section 4, as an application of our main theorem, we define the notion of the n^{th} *Euler class group* of $A[T]$, as mentioned above. We also define the n^{th} *Euler class* of a pair (P, χ), where, P is a projective $A[T]$-

module of rank n (with trivial determinant) and χ is a generator of $\wedge^n(P)$. In this section we also prove our main theorem in a more general form (4.8) and derive several analogoues of results of ([B-RS 3], Section 4) as consequences (see, for example, 4.10, 4.11, 4.12). In Section 5, a *"Quillen-Suslin theory"* for Euler class groups is developed. We prove a *local-global principle* for Euler class groups (5.4), which is an analogue of the Quillen localization theorem. An analogue of local Horrocks theorem is also proved. In Section 6, we define the notion of the n^{th} *weak Euler class group* $E_0^n(A[T])$ which is a certain quotient of $E^n(A[T])$. Section 7 deals with the case when dimension of the base ring is two. In this section we prove a weaker version of the main theorem and apply it to obtain results similar to those in Sections 4, 5 and 6. In Section 2, we define some of the terms used in the paper and quote some results which are used in later sections.

2 Some preliminaries

In this section we define some of the terms used in the paper and state some results for later use.

All rings considered in this paper are commutative and Noetherian and all modules considered are assumed to be finitely generated. For a module M over a ring, $\mu(M)$ will denote the minimal number of generators of M.

Definition 2.1 *Let A be a ring. A row $(a_1, a_2, \cdots, a_n) \in A^n$ is said to be unimodular if there exist $b_1, b_2, \cdots, b_n$ in A such that $a_1b_1 + \cdots + a_nb_n = 1$.*

Definition 2.2 *Let A be a Noetherian ring. Let P be a projective A-module. An element $p \in P$ is said to be unimodular if there exists a linear map $\phi : P \to A$ such that $\phi(p) = 1$.*

We now state a theorem of Serre ([S]).

Theorem 2.3 *Let A be a Noetherian ring with* $\dim A = d$. *Then any projective A-module having* rank $> d$ *has a unimodular element.*

As an immediate consequence we have

Corollary 2.4 *Let A be a Noetherian ring with $\dim A = 1$. Then any projective A-module having trivial determinant is free.*

The following lemma has been proved in ([B]).

Lemma 2.5 *Let A be a ring and $J \subset A$ be an ideal of height r. Let $\overline{P}$, $\overline{Q}$ be projective A/J-modules of rank r and let $\overline{\alpha} : \overline{P} \twoheadrightarrow J/J^2$ and $\overline{\beta} : \overline{Q} \twoheadrightarrow J/J^2$ be surjections. Let $\overline{\psi} : \overline{P} \to \overline{Q}$ be a homomorphism such that $\overline{\beta}\overline{\psi} = \overline{\alpha}$. Then $\overline{\psi}$ is an isomorphism.*

The following lemma is easy to prove and hence we omit the proof.

Lemma 2.6 *Let A be a Noetherian ring and P a finitely generated projective A-module. Let $P[T]$ denote projective $A[T]$-module $P \otimes A[T]$. Let $\alpha(T) : P[T] \twoheadrightarrow A[T]$ and $\beta(T) : P[T] \twoheadrightarrow A[T]$ be two surjections such that $\alpha(0) = \beta(0)$. Suppose further that the projective $A[T]$-modules $\ker \alpha(T)$ and $\ker \beta(T)$ are extended from A. Then there exists an automorphism $\sigma(T)$ of $P[T]$ with $\sigma(0) = id$ such that $\beta(T)\sigma(T) = \alpha(T)$.*

The next lemma follows from the well known Quillen Splitting Lemma ([Q], Lemma 1) and its proof is essentially contained in ([Q], Theorem 1).

Lemma 2.7 *Let A be a Noetherian ring and P be a finitely generated projective A-module. Let $s, t \in A$ be such that $As + At = A$. Let $\sigma(T)$ be an $A_{st}[T]$-automorphism of $P_{st}[T]$ such that $\sigma(0) = id$. Then, $\sigma(T) = \alpha(T)_s\beta(T)_t$, where $\alpha(T)$ is an $A_t[T]$-automorphism of $P_t[T]$ such that $\alpha(T) = id$ modulo the ideal (sT) and $\beta(T)$ is an $A_s[T]$-automorphism of $P_s[T]$ such that $\beta(T) = id$ modulo the ideal (tT).*

Now we state two useful lemmas. The proofs of these can be found in ([B-RS 1]).

Lemma 2.8 *Let A be a Noetherian ring and let I be an ideal of A. Let J, K be ideals of A contained in I such that $K \subset I^2$ and $J + K = I$. Then there exists $c \in K$ such that $I = (J, c)$.*

Lemma 2.9 *Let A be a Noetherian ring containing an infinite field k and let $I \subset A[T]$ be an ideal of height n. Then there exists $\lambda \in k$ such that either $I(\lambda) = A$ or $I(\lambda)$ is an ideal of height n in A, where $I(\lambda) = \{f(\lambda) : f(T) \in I\}$.*

Now we quote a theorem of Eisenbud-Evans ([E-E]), as stated in ([P]).

Theorem 2.10 *Let A be a ring and M be a finitely generated A-module. Let S be a subset of* $\operatorname{Spec} A$ *and $d : S \to N$ be a generalized dimension function. Assume that $\mu_Q(M) \geq 1 + d(Q)$ for all $Q \in S$. Let $(m, a) \in M \oplus A$ be basic at all prime ideals $Q \in S$. Then there exists an element $m' \in M$ such that $m + am'$ is basic at all primes $Q \in S$.*

As a consequence of (2.10), we have the following corollary. For a proof one can look at ([B-RS 3], 2.13).

Corollary 2.11 *Let A be a ring and P be a projective A-module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that* $\operatorname{ht}(I_a) \geq n$*, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq n$ then* $\operatorname{ht} I \geq n$*. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then* $\operatorname{ht} I = n$*.*

The following lemma is an application of (2.8) and (2.11). Its proof is essentially contained in ([B-RS 3], 2.14).

Lemma 2.12 *Let A be a Noetherian ring of dimension $n \geq 2$ and let P be a projective $A[T]$-module of rank n. Let $I \subset A[T]$ be an ideal of height n and let $\overline{\alpha} : P/IP \twoheadrightarrow I/I^2$ be a surjection. Then there exists an ideal $I' \subset A[T]$ and a surjection $\beta : P \twoheadrightarrow I \cap I'$ such that:*
(i) $I + I' = A[T]$.
(ii) $\beta \otimes A[T]/I = \overline{\alpha}$.
(iii) $\operatorname{ht}(I') \geq n$*.*
(iv) Furthermore, given finitely many ideals $I_1, I_2, \cdots, I_r$ of height ≥ 2, I' can be chosen with the additional property that I' is comaximal with each of them.

Definition 2.13 *Let A be a commutative Noetherian ring, P a projective $A[T]$-module. Let $J(A, P) \subset A$ consist of all those $a \in A$ such that P_a is extended from A_a. It follows from* ([Q], Theorem 1)*, that $J(A, P)$ is an ideal and $J(A, P) = \sqrt{J(A, P)}$. This is called the Quillen ideal of P in A.*

Remark 2.14 It is easy to deduce from Quillen-Suslin theorem ([Q], [Su1]) that ht $J(A, P) \geq 1$. If determinant of P is extended from A, then by ([B-R], 3.1), ht $J(A, P) \geq 2$.

The following result is due to Lindel ([L], Theorem 2.6).

Theorem 2.15 *Let A be a commutative Noetherian ring with $\dim A = d$ and $R = A[T_1, \cdots, T_n]$. Let P be a projective R-module of rank $\geq \max(2, d+1)$. Then $E(P \oplus R)$ acts transitively on the set of unimodular elements of $P \oplus R$.*

3 Main theorem

In this section we prove the main theorem. This section is divided into two parts.

3.1 The semilocal case

In this part we prove the main theorem in semilocal situation. We need the following lemmas and propositions.

Lemma 3.1 *Let B be a Noetherian ring with $\dim B = n$ and $J \subset B$ be an ideal contained in the Jacobson radical of B. Let $I \subset B[T]$ be an ideal such that $I + JB[T] = B[T]$. Then any maximal ideal of $B[T]$ containing I has height $\leq n$.*

Proof Suppose $M \subset B[T]$ is a maximal ideal of height $n+1$. Then $M \cap B$ is a maximal ideal of B. Hence $M \cap B$ contains J. Since $I + JB[T] = B[T]$, it follows that I is not contained in M. This proves the lemma. $\square$

The following proposition is implicit in ([N]). We give a proof for the sake of completeness.

Proposition 3.2 *Let A be a semilocal ring and $I \subset A$ be an ideal such that $I = (a_1, \cdots, a_n) + L$, where L is an ideal contained in I^2 and $n \geq 1$. Then, $I = (b_1, \cdots, b_n)$ with $a_i = b_i$ mod L.*

Proof Since $I = (a_1, \cdots, a_n) + L$ and $L \subset I^2$, from (2.8) we get, $I = (a_1, \cdots, a_n, e)$ where $e \in L$ is such that $e(1-e) \in (a_1, \cdots, a_n)$. So, if L is contained in the Jacobson radical of A, we have $I = (a_1, \cdots, a_n)$. Suppose that L is not contained in the Jacobson radical of A and say, $M_1, \cdots, M_r$ are those maximal ideals which do not contain L. After rearranging the M_i's we may assume that a_1 belongs to $M_1, \cdots, M_t$ and does not belong to $M_{t+1}, \cdots, M_r$. By 'Prime avoidance' we can choose $b \in L \cap M_{t+1} \cap \cdots \cap M_r$ such that $b \notin M_1 \cup \cdots \cup M_t$. Then $a_1 + b, a_2, \cdots, a_n$ generate I as they do so locally. $\square$

As a consequence we have the following corollary.

Corollary 3.3 *Let R be a semilocal ring and $K \subset R[T]$ be an ideal containing a monic polynomial. Suppose that $K = (g_1, \cdots, g_n) + (K^2T)$ where $n \geq 2$. Then $K = (k_1, \cdots, k_n)$ such that, k_1 is monic and $k_i = g_i$ mod (K^2T).*

Proof Let $f \in K$ be monic. Adding f^pT $(p \geq 2)$ to g_1 for suitably large p we can assume that g_1 is monic. Let $A = R[T]/(g_1)$ and bar denote reduction modulo (g_1). It is clear that A is a semilocal ring. Now, $\overline{K} = (\overline{g_2}, \cdots, \overline{g_n}) + \overline{(K^2T)}$. From the proof of the above proposition it follows that $\overline{K} = (\overline{g_2} + \overline{h}, \cdots, \overline{g_n})$ where $\overline{h} \in \overline{(K^2T)}$. So, $h = k + g_1 l$, where $k \in (K^2T)$ and $l \in R[T]$. Now it is clear that $K = (g_1, g_2 + k, g_3, \cdots, g_n)$. $\square$

The following lemma is an analogue of Mandal's theorem (2.1 of [M 2], or 2.2 of [M-RS]).

Lemma 3.4 *Let C be a ring and $M \subset C[Y]$ be an ideal which contains a monic polynomial and is such that the ring C_{1+L} is semilocal, where $L = M \cap C$. Suppose that $M/(M^2Y)$ is generated by n elements ($n \geq 2$). Then, any set of n generators of $M/(M^2Y)$ can be lifted to a set of n generators of M.*

Proof Suppose that $M = (F_1, \cdots, F_n) + (M^2Y)$. We can clearly assume that F_1 is monic. Going to $C_{1+L}[Y]$ and applying the corollary above we get that $M_{1+L} = (F_1, G, F_3, \cdots, F_n)$ where $G - F_2 \in (M^2Y)_{1+L}$. Therefore we can find $s \in L$ such that, $M_{1+s} = (F_1, G, F_3, \cdots, F_n)$ and $G - F_2 \in (M^2Y)_{1+s}$. Now we can adapt the proof of ([M 2], 2.1) to get the result. $\square$

With the above lemma in hand one can prove the following analogue of a theorem of Mandal-Raja Sridharan (Theorem 2.3, [M-RS]).

Lemma 3.5 *Let C be a ring. Let $M, N \subset C[Y]$ be ideals such that*
1. M contains a monic polynomial.
2. C_{1+L} is semilocal where $L = M \cap C$.
3. $N = N(0)[Y]$ is an extended ideal.
4. $M + N = C[Y]$.
Let $J = M \cap N$. Suppose $J(0) = (a_1, \cdots, a_n)$ and $M = (F_1(Y), \cdots, F_n(Y)) + M^2$ such that $F_i(0) = a_i$ mod $M(0)^2$. Then $J = (G_1(Y), \cdots, G_n(Y))$ with $G_i(0) = a_i$.

Proof Same as in [M-RS]. $\square$

Now we turn to the main problem. We first state a lemma whose proof is the same as that of ([B-RS 3], 5.3).

Lemma 3.6 *Let R be a semilocal ring of dimension $n \geq 3$, $I \subset R[T]$ be an ideal of height n such that $I + \mathcal{J}R[T] = R[T]$ where $\mathcal{J}$ is the Jacobson radical of R. Let $\omega_I : (R[T]/I)^n \twoheadrightarrow I/I^2$ be a surjection. Suppose that ω_I can be lifted to a surjection $\alpha : R[T]^n \twoheadrightarrow I$. Let $f \in R[T]$ be a unit modulo I and $\theta \in GL_n(R[T]/I)$ be such that det $(\theta) = \overline{f}^2$. Then, the surjection $\omega_I\theta : (R[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\beta : R[T]^n \twoheadrightarrow I$.*

Now we prove the semilocal version of the main question in the following form.

