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A q-analogue of the distance matrix of a tree ☆

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Abstract

We consider a q-analogue of the distance matrix (called the *q-distance matrix*) of an unweighted tree and give formulae for the inverse and the determinant, which generalize the existing formulae for the distance matrix. We obtain the Smith normal form of the q-distance matrix of a tree. The relationship of the q-distance matrix with the Laplacian matrix leads to q-analogue of the Laplacian matrix of a tree, some of whose properties are also given. We study another matrix related to the distance matrix (the *exponential distance matrix*) and show its relationship with the q-Laplacian and the q-distance matrix. A formula for the determinant of the q-distance matrix of a weighted tree is also given.

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1. Introduction

In this paper, we consider a q-analogue of the distance matrix of a tree and call it the q-distance matrix. The inverse and the determinant of the matrix are obtained when the tree is unweighted.

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We also define some related matrices and study their properties. For a weighted tree, we obtain a formula for the determinant of the q-distance matrix.

We refer the reader to the book by Harary [6] for basic definitions and terminology in graph theory. We start with some definitions. A *tree* is a simple connected graph without any circuit. A *weighted tree* is a tree in which each edge is assigned a *weight*, which is a positive number. So, an *unweighted tree* is simply a tree with each edge having weight 1.

Let $\mathbf{e}, \mathbf{0}$ be the column vectors consisting of all ones and all zeros, respectively. Let $J = \mathbf{e}\mathbf{e}^{\mathrm{t}}$ be the matrix of all ones. For a tree T on n vertices, let $\mathbf{d}^{\mathrm{t}} = (d_1, d_2, \ldots, d_n)$, $\boldsymbol{\delta} = 2\mathbf{e} - \mathbf{d}$ and $\mathbf{z} = \mathbf{d} - \mathbf{e}$, where d_i is the degree of the ith vertex of T. Note that $\boldsymbol{\delta} + \mathbf{z} = \mathbf{e}$.

Let T be a tree on n vertices. The distance matrix of a tree T is a $n \times n$ matrix D with $D_{ij} = k$, if the path from the vertex i to the vertex j is of length k; and $D_{ii} = 0$. The *Laplacian matrix*, L, of a tree T is defined by $L = \mathrm{diag}(\mathbf{d}) - A$, where A is the adjacency matrix of T.

The distance matrix of a tree is extensively investigated in the literature. The first known result concerns the determinant of the matrix D (see [5]), which asserts that if T is any tree on n vertices then $\det(D) = (-1)^{n-1}(n-1)2^{n-2}$. Thus, $\det(D)$ is a function dependent on only n, the number of vertices of the tree. The formula for the inverse of the matrix D was obtained in a subsequent paper by Graham and Lovasz [4]. Their result was extended for a weighted tree by Bapat et al. [1]. In Section 2, we extend the result of Graham and Lovasz by considering a new distance matrix, termed the q-distance matrix, denoted $\mathscr{D} = (\mathscr{D}_{ij})$ and defined as follows:

Let T be a tree on n vertices and $D = (D_{ij})$ be its classical distance matrix. For an indeterminate q, we define

$$\mathscr{D}_{ij} = \begin{cases} 1 + q + q^2 + \cdots + q^{k-1}, & \text{if } D_{ij} = k, \\ 0, & \text{if } i = j. \end{cases}$$

For example, the distance matrix $\mathscr{D}$, of a tree T shown in Fig. 1 is given by

$$\mathscr{D} = \begin{bmatrix} 0 & 1 & 1+q & 1+q & 1+q+q^2 & 1+q+q^2 \\ 1 & 0 & 1 & 1 & 1+q & 1+q \\ 1+q & 1 & 0 & 1+q & 1+q+q^2 & 1+q+q^2 \\ 1+q & 1 & 1+q & 0 & 1 & 1 \\ 1+q+q^2 & 1+q & 1+q+q^2 & 1 & 0 & 1+q \\ 1+q+q^2 & 1+q & 1+q+q^2 & 1 & 1+q & 0 \end{bmatrix}.$$

Each element of $\mathscr{D}$ is a polynomial in the indeterminate q. For convenience we denote the matrix simply by $\mathscr{D}$ and suppress the dependence on q in the notation. Observe that $\mathscr{D}$ is an entrywise nonnegative matrix for all $q \geqslant -1$.