Theorem 3.7 *Let R be a semilocal ring with $\dim R = n \geq 3$ and I be an ideal of $R[T]$ of height n such that $I + \mathcal{J}R[T] = R[T]$, where $\mathcal{J}$ is the Jacobson radical of R. Suppose that $I = (f_1, \cdots, f_n) + (I^2T)$. Also suppose that, $IR(T) = (u_1, \cdots, u_n)$ such that $u_i = f_i$ mod $I^2R(T)$. Then, there exist $h_1, \cdots, h_n \in I$ such that $I = (h_1, \cdots, h_n)$ and $h_i = f_i$ mod (I^2T).*

Proof Since $I + \mathcal{J}R[T] = R[T]$, it follows that I is not contained in any ideal which contains a monic polynomial and hence, any monic polynomial of $R[T]$ is unit modulo I. In particular, $I(0) = R$.

We also note that, to prove the theorem, it is enough to prove that $I = (h_1, \cdots, h_n)$ with $h_i = f_i$ mod I^2. Let us briefly explain why this is so. Consider the unimodular rows $(f_1(0), \cdots, f_n(0))$ and $(h_1(0), \cdots, h_n(0))$ over the semilocal ring R. Since $E_n(R)$ acts transitively on the set of unimodular rows over a semilocal ring R, there is a $\overline{\theta} \in E_n(R)$ such that $(h_1(0), \cdots, h_n(0))\overline{\theta} = (f_1(0), \cdots, f_n(0))$. Suppose that $\overline{\theta} = \Pi E_{ij}(r_{ij})$, where $r_{ij} \in R$. Since $I(0) = R$, we can find $f_{ij} \in I$ such that $f_{ij}(0) = r_{ij}$. We consider the elementary matrix $\theta = \Pi E_{ij}(f_{ij}) \in E_n(R[T])$. Then, it is easy to see that $(h_1, \cdots, h_n)\theta$ is a desired set of generators of I.

We give the proof in steps.

Step 1. We have, $I = (f_1, \cdots, f_n) + I^2$. From the given condition we see that there is a monic polynomial $f \in R[T]$ such that $I_f = (u_1, \cdots, u_n)$, where $u_i = f_i$ mod ${I_f}^2$. Let f^k be so chosen such that $f^{2k}u_i \in I$, $1 \leq i \leq n$. Write $f^{2k}u_i = g_i$. Then, $I = (g_1, \cdots, g_n) + I^2$ and $g_i - f^{2k}f_i \in I^2$.

Now $I = (g_1, \cdots, g_n) + I^2$ implies that $(g_1, \cdots, g_n) = I \cap K$ where K is an ideal of $R[T]$ such that $K + I = R[T]$. Since $I_f = (g_1, \cdots, g_n)_f$, we see that $K_f = R[T]_f$, i.e. K contains a monic polynomial. Also note that $K = (g_1, \cdots, g_n) + K^2$.

Since f is a unit modulo I, by (3.6), it is enough to get $I = (m_1, \cdots, m_n)$ where, $m_i - f^{2k}f_i \in I^2$. Therefore it is enough to get $I = (m_1, \cdots, m_n)$ with $m_i - g_i \in I^2$.

Step 2. Using (3.3) we find that $K = (k_1, \cdots, k_n)$ such that $k_i = g_i$ mod K^2. Note that k_1 is a monic polynomial. The row $(k_1, \cdots, k_n)$ is unimodular mod I^2. Since k_1 is monic, it is unit mod I^2. Therefore we can elementarily transform the above row so that $k_n = 1$ mod I^2 (note that this transformation does not affect k_1). Since elementary transformations can be lifted via surjection of rings, we can find $\sigma \in E_n(R[T])$, which is a lift of the above elementary transformation. Let

$$(g_1, \cdots, g_n)\sigma = (h_1, \cdots, h_n).$$

Therefore, we have $I \cap K = (h_1, \cdots, h_n)$, $K = (k_1, \cdots, k_n)$ and $h_i = k_i$ mod K^2. (We are still calling the new set of generators of K as $(k_1, \cdots, k_n)$.)

Write $C = R[T]$ and consider the following ideals in the polynomial extension $C[Y]$:

$$M = (k_1, \cdots, k_{n-1}, Y + k_n), \ N = IC[Y], \ J = M \cap N.$$

Clearly $M+N=C[Y]$. Suppose $L=M\cap C$. Since $k_1\in L$ is a monic polynomial in T, we see that $C/L=R[T]/L$ is integral over $R/(L\cap R)$. Therefore C/L is a semilocal ring as $R/(L\cap R)$ is so. Consequently, C_{1+L} is also semilocal. So conditions 1 - 4 of (3.5) are satisfied.

We have $J(0)=I\cap K=(h_1,\cdots,h_n)$ and $M=(k_1,\cdots,k_{n-1},Y+k_n)$ where $k_i=h_i \bmod M(0)^2$. Therefore, applying (3.5), we get, $J=(G_1(Y),\cdots,G_n(Y))$ such that $G_i(0)=h_i$. Putting $Y=1-k_n$ we get $I=J(1-k_n)=(G_1(1-k_n),\cdots,G_n(1-k_n))$. Now $k_n=1 \bmod I^2$ implies that $G_i(1-k_n)=h_i \bmod I^2$. Writing l_i for $G_i(1-k_n)$ we have $I=(l_1,\cdots,l_n)$ with $l_i=h_i \bmod I^2$. Let $(l_1,\cdots,l_n)\sigma^{-1}=(m_1,\cdots,m_n)$. Then, clearly $m_i=g_i \bmod I^2$ where $I=(m_1,\cdots,m_n)$. This completes the proof. $\square$

3.2 The general case

Now we proceed to prove the main theorem. We begin with a lemma.

Lemma 3.8 *Let A be a Noetherian ring containing the field of rationals with $\dim A=n\geq 2$, $I\subset A[T]$ an ideal of height n. Let P be a projective $A[T]$-module of rank n. Write $J=I\cap J(A,P)$ where $J(A,P)$ is the Quillen ideal of P in A and $B=A_{1+J}$. Suppose that there is a surjection*

$$\overline{\phi}:P\twoheadrightarrow I/(I^2T).$$

Assume further that there exists a surjection

$$\theta:P_{1+J}\twoheadrightarrow I_{1+J}$$

such that θ is a lift of $\overline{\phi}\otimes B$. Then there exists a surjection $\Phi:P\twoheadrightarrow I$ such that Φ is a lift of $\overline{\phi}$.

Proof We choose an element $s\in J$ such that $\theta:P_{1+s}\twoheadrightarrow I_{1+s}$ is surjective. Note that since $s\in J(A,P)$, the projective $A_s[T]$-module P_s is extended. Let $\overline{\phi}_s(0)$ denote the map $(P/TP)_s\twoheadrightarrow I(0)_s$ induced from $\overline{\phi}_s$ by setting $T=0$ and $\gamma=\overline{\phi}_s(0)\otimes A_s[T]:P_s\twoheadrightarrow I_s\ \ (=A_s[T])$. Then the elements $\theta\otimes A_{s(1+sA)}[T]$ and $\gamma\otimes A_{s(1+sA)}[T]$ are unimodular elements of $P^*_{s(1+sA)}$ and they are equal modulo (T). Since dim $A_{s(1+sA)}\leq n-1$, rank $P=n$

and A contains the field of rationals, by ([R], Corollary 2.5), the kernels of the surjections $\theta \otimes A_{s(1+sA)}[T]$ and $\gamma \otimes A_{s(1+sA)}[T]$ are locally free projective modules and hence by Quillen's local-global principle ([Q], Theorem 1), these kernels are projective modules which are extended from $A_{s(1+sA)}$. Hence, by (2.6), there exists an automorphism σ of $P_{s(1+sA)}$ such that $\sigma = \text{id}$ modulo (T) and $(\theta \otimes A_{s(1+sA)}[T])\sigma = \gamma \otimes A_{s(1+sA)}[T]$. Therefore, there exists an element $t \in A$ of the form $1+rs$ such that t is a multiple of $1+s$ and σ is an automorphism of P_{st} with $\sigma = \text{id}$ modulo (T) and $(\theta \otimes A_{st}[T])\sigma = \gamma \otimes A_{st}[T]$. We note that, since $s \in J(A, P)$, it follows that P_{st} is extended from A_{st}.

Since P_{st} is extended, we can adjoin a new variable W and consider the $A_{st}[T, W]$-automorphism of $P_{st}[W]$ given by $\tau(W) = \sigma(TW)$. Since $\tau(0)$ is identity, by (2.7), it follows that $\tau = \alpha_s \beta_t$, where α is an $A_t[T, W]$-automorphism of $P_t[W]$ such that $\alpha = \text{id}$ modulo the ideal (sTW) and β is an $A_s[T, W]$-automorphism of $P_s[W]$ such that $\beta = \text{id}$ modulo the ideal (tTW). Putting $W = 1$ and using a standard patching argument, we see that the surjections $(\theta \otimes A_t[T]).\alpha(1) : P_t \twoheadrightarrow I_t$ and $(\gamma \otimes A_s[T]).\beta(1)^{-1} : P_s \twoheadrightarrow I_s$ patch to yield a surjection $\Phi : P \twoheadrightarrow I$. It is easy to see that Φ is a lift of $\overline{\phi}$. This proves the lemma. $\square$

The following is a restatement of (Lemma 3.6, [B-RS 1]) in our set up. We give the proof for the sake of completeness.

Lemma 3.9 *Let A be a Noetherian ring of dimension $n \geq 3$, $I \subset A[T]$ an ideal of height n and J be any ideal contained in $I \cap A$ such that $\text{ht}\, J \geq 2$. Let P be a projective $A[T]$-module of rank n. Suppose $\psi : P \twoheadrightarrow I/(I^2T)$ is a surjection. Then we can find a lift $\phi \in \text{Hom}_{A[T]}(P, I)$ of ψ, such that the ideal $\phi(P) = I''$ satisfies the following properties:*
(i) $I'' + (J^2T) = I$.
(ii)$I'' = I \cap I'$, where $\text{ht}\,(I') \geq n$.
(iii) $I' + (J^2T) = A[T]$.

Proof We choose any lift $\phi \in \text{Hom}_{A[T]}(P, I)$ of ψ. Since $\phi(P) + (I^2T) = I$, by (2.8) we can choose $b \in (I^2T)$ such that $(\phi(P), b) = I$. Let $C = A[T]/(J^2T)$ and bar denote reduction modulo (J^2T). Now applying (2.11) to the element $(\overline{\phi}, \overline{b})$ of $\overline{P}^* \oplus A[T]/(J^2T)$, we see that, there exists $\beta \in P^*$, such that if $N = (\phi + b\beta)(P)$, then $\text{ht}\,(\overline{N}_{\overline{b}}) \geq n$.

Since $b \in (I^2T)$, the element $\phi + b\beta$ is also a lift of ψ. Therefore, replacing ϕ by $\phi + b\beta$, we may assume that $N = \phi(P)$.

Now as $(N, b) = I$ and $b \in (I^2T)$, it follows that $N = I \cap K$, $(K, b) = A[T]$. Since $b \in I$, $N_b = K_b$. Therefore we have
(1) $\overline{N} = \overline{I} \cap \overline{K}$ with ht $(\overline{K}) =$ ht $(\overline{K}_{\overline{b}}) =$ ht $(\overline{N}_{\overline{b}}) \geq n$.
(2) $(\overline{b}) + \overline{K} = C$.
We claim that $\overline{K} = C$. Assume, to the contrary, that $\overline{K}$ is a proper ideal of C. Since $b \in (I^2T)$, in view of (1) and (2), we have $n \leq$ ht $(\overline{K}) =$ ht $(\overline{K}_{\overline{T}}) \leq$ dim $(C_{\overline{T}}) =$ dim $(A/J^2)[T, T^{-1}] =$ dim $(A/J) + 1 \leq (n-2) + 1 = n - 1$.

This is a contradiction. Thus $\overline{K} = C$ and $\phi(P) + (J^2T) = I$. This proves (i).

We choose, by (2.8), an element $c \in (J^2T)$ such that $(\phi(P), c) = I$. As before, using (2.11), we can add a suitable multiple of c to ϕ and assume that the ideal $\phi(P) = I''$ satisfies (ii) and (iii). This proves the lemma. $\square$

Now we prove the main theorem. The proof of this theorem is motivated by ([B-RS 1], Theorem 3.8).

Theorem 3.10 *Let A be a Noetherian ring of dimension $n \geq 3$, containing the field of rationals. Let $I \subset A[T]$ be an ideal of height n. Suppose that $I = (f_1, \cdots, f_n) + (I^2T)$ and there exist $F_1, \cdots, F_n \in IA(T)$ such that $IA(T) = (F_1, \cdots, F_n)$ and $F_i = f_i$ mod $I^2A(T)$. Then, there exist $g_1, \cdots, g_n$ such that $I = (g_1, \cdots, g_n)$ and $g_i = f_i$ mod (I^2T).*

Proof We give the proof of the theorem in steps.

Step 1. Let $J = I \cap A$. Applying lemma (3.9) we get $k_1, \cdots, k_n \in I$ and an ideal $I' \subset A[T]$ of height n such that,
(i) $(k_1, \cdots, k_n) + (J^2T) = I$ where $k_i = f_i$ mod (I^2T),
(ii) $(k_1, \cdots, k_n) = I \cap I'$,
(iii) $I' + (J^2T) = A[T]$.

Let $J' = I' \cap A$. We claim that dim $(A/(J + J')) = 0$.

Proof of the claim. Since ht $(I) = n$ and ht $(I') \geq n$, we have dim $A/J \leq 1$ and dim $A/J' \leq 1$. Without loss of generality we may assume that dim

$A/J = \dim A/J' = 1$. It suffices to show that there is no prime ideal Q of A containing J and J' and having the property that dim $A/Q = 1$. Suppose, to the contrary, that such a prime ideal exists.

Let $I' = K_1 \cap K_2 \cap \cdots \cap K_r$ be a primary decomposition of I', with $\sqrt{K_l} = P_l$. Then $J' = (K_1 \cap A) \cap \cdots \cap (K_r \cap A)$. Since dim A/J' =dim $A/Q = 1$, it follows that Q is minimal over J'. therefore $Q = P_l \cap A$ for some l. We have $\sqrt{K_l} = P_l \supset I' + QA[T] \supset I' + JA[T]$. But by property (iii) of (3.9), $I' + (J^2T) = A[T]$. This yields a contradiction and proves the claim.

Step 2. Write $B = A_{1+J}$. We have, from (3.9), $I \cap I' = (k_1, \cdots, k_n)$ and the ideals I and I' are comaximal. Going to the ring $B[T]$ we get, $IB[T] \cap I'B[T] = (k_1, \cdots, k_n)B[T]$. Therefore, $I'B[T] = (k_1, \cdots, k_n)B[T] + I'^2B[T]$. Now as $I' + (J^2T) = A[T]$, we have $I'(0) = A$. Note that JB is contained in the Jacobson radical of B.

We claim that we can lift the above set of generators of $I'B[T]/I'^2B[T]$ to a set of generators of $I'B[T]$, i.e., there exist $l_1, \cdots, l_n \in I'B[T]$ such that $I'B[T] = (l_1, \cdots, l_n)$ and $l_i = k_i$ mod $I'^2B[T]$.

Proof of the claim. Since $I'B[T] = (k_1, \cdots, k_n) + I'^2B[T]$ and $I'(0) = A$, it is easy to see, using the Chinese remainder theorem, that there exist $\alpha_1, \cdots, \alpha_n$ such that $I'B[T] = (\alpha_1, \cdots, \alpha_n) + (I'^2T)B[T]$, where $\alpha_i = k_i$ mod $I'^2B[T]$.

Write $R = B_{1+J'B}$. In order to prove the claim, in view of (3.8), it is enough to show that $I'R[T] = (\beta_1, \cdots, \beta_n)$ such that $\beta_i = \alpha_i$ modulo $(I'^2T)R[T]$.

Since $R = B_{1+J'B} = A_{1+J+J'}$, it follows from Step 1 that R is semilocal. Now we consider the ring $R(T)$. Using the subtraction principle ([B-RS 3], 3.3), we get $I'R(T) = (v_1, \cdots, v_n)$ with the property that $v_i = \alpha_i$ modulo $I'^2R(T)$. Therefore, by (3.7), we obtain a set of n generators of $I'R[T]$ with the desired property. Thus the claim is proved.

Step 3. Recall that we have :
(i) $(k_1, \cdots, k_n) + (J^2T) = I$.
(ii) $(k_1, \cdots, k_n) = I \cap I'$, ht $(I') = n$.
(iii) $I' + (J^2T) = A[T]$.

(iv) $(l_1, \cdots, l_n) = I'B[T]$, such that $l_i = k_i \bmod I'^2 B[T]$.

Since $I'B[T] + (J^2T)B[T] = B[T]$, it follows that the row $(l_1, \cdots, l_n)$ is unimodular modulo $(J^2T)B[T]$. Let $D = B[T]/(J^2T)B[T]$ and bar denote reduction modulo $(J^2T)B[T]$. We want to show that $(\overline{l_1}, \cdots, \overline{l_n}) \in Um_n(D)$ can be elementarily transformed to $(\overline{1}, \overline{0}, \cdots, \overline{0})$.

Since JB is contained in the Jacobson radical of B, it is easy to see that JD is contained in the Jacobson radical of D. Therefore, it suffices to show that the row $(\overline{l_1}, \cdots, \overline{l_n})$ can be elementarily completed over the ring D/JD. But this follows from (2.15) because $D/JD \simeq (A/J)[T]$, $\dim A/J \leq 1$ and $n \geq 3$.

Since elementary transformations can be lifted via surjection of rings, it follows that there is an elementary automorphism $\tau \in E_n(B[T])$ such that $(l_1, \cdots, l_n)\tau = (m_1, \cdots, m_n)$ where $(m_1, \cdots, m_n) = (1, 0, \cdots, 0)$ modulo $(J^2T)B[T]$. Applying (2.11), we can find $d_1, \cdots, d_{n-1} \in B[T]$ such that $\operatorname{ht}((m_1 + d_1 m_n, \cdots, m_{n-1} + d_{n-1} m_n)_{m_n}) \geq n - 1$. We write $h_i = m_i + d_i m_n$, $i = 1, 2, \cdots, n-1$. Since $I'B[T] = (h_1, \cdots, h_{n-1}, m_n)$ and $\operatorname{ht}(I') \geq n$, it is easy to verify that $\operatorname{ht}(h_1, \cdots, h_{n-1}) = n - 1$. We have $(h_1, \cdots, h_{n-1}) + (J^2T)B[T] = B[T]$ and we note that $(J^2T)B[T]$ is contained in the Jacobson radical of B. Therefore, it follows from (3.1) that $\dim B[T]/(h_1, \cdots, h_{n-1}) \leq 1$. We take $h_n = h_1 + m_n$. Then $h_n = 1$ modulo $(J^2T)B[T]$.

Note that the automorphism of $B[T]$ which transforms $(l_1, \cdots, l_n)$ to $(h_1, \cdots, h_n)$ is elementary. Let us call it σ.

Step 4. Recall that we have $IB[T] \cap I'B[T] = (k_1, \cdots, k_n)B[T]$ and $I'B[T] = (l_1, \cdots, l_n)$ such that $l_i = k_i \bmod I'^2 B[T]$ for $i = 1, \cdots, n$. From Step 3 we have $\sigma \in E_n(B[T])$ such that $(l_1, \cdots, l_n)\sigma = (h_1, \cdots, h_n)$. Therefore, $I'B[T] = (h_1, \cdots, h_n)$.

Let $(k_1, \cdots, k_n)\sigma = (u_1, \cdots, u_n)$. Then, since σ is elementary, $IB[T] \cap I'B[T] = (u_1, \cdots, u_n)$. Also note that $h_i = u_i$ modulo $I'^2 B[T]$ for $i = 1, \cdots, n$.

Let $C = B[T]$, $R = C[Y]$. Let K_1 be the ideal $(h_1, \cdots, h_{n-1}, Y + h_n)$ of R, $K_2 = IC[Y]$ and $K_3 = K_1 \cap K_2$.

Since from Step 3 we have $\dim B[T]/(h_1, \cdots, h_{n-1}) \leq 1$, we see that all the conditions of ([M-RS], Theorem 2.3) are satisfied. Applying that theorem it follows easily that $K_3 = (H_1(T,Y), \cdots, H_n(T,Y))$ such that $H_i(T,0) = u_i$. Putting $Y = 1 - h_n$, we see that $IB[T] = (H_1(T, 1 - h_n), \cdots, H_n(T, 1 - h_n))$. Since $h_n = 1$ modulo $(I^2T)B[T]$, it follows that $H_i(T, 1 - h_n) = H_i(T, 0)$ $(= u_i)$ modulo $(I^2T)B[T]$.

Let $(H_1(T, 1-h_n), \cdots, H_n(T, 1-h_n))\sigma^{-1} = (w_1, \cdots, w_n)$. Then, $IB[T] = (w_1, \cdots, w_n)$ whereas $w_i = k_i$ modulo $(I^2T)B[T]$ and hence $w_i = f_i$ modulo $(I^2T)B[T]$.

Now we can apply (3.8) to obtain the desired set of generators of I. This proves the theorem. $\square$

It is not hard to see that, adapting the same proof, we can prove the main theorem in the following form.

Theorem 3.11 *Let A, I be as above and P be a projective A-module of rank n with trivial determinant. Suppose there exists a surjection*

$$\phi : P[T] \twoheadrightarrow I/(I^2T).$$

Suppose that $\phi \otimes A(T)$ can be lifted to a surjection $\phi' : P[T] \otimes A(T) \twoheadrightarrow IA(T)$. Then, there is a surjection $\psi : P[T] \twoheadrightarrow I$ which lifts ϕ.

Another similar result will be proved in the next section (Theorem 4.8).

4 The Euler class group of $A[T]$

For the rest of the paper, A will denote a commutative Noetherian ring containing the field of rationals.

Remark 4.1 Let A be a commutative Noetherian ring containing the field of rationals with $\dim A = n \geq 3$. Let us describe the general method of approach to the problems we will be considering. In the following two sections, we will frequently use our main theorem, proved in the last section. In most cases, we will try to find a suitable ideal $I \subset A[T]$ of height n and a surjection $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ in such a manner that the question reduces to finding a set of n generators of I which lifts ω_I. Now, since

A contains $\mathbb{Q}$, it follows from (2.9), that there is a $\lambda \in \mathbb{Q}$ such that either $I(\lambda) = A$ or $I(\lambda)$ is of height n. Therefore, if necessary, we can replace T by $T - \lambda$ and assume that $I(0) = A$ or $I(0)$ is of height n.

To lift ω_I to a surjection from $A[T]^n$ to I, we will consider the induced surjections: $\omega_{I(0)}(= \omega_I \otimes A[T]/(T)) : (A/I(0))^n \twoheadrightarrow I(0)/I(0)^2$ over the ring A and $\omega_I \otimes A(T) : (A(T)/IA(T))^n \twoheadrightarrow IA(T)/I^2A(T)$ over $A(T)$. Since $\dim(A) = \dim(A(T)) = n$ and there is a well-studied description of Euler class group of an arbitrary Noetherian ring which deals with top height ideals only, using results on them (mostly from [B-RS 3]), we will ensure that $\omega_{I(0)}$ and $\omega_I \otimes A(T)$ can be lifted. Then we appeal to the main theorem to conclude that ω_I is liftable. An explicit description of this method is given in the following proposition.

Proposition 4.2 (Addition principle) *Let* $\dim A = n \geq 3$*, and* I*,* J *be two comaximal ideals in* $A[T]$*, each of height* n*. Suppose that* $I = (f_1, \cdots, f_n)$ *and* $J = (g_1, \cdots, g_n)$*. Then* $I \cap J = (h_1, \cdots, h_n)$ *where* $h_i = f_i$ *mod* I^2 *and* $h_i = g_i$ *mod* J^2*.*

Proof Write $K = I \cap J$. Since I and J are comaximal, the generators of I and J induce a set of generators of K/K^2. Say, $K = (k_1, \cdots, k_n) + K^2$ where $k_i = f_i$ mod I^2 and $k_i = g_i$ mod J^2.

Since A contains the field of rationals, by (2.9) we get some $\lambda \in \mathbb{Q}$ such that $K(\lambda) = A$ or $K(\lambda)$ has height n. Therefore, if necessary, we can replace T by $T - \lambda$ and assume that $K(0) = A$ or $\mathrm{ht}(K(0)) = n$.

If $K(0) = A$, by (Remark 3.9, [B-RS 1]), we get $K = (l_1, \cdots, l_n) + (K^2T)$ with $l_i = k_i$ modulo K^2. Now assume that $\mathrm{ht}(K(0)) = n$. Since I and J are comaximal ideals in $A[T]$, it is easy to see that $K(0) = I(0) \cap J(0)$. Therefore, $\mathrm{ht}(I(0)) \geq n$ and $\mathrm{ht}(J(0)) \geq n$. Both of them cannot equal A, as $K(0)$ is proper. If one of them, say $I(0) = A$, then $K(0) = J(0) = (g_1(0), \cdots, g_n(0))$ whereas $g_i(0) = k_i(0)$ modulo $J(0)^2$. Since $I(0) = A$, it follows that $g_i(0) = k_i(0)$ modulo $K(0)^2$. Therefore, again by (3.9, [B-RS 1]) we can get $K = (l_1, \cdots, l_n) + (K^2T)$ with $l_i = k_i$ modulo K^2.

Now assume that both $I(0)$ and $J(0)$ are proper ideals. In this case, by the addition principle (Theorem 3.2, [B-RS 3]) we get $K(0) = (a_1, \cdots, a_n)$ such that $a_i = f_i(0)$ modulo $I(0)^2$ and $a_i = g_i(0)$ modulo $J(0)^2$. Therefore, $a_i = k_i(0)$ modulo $I(0)^2$ and $a_i = k_i(0)$ modulo $J(0)^2$, implying that $a_i = k_i(0)$ modulo $K(0)^2$. Then, as before, by (3.9, [B-RS 1]) we get $K = (l_1, \cdots, l_n) + (K^2T)$ with $l_i = k_i$ modulo K^2.

So, in any case, we can lift the given set of generators of K/K^2 to a set of generators of $K/(K^2T)$. Now we go to the ring $A(T)$. Note that dim $A(T) = n$. Applying the addition principle (3.2, [B-RS 3]), we get, $KA(T) = (\widetilde{k_1}, \cdots, \widetilde{k_n})$ such that $\widetilde{k_i} = f_i$ modulo $I^2A(T)$ and $\widetilde{k_i} = g_i$ modulo $J^2A(T)$. Therefore, $\widetilde{k_i} = k_i$ modulo $I^2A(T)$ and $\widetilde{k_i} = k_i$ modulo $J^2A(T)$ implying that $\widetilde{k_i} = k_i$ modulo $K^2A(T)$.

Now we can appeal to the main theorem (3.10) and obtain the desired set of generators for K. $\square$

Proposition 4.3 (Subtraction principle) *Let* dim $A = n \geq 3$ *and* I, J *be two comaximal ideals in* $A[T]$, *each of height* n. *Suppose that* $I = (f_1, \cdots, f_n)$ *and* $I \cap J = (h_1, \cdots, h_n)$ *such that* $h_i = f_i$ *mod* I^2. *Then,* $J = (g_1, \cdots, g_n)$ *where* $h_i = g_i$ *mod* J^2.