In Section 2, we obtain an expression for $\mathscr{D}^{-1}$ when $q \neq -1$. In Section 3, we use the expression for $\mathscr{D}^{-1}$ to define a generalization, called the q-Laplacian, corresponding to the Laplacian matrix L of a tree. We also define a related matrix, the exponential distance matrix, and examine

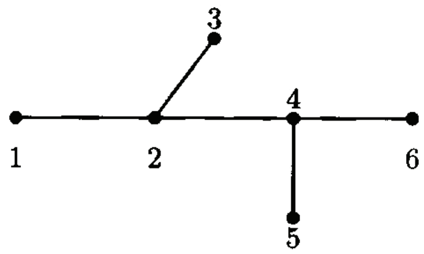

Fig. 1. An unweighted tree on six vertices.

its properties in relation to the Laplacian. Section 4 deals with the invariant factors and Smith normal form of the q-distance matrix. The determinant of the q-distance matrix for a weighted tree is given in Section 5. The formula contains the classical formula of [5] as a special case.

2. q-distance matrix of a tree

In this section, we extend certain results on distance matrices obtained by Graham and Pollak [5] and Graham and Lovasz [4].

Most of the proofs in this paper are based on mathematical induction on the number of vertices of a tree T. So, in the induction step, we start with a tree $\tilde{T}$ having a pendant vertex $k+1$ with vertex k adjacent to it. The tree T is defined as $\tilde{T} \setminus \{k+1\}$. Then, using the matrices $\mathscr{D}, L, \mathbf{z}$ corresponding to the tree T, we define the corresponding matrices $\tilde{\mathscr{D}}$, $\tilde{L}$ and $\tilde{\mathbf{z}}$ of the tree $\tilde{T}$. That is, we have

$$\tilde{\mathscr{D}} = \begin{bmatrix} \mathscr{D} & \mathbf{e} + q\mathscr{D}\mathbf{e}_k \\ \mathbf{e}^t + q\mathbf{e}_k^t\mathscr{D} & 0 \end{bmatrix}, \quad \tilde{L} = \begin{bmatrix} L + \mathbf{e}_k\mathbf{e}_k^t & -\mathbf{e}_k \\ -\mathbf{e}_k^t & 1 \end{bmatrix}, \quad \tilde{\mathbf{z}} = \begin{bmatrix} \mathbf{z} + \mathbf{e}_k \\ 0 \end{bmatrix}. \tag{2.1}$$

We start with the main result of this section.

Theorem 2.1. *Let $\mathscr{D}$ be the q-distance matrix of a tree on n vertices and $q \neq -1$. Then*

$$\mathbf{e} = \frac{1}{n-1}\mathscr{D}(\mathbf{e} - q\mathbf{z}). \tag{2.2}$$

Also, $\mathscr{D}$ is invertible, and

$$\mathscr{D}^{-1} = \frac{1}{(n-1)(1+q)}\mathscr{U} - \frac{1}{1+q}\mathscr{L}, \tag{2.3}$$

where $\mathscr{L} = qL - (q-1)I + q(q-1)\mathrm{diag}(\mathbf{z})$ and $\mathscr{U} = (\mathbf{e} - q\mathbf{z})(\mathbf{e} - q\mathbf{z})^t$.

Proof. We prove the result by induction on n. Let $n = 2$. In this case, the matrices $\mathscr{D}$, L and $\mathbf{z}$ are defined as follows:

$$\mathscr{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So, $\frac{1}{n-1}\mathscr{D}(\mathbf{e} - q\mathbf{z}) = \mathscr{D}\mathbf{e} = \mathbf{e}$. Thus, (2.2) is true for $n = 2$. Also, for $n = 2$ and $q \neq -1$, the right hand side of (2.3) reduces to

$$\begin{aligned}
&\frac{1}{1+q}(qJ - (q-1)J) - \frac{1}{q+1}(qL - (q-1)I) \\
&= \frac{1}{q+1}(-q(I - \mathscr{D}) + (q-1)I + J) \\
&= \frac{1}{q+1}(q\mathscr{D} - I + J) = \frac{1}{q+1}(q\mathscr{D} + \mathscr{D}) \\
&= \mathscr{D} = \mathscr{D}^{-1}.
\end{aligned}$$

Hence, (2.3) holds for $n = 2$. We now assume that both the results are true for $n = k$. Let us prove the result for $n = k + 1$.

We first prove (2.2). That is, we need to show that $\tilde{\mathscr{D}}(\mathbf{e} - q\tilde{\mathbf{z}}) = k\mathbf{e}$. From now on, we will use the expressions for $\tilde{\mathscr{D}}, \tilde{L}, \tilde{\mathbf{z}}$ from (2.1). In this case, we have,

$$\bar{\mathscr{D}}(\mathbf{e}-q\bar{\mathbf{z}})=\begin{bmatrix}\mathscr{D} & \mathbf{e}+q\mathscr{D}\mathbf{e}_k\\ \mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D} & 0\end{bmatrix}\begin{bmatrix}\mathbf{e}-q(\mathbf{z}+\mathbf{e}_k)\\ 1\end{bmatrix}$$
$$=\begin{bmatrix}\mathscr{D}(\mathbf{e}-q\mathbf{z})-q\mathscr{D}\mathbf{e}_k+\mathbf{e}+q\mathscr{D}\mathbf{e}_k\\ (\mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D})(\mathbf{e}-q(\mathbf{z}+\mathbf{e}_k))\end{bmatrix}. \tag{2.4}$$