Proof The method of proof is the same as that of (4.2) and therefore we will just outline the proof here. Let $K = I \cap J$. As above, we can assume that $K(0) = A$ or ht $(K(0)) = n$. First note that $J = (h_1, \cdots, h_n) + J^2$. If $J(0) = A$ or $I(0) = A$ then, as before, we can lift the above set of generators of J/J^2 to a set of generators of $J/(J^2T)$. So assume ht $(K(0)) = n$ and both $I(0)$ and $J(0)$ are proper. Then we can apply the subtraction principle (Theorem 3.3, [B-RS 3]) and conclude that $J(0) = (a_1, \cdots, a_n)$ where $a_i = h_i(0)$ mod $J(0)^2$. Therefore, by (3.9, [B-RS 1]) again, we get $J = (l_1, \cdots, l_n) + (J^2T)$.

Next we go to the ring $A(T)$ and apply the subtraction principle(3.3, [B-RS 3]) there to conclude that $JA(T) = (\widetilde{h_1}, \cdots, \widetilde{h_n})$ with $\widetilde{h_i} = h_i$ modulo $J^2A(T)$. Therefore, using (3.10) we get the desired set of generators for J.

$\square$

Now we proceed to define the n^{th} *Euler class group* of $A[T]$ where A is a commutative Noetherian ring with dim $(A) = n \geq 3$ and which contains the field of rationals.

Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Let α and β be two surjections from $(A[T]/I)^n$ to I/I^2. We say that α and β are *related* if there exists an automorphism σ of $(A[T]/I)^n$ of determinant 1 such that $\alpha\sigma = \beta$. It follows easily that this is an equivalence relation on the set of surjections from $(A[T]/I)^n$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local orientation of* I.

We note that following the remark (4.1) it is not hard to derive that if a surjection α from $(A[T]/I)^n$ to I/I^2 can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$ then so can any β equivalent to α (However, we give a proof below for the convenience of the reader). Therefore, from now on we shall identify a surjection α with the equivalence class $[\alpha]$ to which it belongs.

Proposition 4.4 *Let α and β be two surjections from $(A[T]/I)^n$ to I/I^2 such that there exists $\sigma \in SL_n(A[T]/I)$ with the property that $\alpha\sigma = \beta$. Suppose that α can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$. Then β can also be lifted to a surjection $\phi : A[T]^n \twoheadrightarrow I$.*

Proof Since A contains $\mathbb{Q}$, by (2.9) we can find some $\lambda \in \mathbb{Q}$ such that $I(\lambda) = A$ or $I(\lambda)$ is an ideal of height n. If necessary, we can replace T by $T - \lambda$ and assume that either $I(0) = A$ or $\operatorname{ht} I(0) = n$.

If $I(0) = A$, by (Remark 3.9, [B-RS 1]), we can lift β to a surjection $\gamma : A[T]^n \twoheadrightarrow I/(I^2T)$. On the other hand, if $\operatorname{ht} I(0) = n$, then we consider the surjections $\alpha(0) : (A/I(0))^n \twoheadrightarrow I(0)/I(0)^2$ and $\beta(0) : (A/I(0))^n \twoheadrightarrow I(0)/I(0)^2$ and note that $\alpha(0)\sigma(0) = \beta(0)$. Since $\dim(A/I(0)) = 0$, we have, $SL_n(A/I(0)) = E_n(A/I(0))$ and since elementary matrices can be lifted via a surjection of rings, it follows that $\sigma(0)$ can be lifted to an element $\tau \in E_n(A)$. Composing τ with $\theta(0)$ (which is a lift of $\alpha(0)$), we get a lift of $\beta(0)$ to a surjection from A^n to $I(0)$. So, again by (Remark 3.9, [B-RS 1]), we can lift β to a surjection $\gamma : A[T]^n \twoheadrightarrow I/(I^2T)$.

Now we move to the ring $A(T)$ and consider the induced surjections $\alpha \otimes A(T)$ and $\gamma \otimes A(T)$. Since $\dim(A(T)/IA(T)) = 0$, it follows that $SL_n(A(T)/IA(T)) = E_n(A(T)/IA(T))$. Following the same method as in the above paragraph, we get a lift of γ to a surjection from $A(T)^n$ to $IA(T)$. Now we can apply (3.10) and conclude that β can be lifted to a surjection $\phi : A[T]^n \twoheadrightarrow I$. $\square$

We call a local orientation $[\alpha]$ of I a *global orientation of* I if the surjection $\alpha : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$.

Let G be the free abelian group on the set of pairs (I, ω_I), where $I \subset A[T]$ is an ideal of height n having the property that Spec $(A[T]/I)$ is connected and I/I^2 is generated by n elements, and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I.

Let $I \subset A[T]$ be an ideal of height n. Then I can be decomposed as $I = I_1 \cap \cdots \cap I_r$, where the I_k's are ideals of $A[T]$ of height n, pairwise comaximal and Spec $(A[T]/I_k)$ is connected for each k. The following lemma shows that such a decomposition is unique. We shall say that I_k are the connected components of I.

Lemma 4.5 *The decomposition of I into its connected components, as described above, is unique.*

Proof Suppose that we have another decomposition $I = I'_1 \cap \cdots \cap I'_s$, where I'_i are pairwise comaximal and Spec$(A[T]/I'_i)$ is connected. By set topological arguments , it follows that $r = s$, and after suitably renumbering, that Spec$(A[T]/I_i) =$ Spec$(A[T]/I'_i)$ for every i. Hence $\sqrt{I_i} = \sqrt{I'_i}$. Since I_i are pairwise comaximal, it follows that there exists $f \in I_1$ and $g \in I_2 \cap \cdots \cap I_r$, such that $f + g = 1$. Let S be the multiplicative closed set $\{1, g, g^2, \cdots\}$. It follows easily using the fact that $f + g = 1$, that $I_1 = S^{-1}I \cap A[T]$. Now using the facts that $\sqrt{I_i} = \sqrt{I'_i}$, and $I = I'_1 \cap \cdots \cap I'_r$, it follows easily that $I'_1 \subset S^{-1}I \cap A[T]$. Hence, $I'_1 \subset I_1$. Similarly, $I_1 \subset I'_1$. Therefore, $I'_1 = I_1$. Similarly, $I'_i = I_i$ for every i. This proves the lemma. $\square$

Now assume that $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Let $I = I_1 \cap \cdots \cap I_r$ be the decomposition of I into its connected components. Then, ht $(I_k) = n$ and I_k/I_k^2 is generated by n elements. Let $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ be a surjection. Then ω_I induces surjections $\omega_{I_k} : (A[T]/I_k)^n \twoheadrightarrow I_k/I_k^2$. By (I, ω_I) we mean the element $\Sigma(I_k, \omega_{I_k})$ of G.

Let H be the subgroup of G generated by set of pairs (I, ω_I), where I is an ideal of $A[T]$ of height n generated by n elements and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ has the property that ω_I can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$ (in other words, a global orientation of I). We define the n^{th} *Euler class group* of $A[T]$, denoted by $E^n(A[T])$, to be G/H.

By a slight abuse of notation, we will write $E(A[T])$ for $E^n(A[T])$ throughout the paper.

Let P be a projective $A[T]$-module of rank n having trivial determinant. Let $\chi : A[T] \simeq \wedge^n P$ be an isomorphism. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(A[T])$ as follows:

Let $\lambda : P \twoheadrightarrow I_0$ be a surjection, where I_0 is an ideal of $A[T]$ of height n. Let bar denote reduction modulo I_0. We obtain an induced surjection $\overline{\lambda} : P/I_0P \twoheadrightarrow I_0/I_0^2$. Note that, since P has trivial determinant and $\dim(A[T]/I_0) \leq 1$, by (2.4), P/I_0P is a free $A[T]/I_0$-module of rank n. We choose an isomorphism $\overline{\gamma} : (A[T]/I_0)^n \simeq P/I_0P$, such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let ω_{I_0} be the surjection $\overline{\lambda}\overline{\gamma} : (A[T]/I_0)^n \twoheadrightarrow I_0/I_0^2$. Let $e(P, \chi)$ be the image in $E(A[T])$ of the element (I_0, ω_{I_0}). We say that (I_0, ω_{I_0}) is *obtained* from the pair (λ, χ).

Lemma 4.6 *The assignment sending the pair (P, χ) to the element $e(P, \chi)$, as described above, is well defined.*

Proof Let $\mu : P \twoheadrightarrow I_1$ be another surjection where $I_1 \subset A[T]$ is an ideal of height n. Let (I_1, ω_{I_1}) be obtained from (μ, χ).

Applying (2.12) we can find an ideal $K \subset A[T]$ of height n such that K is comaximal with I_0, I_1 and there is a surjection $\nu : A[T]^n \twoheadrightarrow I_0 \cap K$ such that $\nu \otimes A[T]/I_0 = \omega_{I_0}$. Since K and I_0 are comaximal, ν induces a local orientation ω_K of K. Clearly, $(I_0, \omega_{I_0}) + (K, \omega_K) = 0$ in $E(A[T])$.

Let $L = K \cap I_1$. Note that ω_K and ω_{I_1} together induce local orientation ω_L of L. We wish to show that $(L, \omega_L) = 0$ in $E(A[T])$ which proves the lemma because $(L, \omega_L) = (K, \omega_K) + (I_1, \omega_{I_1})$ in $E(A[T])$.

It is easy to see that applying (2.9) we can assume that each of the ideals $K(0), I_0(0), I_1(0)$ in A is either of height n or equal to A. We take the case when $\text{ht}(K(0)) = \text{ht}(I_0(0)) = \text{ht}(I_1(0)) = n$ (other cases can be handled similarly).

Since $e(P/TP, \chi \otimes A[T]/(T))$ is well defined in $E(A)$ (see [B-RS 3], Section 4), it follows that $(L(0), \omega_{L(0)}) = 0$ in $E(A)$. Therefore, we can lift ω_L to a surjection $A[T]^n \twoheadrightarrow L/(L^2T)$. On the other hand, using the fact that $e(P \otimes A(T), \chi \otimes A(T))$ is well defined in $E(A(T))$, it follows that $\omega_L \otimes A(T)$ is a global orientation of $LA(T)$. Therefore, in view of (4.1), using (3.10) we see that ω_L is a global orientation. This proves the lemma. $\square$

We define the *Euler class* of (P, χ) to be $e(P, \chi)$.

Theorem 4.7 *Let A be of dimension $n \geq 3$, $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements and let $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ be a local orientation of I. Suppose that the image of (I, ω_I) is zero in the Euler Class group $E(A[T])$ of $A[T]$. Then, I is generated by n elements and ω_I can be lifted to a surjection $\theta : A[T]^n \twoheadrightarrow I$.*

Proof Without loss of generality we can assume that either $I(0) = A$ or $\text{ht}\,(I(0)) = n$. Suppose $I(0) \neq A$. Now (I, ω_I) gives an element $(I(0), \omega_{I(0)})$ of $E(A)$. Since $(I, \omega_I) = 0$ in $E(A[T])$, we have $(I(0), \omega_{I(0)}) = 0$ in $E(A)$. Therefore, by (Theorem 4.2, [B-RS 3]), $\omega_{I(0)}$ can be lifted to a set of generators of $I(0)$, which in turn implies that ω_I can be lifted to a set of generators of $I/(I^2T)$. If $I(0) = A$, we can also lift ω_I to a set of generators of $I/(I^2T)$.

In $E(A(T))$ also, the element $(IA(T), \omega_{IA(T)})$ is zero, which, by (Theorem 4.2, [B-RS 3]) implies that $\omega_{IA(T)}$ can be lifted to a set of generators of $IA(T)$.

Using (3.10), the theorem follows. $\square$

Now we prove our main theorem in a more general form.

Theorem 4.8 *Let A be a Noetherian ring containing $\mathbb{Q}$ with* $\dim A = n \geq 3$ *and $I \subset A[T]$ be an ideal of height n. Let P be a projective $A[T]$-module of rank n whose determinant is trivial. Assume that we are given a surjection*

$$\psi : P \twoheadrightarrow I/(I^2T).$$

Assume further that $\psi \otimes A(T)$ can be lifted to a surjection

$$\psi' : P \otimes A(T) \twoheadrightarrow IA(T).$$

Then, there exists a surjection $\Psi : P \twoheadrightarrow I$ such that Ψ is a lift of ψ.

Proof We fix an isomorphism $\chi : A[T] \simeq \wedge^n P$.

Let $J(A, P)$ denote the Quillen ideal of P in A. Let $J = J(A, P) \cap I$. Since the determinant of P is trivial, we have, $\text{ht}\, J(A, P) \geq 2$. So it follows that, $\operatorname{ht} J \geq 2$. Therefore we can apply lemma (3.9) and obtain a lift $\phi \in \operatorname{Hom}_{A[T]}(P, I)$ of ψ and an ideal $I' \subset A[T]$ of height n such that (1) $I' + (J^2T) = A[T]$, (2) $\phi : P \twoheadrightarrow I \cap I'$ is a surjection and (3) $\phi(P) + (J^2T) = I$.

It follows that $e(P, \chi) = (I \cap I', \omega_{I \cap I'})$ in $E(A[T])$ where the local orientation $\omega_{I \cap I'}$ is obtained by composing $\phi \otimes A[T]/(I \cap I')$ with a suitable isomorphism $\overline{\lambda} : (A[T]/I \cap I')^n \simeq P/(I \cap I')P$, as described above in the definition of an Euler class.

Therefore, $e(P, \chi) = (I, \omega_I) + (I', \omega_{I'})$. We note that since $I'(0) = A$, we can lift $\omega_{I'}$ to a surjection from $A[T]^n \twoheadrightarrow I'/({I'}^2T)$. Moreover, considering the equation $e(P \otimes A(T), \chi \otimes A(T)) = (IA(T), \omega_I \otimes A(T)) + (I'A(T), \omega_{I'} \otimes A(T))$ in $E(A(T))$ and using the condition of the theorem it is easy to deduce that $(I'A(T), \omega_{I'} \otimes A(T)) = 0$ in $E(A(T))$ (Actually, the condition of the theorem tells that $e(P \otimes A(T), \chi \otimes A(T)) = (IA(T), \omega_I \otimes A(T))$). As a result, by (3.10), $(I', \omega_{I'}) = 0$ in $E(A[T])$. Therefore, by (4.7), $I' = (l_1, \cdots, l_n)$ such that this set of generators is a lift of $\omega_{I'}$.

Let us write $B = A_{1+J}$. Note that we have $I' + (J^2T) = A[T]$ and JB is contained in the Jacobson radical of B. Therefore, proceeding as in Step 3 of (3.10), and using (2.15) we can alter the above set of generators of $I'B[T]$ by an elementary transformation $\sigma \in E_n(B[T])$ and assume that
(i) $\operatorname{ht}(l_1, \cdots, l_{n-1}) = n - 1$,
(ii) $\dim B[T]/(l_1, \cdots, l_{n-1}) \leq 1$ and
(iii) $l_n = 1$ modulo $(J^2T)B[T]$.

We set $C = B[T]$, $R = C[Y]$, $K_1 = (l_1, \cdots, l_{n-1}, Y + l_n)$, $K_2 = IC[Y]$, $K_3 = K_1 \cap K_2$. Let us denote P_{1+J} by P'.

We claim that there exists a surjection $\eta(Y) : P'[Y] \twoheadrightarrow K_3$ such that $\eta(0) = \phi \otimes B[T]$.

We first show that the theorem follows from the claim. Specializing η at $Y = 1 - l_n$, we obtain a surjection $\theta : P' \twoheadrightarrow IB[T]$.

Since $l_n = 1$ modulo $(J^2T)B[T]$, the following equalities hold modulo (J^2T):

$$\theta = \eta(1 - l_n) = \eta(0) = \phi.$$

Therefore θ lifts $\psi \otimes B[T]$. Now using lemma (3.8), the theorem follows.

Proof of the claim: Recall that we chose an isomorphism $\overline{\lambda} : (A[T]/I \cap I')^n \simeq P/(I \cap I')P$ such that $\wedge^n \overline{\lambda} = \chi \otimes A[T]/(I \cap I')$. This induces an isomorphism

$\overline{\mu} : (A[T]/I')^n \simeq P/I'P$ such that $\wedge^n \overline{\mu} = \chi \otimes A[T]/I'$. Also note that $\phi \otimes A[T]/I' = \omega_{I'} \overline{\mu}^{-1}$.

Since $C[Y]/K_1 \simeq B[T]/(l_1, \cdots, l_{n-1})$, we have $\dim C[Y]/K_1 \leq 1$. Therefore, the projective $C[Y]/K_1$-module $P'[Y]/K_1P'[Y]$ is free of rank n. We choose an isomorphism $\tau(Y) : (C[Y]/K_1)^n \simeq P'[Y]/K_1P'[Y]$ such that $\wedge^n \tau(Y) = \chi \otimes C[Y]/K_1$. Since $\wedge^n \overline{\mu} = \chi \otimes B[T]/I'B[T]$, it follows that $\tau(0)$ and $\overline{\mu}$ differ by an element of $SL_n(B[T]/I'B[T])$. Since $I'B[T]+(J^2T)B[T] = B[T]$ and JB is contained in the Jacobson radical of B, we have, by (3.1), $\dim (B[T]/I'B[T]) = 0$. Hence, $SL_n(B[T]/I'B[T]) = E_n(B[T]/I'B[T])$. Since elementary transformations can be lifted via surjection of rings, we see that, we may alter $\tau(Y)$ by an element of $SL_n(C[Y]/K_1)$ and assume that $\tau(0) = \overline{\mu}$. Let $\alpha(Y) : (C[Y]/K_1)^n \twoheadrightarrow K_1/K_1^2$ denote the surjection induced by the set of generators $(l_1, \cdots, l_{n-1}, Y + l_n)$ of K_1. Thus, we obtain a surjection

$$\beta(Y) = \alpha(Y)\tau(Y)^{-1} : P'[Y]/K_1P'[Y] \twoheadrightarrow K_1/K_1^2.$$

Since $\tau(0) = \overline{\mu}$, $\phi \otimes B[T]/I'B[T] = \omega_{I'}\overline{\mu}^{-1}$ and $\alpha(0) = \omega_{I'}$, we have $\beta(0) = \phi \otimes B[T]/I'B[T]$. Therefore, applying ([M-RS], Theorem 2.3), we obtain $\eta(Y) : P'[Y] \twoheadrightarrow K_3$ such that $\eta(0) = \phi \otimes B[T]$.

Thus, the claim is proved and hence the theorem. $\square$

To derive some corollaries of the above two theorems, we need the following lemma.

Lemma 4.9 *Let A be a ring, $I \subset A[T]$ be an ideal and P be a projective $A[T]$-module. Suppose that we are given surjections $\alpha : P \twoheadrightarrow I/I^2$ and $\beta : P \twoheadrightarrow I(0) = I/I \cap (T)$ such that $\alpha \otimes_{A[T]/I} A/I(0) = \beta \otimes_A A/I(0)$. Then there is a surjection $\theta : P \twoheadrightarrow I/(I^2T)$ such that θ lifts α and β.*

Proof We choose lifts $\psi_1, \psi_2 \in \mathrm{Hom}_{A[T]}(P, I)$ of α and β respectively. We note that $(\psi_1 - \psi_2)(P) \subset I^2 + (T)$.

Since
$$\frac{I^2 + (T)}{I^2 \cap (T)} = \frac{I^2}{I^2 \cap (T)} \oplus \frac{(T)}{I^2 \cap (T)},$$
considering $(\psi_1 - \psi_2)$ as a map from P to $I^2 + (T)/I^2 \cap (T)$ we can decompose it as $(\psi_1 - \psi_2) = (\eta_1, \eta_2)$.

Write $\theta_1 = \psi_1 - \eta_1$, $\theta_2 = \psi_2 - \eta_2$. Note that θ_1 and θ_2 lift α and β respectively. It is also easy to see that $\theta_1 = \theta_2$. We call it θ.

So we have a map $\theta : P \longrightarrow I/I^2 \cap (T)$ which lifts both α and β. Write $\theta(P) = K$ and consider the ideal $K + (I^2T)$. Since $K + I \cap (T) = I$ and $K + I^2 = I$, it follows that a maximal ideal M of $A[T]$ contains I if and only if it contains $K + (I^2T)$. Note that since $K + I^2 = I$, for every maximal ideal M of $A[T]$ containing I we have $K_M = I_M$. Therefore, it follows that $K + (I^2T) = I$. In other words, θ is a surjection. Since θ lifts both α and β, this proves the lemma. $\square$

Corollary 4.10 *Let A be of dimension $n \geq 3$. Let P be a projective $A[T]$-module of rank n having trivial determinant and χ be a trivialization of $\wedge^n P$. Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Let ω_I be a local orientation of I. Suppose that $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$. Then, there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I, ω_I) is obtained from (α, χ).*

Proof Since determinant of P is trivial, P/IP is a free $A[T]/I$-module of rank n. We can choose an isomorphism $\lambda : P/IP \simeq (A[T]/I)^n$ such that $\wedge^n \lambda = (\chi \otimes A[T]/I)^{-1}$. Therefore, we get a surjection $\omega_I \lambda : P/IP \twoheadrightarrow I/I^2$.

We can assume that either $I(0) = A$ or $\mathrm{ht}\,(I(0)) = n$. First suppose that $I(0) = A$. Then, it follows using (4.9), that there is a surjection from P to $I/(I^2T)$ which lifts $\omega_I \lambda : P/IP \twoheadrightarrow I/I^2$.

If $\mathrm{ht}\,(I(0)) = n$, then since $e(P/TP, \chi \otimes A[T]/(T)) = (I(0), \omega_{I(0)})$ in $E(A)$, it follows from (4.3, [B-RS 3]), that there is a surjection $\alpha : P/TP \twoheadrightarrow I(0)$ such that $(I(0), \omega_{I(0)})$ is obtained from $(\alpha, \chi \otimes A[T]/(T))$. Therefore, it follows from (4.9) that we have a surjection $P \twoheadrightarrow I/(I^2T)$ which is a lift of $\omega_I \lambda$.

So, in any case, we have a surjection $\gamma : P \twoheadrightarrow I/(I^2T)$ which lifts $\omega_I \lambda$.

Since $e(P \otimes A(T), \chi \otimes A(T)) = (IA(T), \omega_I \otimes A(T))$ in $E(A(T))$, it follows from (4.3, [B-RS 3]) that there is a surjection $\Gamma : P \otimes A(T) \twoheadrightarrow IA(T)$ such that $(IA(T), \omega_I \otimes A(T))$ is obtained from $(\Gamma, \chi \otimes A(T))$. This actually means that Γ is a lift of $\gamma \otimes A(T)$.

Therefore, by (4.8), the result follows. $\square$

Corollary 4.11 *Let A be as above. Let P be a projective $A[T]$-module of rank n having trivial determinant and χ be a trivialization of $\wedge^n P$. Then, $e(P,\chi) = 0$ if and only if P has a unimodular element. In particular, if P has a unimodular element then P maps onto any ideal of $A[T]$ of height n generated by n elements.*

Proof Let $\alpha : P \twoheadrightarrow I$ be a surjection where I is an ideal in $A[T]$ of height n. Let $e(P,\chi) = (I,\omega_I)$ in $E(A[T])$, where (I,ω_I) is obtained from the pair (α,χ).

First assume that $e(P,\chi) = 0$. Then, $e(P \otimes A(T), \chi \otimes A(T)) = 0$ in $E(A(T))$. Therefore, by (4.4, [B-RS 3]), $PA(T)$ has a unimodular element. Consequently, by (Theorem 3.4, [B-RS 4]), it follows that P has a unimodular element.

Now we assume that P has a unimodular element. But then, following (4.1), it is easy to see that $(I,\omega_I) = 0$ in $E(A[T])$.

The last assertion of the corollary follows from (4.10). $\square$

Corollary 4.12 *Let $\dim A = n \geq 2$ and $I \subset A[T]$ be an ideal of height n. Let P be a projective $A[T]$-module of rank n with trivial determinant and $\alpha : P \twoheadrightarrow I$ be a surjection. Suppose that P has a unimodular element. Then I is generated by n elements.*

Proof If $n = 2$, then by (2.4) P is a free module and hence I is generated by n elements. Therefore, in what follows, we assume $n \geq 3$.

Let us fix an isomorphism $\chi : A[T] \simeq \wedge^n P$. Suppose that $(I,\omega_I) \in E(A[T])$ is obtained from the pair (α,χ). Then we have $e(P,\chi) = (I,\omega_I)$ in $E(A[T])$. Since P has a unimodular element, it follows from (4.11) that $e(P,\chi) = 0$. Now the corollary follows from (4.7). $\square$

Now we prove the "Subtraction Principle" in a more general form in the following corollary.

Corollary 4.13 *Let A be a Noetherian ring with* $\dim A = n \geq 3$. *Let P and Q be projective $A[T]$-modules of rank n and $n-1$ respectively, such that their determinants are free. Let $\chi : \wedge^n(P) \simeq \wedge^n(Q \oplus A[T])$ be an isomorphism. Let*

I_1, $I_2 \subset A[T]$ be comaximal ideals, each of height n. Let $\alpha : P \twoheadrightarrow I_1 \cap I_2$ and $\beta : Q \oplus A[T] \twoheadrightarrow I_2$ be surjections. Let bar denote reduction modulo I_2 and $\overline{\alpha} : \overline{P} \twoheadrightarrow I_2/I_2^2$, $\overline{\beta} : \overline{Q \oplus A[T]} \twoheadrightarrow I_2/I_2^2$ be surjections induced from α and β respectively. Suppose that there exists an isomorphism $\delta : \overline{P} \simeq \overline{Q \oplus A[T]}$ such that (i) $\overline{\beta}\delta = \overline{\alpha}$, (ii) $\wedge^n(\delta) = \overline{\chi}$. Then, there exists a surjection $\theta : P \twoheadrightarrow I_1$ such that $\theta \otimes A[T]/I_1 = \alpha \otimes A[T]/I_1$.

Proof Let us fix an isomorphism $\sigma : A[T] \simeq \wedge^n(P)$. Let $(I_1 \cap I_2, \omega_{I_1 \cap I_2})$ be obtained from (α, σ). Then $e(P, \sigma) = (I_1 \cap I_2, \omega_{I_1 \cap I_2}) = (I_1, \omega_{I_1}) + (I_2, \omega_{I_2})$ in $E(A[T])$.

On the other hand, let $(I_2, \widetilde{\omega_{I_2}})$ be obtained from $(\beta, \chi\sigma)$. It is easy to see, from the conditions stated in the proposition, that $(I_2, \omega_{I_2}) = (I_2, \widetilde{\omega_{I_2}})$ in $E(A[T])$. But $Q \oplus A[T]$ has a unimodular element. Therefore, it follows that $(I_2, \omega_{I_2}) = 0$. Consequently, $e(P, \sigma) = (I_1, \omega_{I_1})$. Therefore, the result follows from (4.10). $\square$

5 A "Quillen-Suslin theory" for the Euler class groups

In this section we investigate some questions concerning the relations among the Euler class groups $E(A)$, $E(A[T])$ and $E(A(T))$. The motivation for these questions comes from the Quillen-Suslin theory for projective modules.