We calculate the two blocks separately using the induction hypothesis. The first block is given by

$$\mathscr{D}(\mathbf{e}-q\mathbf{z})-q\mathscr{D}\mathbf{e}_k+\mathbf{e}+q\mathscr{D}\mathbf{e}_k=(k-1)\mathbf{e}+\mathbf{e}=k\mathbf{e}. \tag{2.5}$$

Note that $\mathbf{e}^{\mathrm{t}}\mathbf{z}=\mathbf{e}^{\mathrm{t}}(\mathbf{d}-\mathbf{e})=2(k-1)-k=k-2$. So,

$$\mathbf{e}^{\mathrm{t}}(\mathbf{e}-q\mathbf{z})=k-q(k-2). \tag{2.6}$$

Therefore, using (2.6), the second block reduces to

$$(\mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D})(\mathbf{e}-q(\mathbf{z}+\mathbf{e}_k))=k-q(k-2)-q+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D}(\mathbf{e}-q\mathbf{z})-q^2\mathbf{e}_k^{\mathrm{t}}\mathscr{D}\mathbf{e}_k$$
$$=k-q(k-1)+q\mathbf{e}_k^{\mathrm{t}}(k-1)\mathbf{e}-q^2\cdot 0=k. \tag{2.7}$$

Therefore, by substituting the results from (2.5) and (2.7) in (2.4), the proof of (2.2) is complete, as

$$\bar{\mathscr{D}}(\mathbf{e}-q\bar{\mathbf{z}})=\begin{bmatrix}k\mathbf{e}\\ k\end{bmatrix}=k\mathbf{e}.$$

Under the assumption that $q\neq -1$, we now prove that the matrix $\bar{\mathscr{D}}^{-1}$ is indeed given by (2.3). By the induction hypothesis, we assume that $\mathscr{D}$ is an invertible matrix and use it to show that $\bar{\mathscr{D}}$ is invertible. From (2.1), note that $\bar{\mathscr{D}}$ is a block matrix and is given by

$$\bar{\mathscr{D}}=\begin{bmatrix}\mathscr{D} & \mathbf{e}+q\mathscr{D}\mathbf{e}_k\\ \mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D} & 0\end{bmatrix}.$$

Thus, if $\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$ is the inverse of $\bar{\mathscr{D}}$, then we need to show that

$$A_{11}=\mathscr{D}^{-1}+\mathscr{D}^{-1}(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)W^{-1}(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)^{\mathrm{t}}\mathscr{D}^{-1}\quad\text{and} \tag{2.8}$$
$$A_{12}=-\mathscr{D}^{-1}(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)W^{-1}, \tag{2.9}$$

where $W=0-(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)^{\mathrm{t}}\mathscr{D}^{-1}(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)=-(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)^{\mathrm{t}}\mathscr{D}^{-1}(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)$ is a 1×1 matrix. From the induction hypothesis and (2.2), observe that $\mathscr{D}^{-1}\mathbf{e}=\frac{1}{k-1}(\mathbf{e}-q\mathbf{z})$. Therefore, using (2.6), we get

$$-W=(\mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D})\mathscr{D}^{-1}(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)=\mathbf{e}^{\mathrm{t}}\mathscr{D}^{-1}\mathbf{e}+q\mathbf{e}^{\mathrm{t}}\mathbf{e}_k+q\mathbf{e}_k^{\mathrm{t}}\mathbf{e}+q^2\mathbf{e}_k^{\mathrm{t}}\mathscr{D}\mathbf{e}_k$$
$$=\frac{1}{k-1}(\mathbf{e}^{\mathrm{t}}(\mathbf{e}-q\mathbf{z}))+2q+q^2\cdot 0=\frac{k}{k-1}(1+q). \tag{2.10}$$

We will prove (2.8) and (2.9) in two steps.