Remark 5.1 Let A be a Noetherian ring with $\dim A = n \geq 3$. Let $I \subset A[T]$ be an ideal of height n and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ be a local orientation of I. Let $\overline{f} \in A[T]/I$ be a unit. Composing ω_I with an automorphism of $(A[T]/I)^n$ with determinant $\overline{f}$, we obtain another local orientation of I which we denote by $\overline{f}\omega_I$. On the other hand, let ω_I, $\widetilde{\omega_I}$ be two local orientations of I. Then, it is easy to see from (2.5), that $\widetilde{\omega_I} = \overline{f}\omega_I$ for some unit $\overline{f} \in A[T]/I$.

Following is an improvement of (3.6).

Lemma 5.2 *Let A be a Noetherian ring with* $\dim A = n \geq 3$*, $I \subset A[T]$ an ideal of height n and $\omega_I : (A[T]/I)^n \twoheadrightarrow I/I^2$ a surjection. Suppose that ω_I can* *be lifted to a surjection $\alpha : A[T]^n \twoheadrightarrow I$. Let $f \in A[T]$ be a unit modulo I. Let $\theta \in GL_n(A[T]/I)$ with determinant $\overline{f^2}$. Then, the surjection $\omega_I\theta : (A[T]/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\beta : A[T]^n \twoheadrightarrow I$.*

Proof Without loss of generality we can assume that either $I(0) = A$ or $I(0)$ has height n.

Suppose that $\text{ht}\,(I(0)) = n$. Now ω_I induces a surjection, say, $\omega_{I(0)} : (A/I(0))^n \twoheadrightarrow I(0)/I(0)^2$ which can be lifted to $\alpha(0) : A^n \twoheadrightarrow I(0)$. Note that $f(0) \in A$ is a unit modulo $I(0)$ and $\theta(0)$ ($\in GL_n(A/I(0))$) has determinant $\overline{f(0)^2}$. Therefore, by ([B-RS 3], 5.3), $\omega_{I(0)}\theta(0)$ can be lifted to a surjection $\overline{\beta} : A^n \twoheadrightarrow I(0)$. Consequently, we can lift $\omega_I\theta$ to a surjection $\psi : A[T]^n \twoheadrightarrow I/(I^2T)$.

On the other hand, if $I(0) = A$, we can always lift $\omega_I\theta$ to a surjection $\psi : A[T]^n \twoheadrightarrow I/(I^2T)$.

Similarly we can go to $A(T)$ and apply (5.3, [B-RS 3]) there to find that $\omega_I\theta \otimes A(T)/IA(T) : (A(T)/IA(T))^n \twoheadrightarrow IA(T)/I^2A(T)$ can be lifted to a surjection $\phi : A(T)^n \twoheadrightarrow IA(T)$.

Now the lemma is a consequence of our main theorem (3.10). $\square$

Applying (5.2), we obtain the following lemma. (The method of proof is same as in ([B-RS 3], 5.4).)

Lemma 5.3 *Let A be a Noetherian ring with $\dim A = n \geq 3$, $I \subset A[T]$ an ideal of height n and ω_I be a local orientation of I. Let $\overline{f}$ be a unit modulo I. Then $(I, \omega_I) = (I, \overline{f^2}\omega_I)$ in $E(A[T])$.*

Proof If $(I, \omega_I) = 0$ in $E(A[T])$, then the result follows from (5.2). Therefore, let us assume that $(I, \omega_I) \neq 0$ in $E(A[T])$. Then, by (2.12), there exists an ideal $I_1 \subset A[T]$ of height n which is comaximal with I and a surjection $\alpha : A[T]^n \twoheadrightarrow I \cap I_1$ such that $\alpha \otimes A[T]/I = \omega_I$. Let $\alpha \otimes A[T]/I_1 = \omega_{I_1}$. By the Chinese remainder theorem, we can choose $g \in A[T]$ such that $g = f^2$ modulo I and $g = 1$ modulo I_1. Applying (5.2), we see that there exists a surjection $\gamma : A[T]^n \twoheadrightarrow I \cap I_1$ such that $\gamma \otimes A[T]/I = \overline{f^2}\omega_I$ and $\gamma \otimes A[T]/I_1 = \omega_{I_1}$. From the surjection α we get $(I, \omega_I) + (I_1, \omega_{I_1}) = 0$ in

$E(A[T])$. From the surjection γ we get $(I, \overline{f^2\omega_I}) + (I_1, \omega_{I_1}) = 0$ in $E(A[T])$. Therefore, $(I, \omega_I) = (I, \overline{f^2\omega_I})$ in $E(A[T])$. This completes the proof. $\square$

A consequence of Quillen's 'local-global principle' ([Q]) is that the following sequence of groups is exact.

$$0 \longrightarrow \widetilde{K_0}(A) \longrightarrow \widetilde{K_0}(A[T]) \longrightarrow \Pi_m \widetilde{K_0}(A_m[T]),$$

where the direct product runs over all maximal ideals m of A. The following theorem shows that a 'local-global principle' holds for Euler class groups also.

Theorem 5.4 *Let A be a Noetherian ring containing $\mathbb{Q}$ with* $\dim A = n \geq 3$. *Then the following sequence of groups is exact.*

$$0 \longrightarrow E(A) \longrightarrow E(A[T]) \longrightarrow \Pi_m E(A_m[T]),$$

where the direct product runs over all maximal ideals m of A such that $\operatorname{ht}(m) = n$.

To prove this theorem we need the following lemma.

Lemma 5.5 *Let R be a semilocal ring (containing $\mathbb{Q}$) of dimension $n \geq 3$, $I_1 \subset R[T]$ be an ideal of height n such that $I_1 + \mathcal{J}R[T] = R[T]$ where $\mathcal{J}$ is the Jacobson radical of R. Suppose that ω_{I_1} is a local orientation of I_1 given by $I_1 = (f_1, \cdots, f_n) + I_1^2$ and $\omega_{I_1} \otimes R_m[T]$ can be lifted to a set of generators of $I_1 \otimes R_m[T]$ for all maximal ideals m of R of height n (hence for all maximal ideals of R). Then there is a set of generators of I_1 which lifts ω_{I_1}.*

Proof We proceed by induction on the number of maximal ideals in the base ring. Clearly, when the base ring is local, we have nothing to prove.

Suppose that *max*$(R) = \{m_1, \cdots, m_k\}$. Let $I_1 = I_1' \cap \cdots \cap I_r'$ be a primary decomposition of I_1. Let $P_i = \sqrt{I_i'} \cap R$. Since $I_1 + \mathcal{J}R[T] = R[T]$, it follows that $\sqrt{I_i'}$ is a maximal ideal of $R[T]$ of height n whereas P_i is a prime ideal of R of height $n-1$.

Suppose that, from the family of prime ideals $\{P_1, \cdots, P_r\}$ (after renumbering), $\{P_1, \cdots P_s\}$ is the collection such that P_i is not contained in m_1. Note that this collection can be empty.

We write $T_1 = R - (P_1 \cup \cdots \cup P_s)$ and $T = R - (P_1 \cup \cdots \cup P_s \cup m_1)$; $S_1 = T_1^{-1}R$ and $S = T^{-1}R$. Note that $T \subset T_1$ and hence S_1 is a localization of S.

Since S_1 is a semilocal ring such that all the maximal ideals of S_1 are of height $n-1$, it follows by a theorem of Mandal ([M 1]), that $(I_1, \omega_{I_1}) = 0$ in $E(S_1[T])$. Therefore, there exists $t_1 \in T_1$ such that $(I_1, \omega_{I_1}) = 0$ in $E(S_{t_1}[T])$. Since $t_1 \notin P_1 \cup \cdots \cup P_s$, we have $(t_1) + I_1S[T] = S[T]$. Now adapting the proof of (5.6, [B-RS 3]) we can find an ideal $I_2 \subset S[T]$ of height n such that (1) $t_1^p \in I_2$ for some positive integer p, (2) $I_1S[T] \cap I_2 = (g_1, \cdots, g_n)$, and (3) $(I_1S[T], \omega'_{I_1}) + (I_2, \omega_{I_2}) = 0$ in $E(S[T])$ where the local orientations are induced by $g_1, \cdots, g_n$. Note that, by (5.3), $(I_1S[T], \omega_{I_1} \otimes S[T]) = (I_1S[T], \omega'_{I_1})$ and therefore, $(I_1S[T], \omega_{I_1} \otimes S[T]) + (I_2, \omega_{I_2}) = 0$ in $E(S[T])$.

Let $J_2 = I_2 \cap S$. Note that since $t_1^p \in J_2$, J_2 is not contained in $P_1 \cup \cdots \cup P_s$ and hence S_{1+J_2} is a local ring, with maximal ideal $m_1S_{1+J_2}$. Therefore, it follows that $(I_2, \omega_{I_2}) = 0$ in $E(S_{1+J_2}[T])$. We claim that this implies $(I_2, \omega_{I_2}) = 0$ in $E(S[T])$.

Proof of the claim : We note that since S is semilocal, we can apply (3.2) and ([B-RS 1], 3.9) to adjust ω_{I_2} so that it is induced by a set of generators of $I_2/(I_2^2T)$, say, given by, $I_2 = (G_1, \cdots, G_n) + (I_2^2T)$ (therefore, $G_i = g_i$ mod I_2^2). We have obtained, in the above paragraph, $I_2S_{1+J_2}[T] = (F_1, \cdots, F_n)$ such that $F_i = g_i$ mod $I_2^2S_{1+J_2}[T]$. Therefore, it follows that, $I_2S_{1+J_2}(T) = (F_1, \cdots, F_n)$ such that $F_i = G_i$ mod $I_2^2S_{1+J_2}(T)$. Applying the main theorem (3.10), we get, $I_2S_{1+J_2}[T] = (H_1, \cdots, H_n)$ such that $H_i = G_i$ mod $(I_2^2T)S_{1+J_2}[T]$. Now we can apply (3.8) and conclude that $(I_2, \omega_{I_2}) = 0$ in $E(S[T])$. This proves the claim.

So it follows that $(I_1S[T], \omega_{I_1} \otimes S[T]) = 0$ in $E(S[T])$.

We can repeat the same arguments as above and find $t \in T$ and an ideal $I_3 \subset R[T]$ of height n such that (1) $t^q \in I_3$ for some positive integer q, (2) $I_1 \cap I_3 = (h_1, \cdots, h_n)$, and (3) $(I_1, \omega_{I_1}) + (I_3, \omega_{I_3}) = 0$ in $E(R[T])$ where the local orientation ω_{I_3} is induced by $h_1, \cdots, h_n$.

Let $J_3 = I_3 \cap R$. Since $t^q \in J_3$ and $t \notin P_1 \cup \cdots \cup P_s \cup m_1$, we see that $max(R_{1+J_3}) \subset \{m_2, \cdots, m_k\}$. Therefore, by the induction hypothesis it follows that ω_{I_3} can be lifted to a set of generators of $I_3R_{1+J_3}[T]$. Therefore, as above, this implies that $(I_3, \omega_{I_3}) = 0$ in $E(R[T])$. Consequently, $(I_1, \omega_{I_1}) = 0$ in $E(R[T])$. This proves the lemma.

□

The following is an alternative proof of the above lemma in the case when R is a domain. We do not assume in this proof that I_1 is comaximal with $\mathcal{J}R[T]$.

Proof Suppose that *max* $(R) = \{m_1, \cdots, m_k\}$. Since $\omega_{I_1} \otimes R_{m_1}[T]$ can be lifted to a set of generators of $I_1 R_{m_1}[T]$, we can find $a \in R - m_1$ such that $I_{1_a} = (g_1, \cdots, g_n)$ with $g_i = f_i$ modulo $I_{1_a}^2$. Let $J_1 = I_1 \cap R$. Let $b \in J_1^2$ and $c = ab$. Let bar denote reduction modulo c. Note that dim $\overline{R} \leq n-1$. Adapting the method of Bhatwadekar-Rao (see [B-RS 5], 2.5), we obtain, $\overline{I_1} = (\overline{h_1}, \cdots, \overline{h_n})$ such that $\overline{h_i} = \overline{f_i}$ mod $\overline{I_1}^2$. By adding suitable multiples of c to $h_1, \cdots, h_n$, we may assume by the Eisenbud-Evans theorem (2.11), that $(h_1, \cdots, h_n) = I_1 \cap I_2$, where ht $(I_2) = n$ and $I_2 + (c) = R[T]$. Note that $(h_1, \cdots, h_n)$ induces ω_{I_1} and a local orientation ω_{I_2} of I_2 such that $(I_1, \omega_{I_1}) + (I_2, \omega_{I_2}) = 0$ in $E(R[T])$.

Applying the subtraction principle (4.3), we see that $I_{2_a} = (k_1, \cdots, k_n)$ with $k_i = h_i$ mod ${I_{2_a}}^2$. We note that a is a unit modulo I_2. Therefore, adapting the proof of (5.6, [B-RS 3]) we can find an ideal $I_3 \subset R[T]$ of height n such that (1) I_3 contains a power of a, (2) $I_2 \cap I_3 = (l_1, \cdots, l_n)$, and (3) $(I_2, \omega'_{I_2}) + (I_3, \omega_{I_3}) = 0$ in $E(R[T])$.

Since by (5.3), we have $(I_2, \omega_{I_2}) = (I_2, \omega'_{I_2})$ in $E(R[T])$, it follows that $(I_2, \omega_{I_2}) + (I_3, \omega_{I_3}) = 0$ in $E(R[T])$.

Let $J_3 = I_3 \cap R$. Since $a \in R - m_1$ whereas J_3 contains a power of a, it follows that m_1 does not belong to *max* (R_{1+J_3}). Therefore, by the induction hypothesis, $(I_3, \omega_{I_3}) \otimes R_{1+J_3}[T] = 0$ in $E(R_{1+J_3}[T])$. As in the first proof (see the claim and its proof), this implies that $(I_3, \omega_{I_3}) = 0$ in $E(R[T])$. Therefore, it follows that $(I_1, \omega_{I_1}) = 0$ in $E(R[T])$. This proves the lemma.

□

Proof of Theorem 5.4 : Let $I \subset A[T]$ be an ideal of height n and ω_I be a local orientation of I such that $\omega_I \otimes A_m[T]$ is a global orientation for all maximal ideals m of A of height n. We show that there exists an ideal $J \subset A$ of height n and a local orientation $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ such that $(J[T], \omega_J \otimes A[T]) = (I, \omega_I)$ in $E(A[T])$.

Since A contains $\mathbb{Q}$, we can assume that either $I(0)$ is an ideal of height n or $I(0) = A$.

Case 1. When $I(0)$ is proper.

Applying (2.12) we can find an ideal $K \subset A$ of height n which is comaximal with $I \cap A$ and a local orientation ω_K of K such that $(I(0), \omega_{I(0)}) + (K, \omega_K) = 0$ in $E(A)$.

Let $L = I \cap K[T]$. Since the ideals I and $K[T]$ are comaximal, ω_I and ω_K induce $\omega_L : (A[T]/L)^n \twoheadrightarrow L/L^2$ and we have the following equation in $E(A[T])$:

$$(L, \omega_L) = (I, \omega_I) + (K[T], \omega_K \otimes A[T]).$$

Since $(L(0), \omega_{L(0)}) = (I(0), \omega_{I(0)}) + (K, \omega_K) = 0$, it follows that we can lift ω_L to a set of generators of $L/(L^2T)$. We proceed to prove that $(L, \omega_L) = 0$. Note that since $K[T]$ is extended and $L = I \cap K[T]$, it follows that $\omega_L \otimes A_m[T]$ is a global orientation of $L \otimes A_m[T]$ for all maximal ideals m of A of height n.

Since ω_L is actually induced by a set of generators of $L/(L^2T)$, adapting the proof of (3.10) and applying the above lemma , it is easy to see that this set of generators of $L/(L^2T)$ can be lifted to a set of generators of L.

Thus, $(L, \omega_L) = 0$ in $E(A[T])$ and this, in turn, implies that $(I, \omega_I) = (I(0)[T], \omega_{I(0)} \otimes A[T])$ in $E(A[T])$.

Case 2. In this case, $I(0) = A$.

Then we can lift ω_I to a set of generators of $I/(I^2T)$. Proceeding as we did for L in Case 1 it follows that $(I, \omega_I) = 0$ in $E(A[T])$.

Therefore, the proof of the theorem is complete. $\square$

As a consequence, we get the following interesting corollary.

Corollary 5.6 *Let A be a Noetherian ring containing $\mathbb{Q}$ with* $\dim A = n \geq 3$. *P be a projective $A[T]$-module of rank n with trivial determinant. Suppose that P/TP has a unimodular element. Assume further that the projective $A_m[T]$-module $P \otimes A_m[T]$ has a unimodular element for every maximal ideal m of A of* *height n. Then, P has a unimodular element. (Taking $P = Q[T]$, we see that the condition that P/TP has a unimodular element, is necessary.)*

Let A be a Noetherian ring containing $\mathbb{Q}$ with $\dim A = n \geq 3$. Note that we have a canonical map $\Phi : E(A) \longrightarrow E(A[T])$. It is easy to see that Φ is injective.

However, there is an example due to Bhatwadekar, Mohan Kumar and Srinivas ([B-RS 1], Example 6.4), where they have constructed a normal affine domain A over $\mathbb{C}$ of dimension 3, an ideal $I \subset A[T]$ of height 3 such that $I(0) = A$ and a surjection $\widetilde{\phi} : A[T]^3 \twoheadrightarrow I/(I^2T)$, and it has been shown that $\widetilde{\phi}$ cannot be lifted to a surjection from $A[T]^3$ to I (in fact, I is not a surjective image of any projective $A[T]$-module of rank 3 which is *extended from* A). It also follows from their example that for any local orientation ω_I of I, the element (I, ω_I) in $E(A[T])$ does not come from $E(A)$. Therefore, the canonical map $\Phi : E(A) \longrightarrow E(A[T])$ is not surjective in general.

We note that in their example the affine domain in question is normal, but not regular. Therefore, one can ask the following natural question.

Question 1. Let A be a regular ring containing $\mathbb{Q}$ with $\dim A = n \geq 3$. Is the canonical map $\Phi : E(A) \longrightarrow E(A[T])$ an isomorphism?

The following proposition gives a partial answer to the above question.

Proposition 5.7 *Let A be a smooth affine domain containing the field of rationals with* $\dim A = n \geq 3$. *Then the canonical map $\Phi : E(A) \longrightarrow E(A[T])$ is an isomorphism.*

Proof Let $(I, \omega_I) \in E(A[T])$ where I is an ideal of $A[T]$ of height n and ω_I is a local orientation of I. By (2.9), without loss of generality we may assume that either $I(0) = A$ or $\operatorname{ht}(I(0)) = n$.

If $I(0) = A$, then we can lift ω_I to a set of generators of $I/(I^2T)$. Then, by ([B-RS 1], Theorem 3.8), it follows that ω_I is a global orientation.

Now suppose that $\operatorname{ht} I(0) = n$. We consider the element $(I(0), \omega_{I(0)}) \in E(A)$, induced by (I, ω_I). Applying (2.12), we can find an ideal $K \subset A$ of height n which is comaximal with $I \cap A$ and a local orientation ω_K of K such that $(I(0), \omega_{I(0)}) + (K, \omega_K) = 0$ in $E(A)$.

Let $L = I \cap K[T]$. Since the ideals I and $K[T]$ are comaximal, ω_I and ω_K induce $\omega_L : (A[T]/L)^n \twoheadrightarrow L/L^2$ and we have the following equation in $E(A[T])$:

$$(L, \omega_L) = (I, \omega_I) + (K[T], \omega_K \otimes A[T]).$$

Since $(L(0), \omega_{L(0)}) = (I(0), \omega_{I(0)}) + (K, \omega_K) = 0$ in $E(A)$, it follows that we can lift ω_L to a set of generators of $L/(L^2T)$. Now, by ([B-RS 1], 3.8), ω_L is a global orientation. Thus, $(L, \omega_L) = 0$ and hence it follows that $(I, \omega_I) = (I(0)[T], \omega_{I(0)} \otimes A[T])$ in $E(A[T])$.

Therefore, Φ is a surjection. $\square$

Let A be a Noetherian ring containing $\mathbb{Q}$ with $\dim A = n \geq 3$. Since the ring extension $A[T] \to A(T)$ is flat, we see that there is a canonical map $\Gamma : E(A[T]) \longrightarrow E(A(T))$. We end this section discussing the following interesting question, which is, in fact, an analogue of the Affine Horrocks theorem.

Question 2. Is the canonical map $\Gamma : E(A[T]) \longrightarrow E(A(T))$ injective?

The following proposition gives a partial answer to the above question.

Proposition 5.8 *Let A be a Noetherian ring of dimension $n \geq 3$ containing $\mathbb{Q}$. Then the canonical map $\Gamma : E(A[T]) \longrightarrow E(A(T))$ is injective in the following cases:*
(1) ht $\mathcal{J} \geq 1$, *where $\mathcal{J}$ denotes the Jacobson radical of A.*
(2) A is an affine domain over an algebraically closed field k of characteristic zero.

Proof (1) Let $(I, \omega_I) \in E(A[T])$ be such that, $(IA(T), \omega_I \otimes A(T)) = 0$ in $E(A(T))$. Now since ht $\mathcal{J} \geq 1$, it follows that ht $(\mathcal{J}, T) \geq 2$. Therefore, using (2.12), we can find an ideal $K \subset A[T]$ of height n and a local orientation ω_K such that, K is comaximal with $I \cap (\mathcal{J}, T)$ and $(I, \omega_I) + (K, \omega_K) = 0$ in $E(A[T])$. Since $K + (\mathcal{J}, T) = A[T]$, it is easy to see that $K(0) = A$. Therefore, we can lift ω_K to a set of generators of $K/(K^2T)$. Now since $(KA(T), \omega_K \otimes A(T)) = 0$ in $E(A(T))$, it follows from (3.10), that $(K, \omega_K) = 0$ in $E(A[T])$. Therefore, $(I, \omega_I) = 0$ in $E(A[T])$. This proves (1).

(2) From (1) it follows that Γ is injective when A is local. Therefore, in view of (5.4), it is easy to see that to prove the injectivity of Γ (for any Noetherian ring A containing $\mathbb{Q}$ with $\dim A \geq 3$), it is enough to prove the injectivity of the canonical map from $E(A)$ to $E(A(T))$. We note that this is exactly how the Quillen-Suslin theorem is proved.

Let A be an affine domain over an algebraically closed field k of characteristic 0. Let $(I, \omega_I) \in E(A)$ be such that $(I \otimes A(T), \omega_I \otimes A(T)) = 0$ in $E(A(T))$.

Suppose that $I = (a_1, \cdots, a_n) + I^2$ and this set of generators of I/I^2 corresponds to ω_I. By (2.8), we see that there exists $a \in I$ such that $I_{1-a} = (a_1, \cdots, a_n)$. Write $b = 1 - a$. Note that if b is a unit in A, we are done. Therefore assume that b is not a unit in A. Then, since k is algebraically closed, b is transcendental over k. We consider the multiplicatively closed set $S = \{1 + cb \,|\, c \in k[b]\}$. Note that $a \in S$.

We consider the surjection $\alpha : A_b^n \twoheadrightarrow I_b$ which sends e_i to a_i and the surjection $\beta : A_S^n \twoheadrightarrow I_S$ which sends e_1 to 1 and e_i to 0 for $i \geq 2$. We note that A_{bS} is an affine domain over the C_1-field $k(b)$ of dimension $n - 1$. Therefore, by a result of Suslin ([Su2]), the unimodular row $(a_1, \cdots, a_n)$ over A_{bS} is completable to a matrix $\sigma \in SL_n(A_{bS})$ and hence by patching we obtain a surjection $\gamma : P \twoheadrightarrow I$ where P is a projective A-module of rank n with trivial determinant.

We fix an isomorphism $\chi : A \simeq \wedge^n(P)$. Then (γ, χ) induces an element $(I, \widetilde{\omega}_I)$ in $E(A)$. It follows from (5.1), that $\omega_I = \overline{c}\widetilde{\omega}_I$ for some unit $\overline{c} \in A/I$. Since k is an algebraically closed field of characteristic zero and $\dim (A/I) = 0$, we can find a unit $\overline{d} \in A/I$ such that $\overline{d^{n-1}} = \overline{c}$. Now by ([B-RS 3], 5.1), there exists a projective A-module P_1 of rank n with trivial determinant, an isomorphism $\chi_1 : A \simeq \wedge^n(P_1)$ and a surjection $\delta : P_1 \twoheadrightarrow I$ such that $e(P_1, \chi_1) = (I, \overline{d^{n-1}}\widetilde{\omega}_I)$ in $E(A)$. Thus, $e(P_1, \chi_1) = (I, \omega_I)$ in $E(A)$. Now since $(IA(T), \omega_I \otimes A(T)) = 0$ in $E(A(T))$, it follows from ([B-RS 3], 4.4), that $P_1 \otimes A(T)$ has a unimodular element. Therefore, by ([B-RS 4], 3.4), P_1 has a unimodular element. Hence $(I, \omega_I) = 0$ in $E(A)$. This completes the proof. $\square$

6 The weak Euler class group of $A[T]$

Let A be a Noetherian ring of dimension $n \geq 3$ containing the field of rationals. We define the n^{th} *weak Euler class group* $E_0^n(A[T])$ of $A[T]$, in the following way:

Let S be the set of ideals $\mathcal{I} \subset A[T]$ with the properties: i) $\operatorname{ht}(\mathcal{I}) = n$, ii) $\mathcal{I}/\mathcal{I}^2$ is generated by n elements and iii) $\operatorname{Spec}(A[T]/\mathcal{I})$ is connected. Let G be the free abelian group on S.

Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Now I can be decomposed as $I = \cap_{i=1}^{k} \mathcal{I}_i$ where $\mathcal{I}_i$'s are pairwise comaximal and $\operatorname{Spec}(A[T]/\mathcal{I}_i)$ is connected for each i. We associate to I, the element $\Sigma \mathcal{I}_i$ of G. By abuse of notation we denote this element by (I).

Let H be the subgroup of G generated by elements of the type (I), where $I \subset A[T]$ is an ideal of height n such that I is generated by n elements.

We define $E_0^n(A[T]) = G/H$.

By a slight abuse of notation, we will write $E_0(A[T])$ for $E_0^n(A[T])$ in what follows.

We note that there is a canonical surjective group homomorphism from $E(A[T])$ to $E_0(A[T])$ obtained by forgetting the orientations.

We first prove some general results on $E(A[T])$ in the form of the following lemmas. We will need them to prove results on $E_0(A[T])$.

The proof of the following lemma is contained in ([B-RS 3], 2.7, 2.8 and 5.1) and hence we omit the proof.

Lemma 6.1 *Let A be a Noetherian ring with* $\dim A = n \geq 3$. *Let P be a projective $A[T]$-module of rank n having trivial determinant and $\chi : A[T] \simeq \wedge^n P$ be a trivialization. Let $\alpha : P \twoheadrightarrow I$ be a surjection, where $I \subset A[T]$ is an ideal of height n and let (I, ω_I) be obtained from (α, χ). Let $f \in A[T]$ be a unit modulo I. Then, there exists a projective $A[T]$-module P_1 of rank n having trivial determinant, a trivialization χ_1 of $\wedge^n P_1$, and a surjection $\beta : P_1 \twoheadrightarrow I$ such that :*
i) P is stably isomorphic to P_1, ii) $(I, \overline{f^{n-1}}\omega_I)$ is obtained from (β, χ_1).

The following lemma is essentially a translation of (2.12) in the language of Euler class groups.

Lemma 6.2 *Let A be a Noetherian ring of dimension $n \geq 3$, $I \subset A[T]$ an ideal of height n and ω_I a local orientation of I. Suppose that $(I, \omega_I) \neq 0$ in $E(A[T])$. Then, there exists an ideal I_1 of $A[T]$ of height n and a local orientation ω_{I_1} of I_1 such that $(I, \omega_I) + (I_1, \omega_{I_1}) = 0$ in $E(A[T])$. Further, given any finite set of ideals $K_1, \cdots, K_t$ of $A[T]$ with $\operatorname{ht}(K_i) \geq 2$, I_1 can be chosen with the additional property that it is comaximal with each K_i.*

Adapting the proof of (3.7, [B-RS 2]) and using the Eisenbud-Evans theorem (2.11) in place of "Swan's Bertini" theorem, the following lemma can be easily deduced.

Lemma 6.3 *Let A be a Noetherian ring of even dimension $n \geq 4$. Let P be a stably free $A[T]$-module of rank n and $\chi : A[T] \simeq \wedge^n(P)$ be an isomorphism. Suppose that $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$, where $I \subset A[T]$ is an ideal of height n and ω_I is a local orientation of I. Then there is an ideal $I_1 \subset A[T]$ generated by n elements and a local orientation ω_{I_1} of I_1 such that $(I, \omega_I) = (I_1, \omega_{I_1})$ in $E(A[T])$. Moreover, I_1 can be chosen to be comaximal with any given ideal of $A[T]$ of height n.*

The following three propositions can be proved by using (5.3, 6.1, 6.2, 6.3) of this paper and adapting the proofs of ([B-RS 2], 3.8, 3.9, 3.10).

Proposition 6.4 *Let A be a Noetherian ring of even dimension $n \geq 4$. Let I_1, I_2 be two comaximal ideals of $A[T]$, each of height n. Let $I_3 = I_1 \cap I_2$. If any two of I_1, I_2 and I_3 are surjective images of stably free projective $A[T]$-modules of rank n, then so is the third.*

Proposition 6.5 *Let A be a Noetherian ring of even dimension $n \geq 4$. Let $I \subset A[T]$ be an ideal of height n such that I/I^2 is generated by n elements. Then $(I) = 0$ in $E_0(A[T])$ if and only if I is the surjective image of a stably free projective $A[T]$-module of rank n.*

Proposition 6.6 *Let A be a Noetherian ring of even dimension $n \geq 4$. Let P be a projective $A[T]$-module of rank n with trivial determinant. Suppose that P maps* *onto an ideal $I \subset A[T]$ of height n. Then $(I) = 0$ in $E_0(A[T])$ if and only if $[P] = [Q \oplus A[T]]$ in $K_0(A[T])$ for some projective $A[T]$-module Q of rank $n-1$.*

Proposition 6.7 *Let A be a Noetherian ring of even dimension $n \geq 4$, $I \subset A[T]$ an ideal of height n such that I/I^2 is generated by n elements. Let $\widetilde{\omega_I} : (A[T]/I)^n \twoheadrightarrow I/I^2$ be a surjection. Suppose that the element $(I, \widetilde{\omega_I})$ of $E(A[T])$ belongs to the kernel of the canonical homomorphism $E(A[T]) \twoheadrightarrow E_0(A[T])$. Then, there exists a stably free $A[T]$-module P_1 of rank n and a trivialization χ_1 of $\wedge^n P$ such that $e(P_1, \chi_1) = (I, \widetilde{\omega_I})$ in $E(A[T])$.*

Proof (We follow the same proof as in (6.5, [B-RS 3]).) Since $(I) = 0$ in $E_0(A[T])$, by (6.5), there exists a stably free $A[T]$-module P of rank n and a surjection $\alpha : P \twoheadrightarrow I$. Let $\chi : A[T] \simeq \wedge^n(P)$ be an isomorphism. Suppose that (I, ω_I) is obtained from (α, χ). By (5.1), there exists $f \in A[T]$ such that $\overline{f} \in A[T]/I$ is a unit and $\widetilde{\omega_I} = \overline{f}\omega_I$. By (6.1), there exists a projective $A[T]$-module P_1 such that P_1 is stably isomorphic to P and an isomorphism $\chi_1 : A[T] \simeq \wedge^n(P_1)$, such that $e(P_1, \chi_1) = (I, \overline{f^{n-1}}\omega_I)$ in $E(A[T])$. Since n is even, by (5.3) we have $(I, \overline{f^{n-1}}\omega_I) = (I, \overline{f}\omega_I)$ in $E(A[T])$. Hence, $e(P_1, \chi_1) = (I, \widetilde{\omega_I})$ in $E(A[T])$. $\square$

7 The case of dimension two

In this section we briefly outline results similar to those in the previous sections in the case when dimension of the base ring is two.

We first note that there is an example ([B-RS 1], Example 3.15) which shows that the main theorem (Theorem 3.10) is not true if $\dim A = 2$. However, we have the following

Theorem 7.1 *Let A be a Noetherian ring of dimension 2 (containing $\mathbb{Q}$) and $I \subset A[T]$ be an ideal of height 2 such that $I = (f_1, f_2) + (I^2T)$. Suppose that there exist $F_1, F_2 \in IA(T)$ such that $IA(T) = (F_1, F_2)$ and $F_i = f_i \bmod I^2A(T)$ for $i = 1, 2$. Then, there exist h_1, h_2 and $\theta \in SL_2(A[T]/I)$ such that (i) $I = (h_1, h_2)$, (ii) $(\overline{f_1}, \overline{f_2})\theta = (\overline{h_1}, \overline{h_2})$ (bar denoting modulo I^2) and (iii) $f_i(0) = h_i(0)$ for $i = 1, 2$.*

Proof Since a unimodular row of length two (over any ring) is always completable to a matrix with determinant 1, it follows easily using a standard patching argument that there is a projective $A[T]$-module P with trivial determinant mapping onto I. Let $\alpha : P \twoheadrightarrow I$ be the surjection. Fix an isomorphism $\chi : A[T] \simeq \wedge^2 P$. Since P/IP is free, α and χ induce a set of generators of I/I^2, say, $I = (g_1, g_2) + I^2$.

It follows from (2.5) that there is a matrix $\overline{\sigma} \in GL_2(A[T]/I)$ with determinant (say) $\overline{f}$ such that $(\overline{f_1}, \overline{f_2}) = (\overline{g_1}, \overline{g_2})\overline{\sigma}$. Now following ([B-RS 3], 2.7, 2.8), we see that, there exists a projective $A[T]$-module P_1 of rank 2 having trivial determinant, a trivialization χ_1 of $\wedge^2 P_1$, and a surjection $\beta : P_1 \twoheadrightarrow I$ such that if the set of generators of I/I^2 induced by β and χ_1 is $\overline{h_1}, \overline{h_2}$, then $(\overline{h_1}, \overline{h_2}) = (\overline{g_1}, \overline{g_2})\overline{\delta}$, where $\overline{\delta} \in GL_2(A[T]/I)$ has determinant $\overline{f}$. Therefore, it follows that the two set of generators, $(\overline{f_1}, \overline{f_2})$ and $(\overline{h_1}, \overline{h_2})$ of I/I^2 are connected by a matrix in $SL_2(A[T]/I)$.

From the above discussion it is clear that, $e(P_1 \otimes A(T), \chi_1 \otimes A(T)) = (IA(T), \omega_I \otimes A(T))$ in $E(A(T))$, where $\omega_I : (A[T]/I)^2 \twoheadrightarrow I/I^2$ is the surjection corresponding to the generators $(\overline{f_1}, \overline{f_2})$. Therefore, from the given condition of the theorem it follows that $P_1 \otimes A(T)$ has a unimodular element and hence is free. Therefore, by the Affine Horrocks theorem, P_1 is a free $A[T]$-module. This proves (i) and (ii).

To prove (iii), note that $I(0) = (f_1(0), f_2(0)) = (h_1(0), h_2(0))$ and there is some $\gamma \in SL_2(A/I(0))$ such that $(\widetilde{f_1(0)}, \widetilde{f_2(0)}) = (\widetilde{h_1(0)}, \widetilde{h_2(0)})\gamma$, where tilde denotes reduction modulo $I(0)^2$. Applying ([B-RS 3], Lemma 2.3), we get $\Gamma \in SL_2(A)$ such that $(f_1(0), f_2(0)) = (h_1(0), h_2(0))\Gamma$. Changing (h_1, h_2) by this Γ, we get the desired set of generators of I.

$\square$

As applications of the above theorem, we can prove the following addition and subtraction principles. The method of proof is the same as that used in Section 4 and hence omitted.

Corollary 7.2 *Let A be a Noetherian ring containing $\mathbb{Q}$ with $\dim A = 2$ and I_1, I_2 be two comaximal ideals in $A[T]$, each of height 2. Suppose that $I_1 = (f_1, f_2)$ and $I_2 = (g_1, g_2)$. Then there exist $h_1, h_2 \in I_1 \cap I_2$ and $\sigma_i \in SL_2(A[T]/I_i)$, $i = 1, 2$, such that, $I_1 \cap I_2 = (h_1, h_2)$ and $((h_1, h_2) \otimes A[T]/I_1)\sigma_1 = (f_1, f_2) \otimes A[T]/I_1$ and $((h_1, h_2) \otimes A[T]/I_2)\sigma_2 = (g_1, g_2) \otimes A[T]/I_2$.*

Corollary 7.3 *Let A be a Noetherian ring containing $\mathbb{Q}$ with $\dim A = 2$ and I_1, I_2 be two comaximal ideals in $A[T]$, each of height 2. Suppose that $I_1 = (f_1, f_2)$ and $I_1 \cap I_2 = (h_1, h_2)$ such that $h_i = f_i$ mod I_1^2. Then, there exist $g_1, g_2 \in I_2$ and $\sigma \in SL_2(A[T]/I_2)$ such that $I_2 = (g_1, g_2)$ and $((h_1, h_2) \otimes A[T]/I_2)\sigma = (g_1, g_2) \otimes A[T]/I_2$.*

Remark 7.4 For a two dimensional ring A containing $\mathbb{Q}$, we can define the notions of the *Euler class group* and the *weak Euler class group* of $A[T]$ in exactly the same way as we did in previous sections. The only difference is that, for an ideal I of $A[T]$ of height 2, a local orientation $[\alpha]$ will be called a global orientation if there is a surjection $\theta : A[T]^2 \twoheadrightarrow I$ and some $\sigma \in SL_2(A[T]/I)$ such that $\alpha\sigma = \theta \otimes A[T]/I$. For a rank 2 projective $A[T]$-module P having trivial determinant, the *Euler class* of P is defined as in Section 4.

Theorem 7.5 *Let A be a Noetherian ring containing $\mathbb{Q}$ with $\dim A = 2$ and $I \subset A[T]$ be an ideal of height 2 such that I/I^2 is generated by 2 elements. Let $\omega_I : (A[T]/I)^2 \twoheadrightarrow I/I^2$ be a local orientation of I. Suppose that the image of (I, ω_I) is zero in $E(A[T])$. Then, ω_I is a global orientation of I.*

Proof Same as 4.7. □

Theorem 7.6 *Let A be a Noetherian ring containing $\mathbb{Q}$ with $\dim A = 2$ and $I \subset A[T]$ be an ideal of height 2 such that I/I^2 is generated by 2 elements. Let $\omega_I : (A[T]/I)^2 \twoheadrightarrow I/I^2$ be a local orientation of I. Let P be a projective $A[T]$-module of rank 2 having trivial determinant and χ be a trivialization of $\wedge^2 P$. Suppose that $e(P, \chi) = (I, \omega_I)$ in $E(A[T])$. Then, there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I, ω_I) is obtained from (α, χ).*

Proof It follows adapting the method of Murthy ([Mu], Theorem 1.3) that there is a projective $A[T]$-module Q of rank 2, stably isomorphic to P, togetherwith an isomorphism $\chi_1 : A[T] \simeq \wedge^2 Q$ and a surjection $\beta : Q \twoheadrightarrow I$ such that (β, χ_1) induces (I, ω_I). Now one can follow the proof of ([B-RS 3], Theorem 7.2), and the "Symplectic" cancellation theorem of Bhatwadekar ([B], Theorem 4.8) to prove the theorem.

□

Remark 7.7 Let A be as above. Let $I \subset A[T]$ be an ideal of height 2 such that I/I^2 is generated by 2 elements and ω_I be a local orientation of I. It is clear from (7.1) that there exists a projective $A[T]$-module P of rank 2 togetherwith an isomorphism $\chi : A[T] \simeq \wedge^2 P$ and a surjection $\alpha : P \twoheadrightarrow I$ such that (I, ω_I) is obtained from (α, χ). As an immediate consequence of this observation, we see that the 'local-global principle' (5.4), holds when $\dim A = 2$ (actually it reduces to the Quillen localization theorem). Since projective $A[T]$-modules are extended when A is regular (containing $\mathbb{Q}$), it follows that Question 1 in Section 4 has an affirmative answer in the two dimensional case. As for Question 2 of Section 4, we see that it reduces to the Affine Horrocks theorem.

The theory of the weak Euler class group described in Section 6 also follows in a like manner in the two dimensional case.

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