Step 1: Using (2.10) and the induction hypothesis,

$$A_{11}=\mathscr{D}^{-1}+\mathscr{D}^{-1}(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)W^{-1}(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)^{\mathrm{t}}\mathscr{D}^{-1}$$
$$=\mathscr{D}^{-1}-\frac{k-1}{k(1+q)}[(\mathscr{D}^{-1}\mathbf{e}+q\mathbf{e}_k)(\mathbf{e}^{\mathrm{t}}\mathscr{D}^{-1}+q\mathbf{e}_k^{\mathrm{t}})]$$

$$
\begin{aligned}
&= \frac{\mathscr{U}}{(k-1)(1+q)} - \frac{\mathscr{L}}{1+q} - \frac{k-1}{k(1+q)} \\
&\quad \times \left[\frac{(\mathbf{e}-q\mathbf{z})(\mathbf{e}-q\mathbf{z})^{\mathrm{t}}}{(k-1)^2} + \frac{q(\mathbf{e}-q\mathbf{z})\mathbf{e}_k^{\mathrm{t}}}{k-1} + \frac{q\mathbf{e}_k(\mathbf{e}-q\mathbf{z})^{\mathrm{t}}}{k-1} + q^2\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}} \right] \\
&= \frac{\mathscr{U}}{k(1+q)} - \frac{\mathscr{L}}{1+q} - \frac{q((\mathbf{e}-q\mathbf{z})\mathbf{e}_k^{\mathrm{t}} + \mathbf{e}_k(\mathbf{e}-q\mathbf{z})^{\mathrm{t}})}{k(1+q)} - \frac{(k-1)q^2\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}}}{k(1+q)}
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
A_{12} &= -\mathscr{D}^{-1}(\mathbf{e}+q\mathscr{D}\mathbf{e}_k)W^{-1} = \left(\frac{1}{k-1}(\mathbf{e}-q\mathbf{z}) + q\mathbf{e}_k\right)\frac{k-1}{k(1+q)} \\
&= \frac{1}{k(1+q)}(\mathbf{e}-q\mathbf{z}+q(k-1)\mathbf{e}_k).
\end{aligned} \tag{2.12}
$$

Step 2: We now determine the matrices $\bar{\mathscr{L}}$ and $\bar{\mathscr{U}}$. Using (2.1) and (2.3), we have

$$
\begin{aligned}
\bar{\mathscr{L}} &= q\bar{L} - (q-1)\bar{I} + q(q-1)\mathrm{diag}(\bar{z}) \\
&= q\left[\begin{array}{c|c} L+\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}} & -\mathbf{e}_k \\ -\mathbf{e}_k^{\mathrm{t}} & 1 \end{array}\right] - \left[\begin{array}{c|c} I & \mathbf{0} \\ \mathbf{0}^{\mathrm{t}} & 1 \end{array}\right] + q(q-1)\left[\begin{array}{c|c} \mathrm{diag}(\mathbf{z}) & \mathbf{0} \\ \mathbf{0}^{\mathrm{t}} & 0 \end{array}\right] \\
&= \left[\begin{array}{c|c} \mathscr{L}+q^2\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}} & -q\mathbf{e}_k \\ -q\mathbf{e}_k^{\mathrm{t}} & 1 \end{array}\right]
\end{aligned} \tag{2.13}
$$

and

$$
\begin{aligned}
\bar{\mathscr{U}} &= (\bar{\mathbf{e}}-q\bar{\mathbf{z}})(\bar{\mathbf{e}}-q\bar{\mathbf{z}})^{\mathrm{t}} = \begin{bmatrix} \mathbf{e}-q(\mathbf{z}+\mathbf{e}_k) \\ 1 \end{bmatrix} \left[(\mathbf{e}-q(\mathbf{z}+\mathbf{e}_k))^{\mathrm{t}} \mid 1\right] \\
&= \left[\begin{array}{c|c} \mathscr{U} - q((\mathbf{e}-q\mathbf{z})\mathbf{e}_k^{\mathrm{t}} + \mathbf{e}_k(\mathbf{e}-q\mathbf{z})^{\mathrm{t}}) + q^2\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}} & (\mathbf{e}-q\mathbf{z})-q\mathbf{e}_k \\ ((\mathbf{e}-q\mathbf{z})-q\mathbf{e}_k)^{\mathrm{t}} & 1 \end{array}\right]
\end{aligned} \tag{2.14}
$$

Thus, using (2.13) and (2.14), the first block of the matrix $\bar{\mathscr{D}}^{-1}$ is given by

$$
\begin{aligned}
\frac{1}{k(1+q)}\bar{\mathscr{U}} - \frac{1}{1+q}\bar{\mathscr{L}} &= \frac{\mathscr{U}}{k(1+q)} - \frac{\mathscr{L}}{1+q} \\
&\quad - \frac{q((\mathbf{e}-q\mathbf{z})\mathbf{e}_k^{\mathrm{t}} + \mathbf{e}_k(\mathbf{e}-q\mathbf{z})^{\mathrm{t}})}{k(1+q)} - \frac{(k-1)q^2\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}}}{k(1+q)}
\end{aligned} \tag{2.15}
$$

and the second block of the matrix $\bar{\mathscr{D}}^{-1}$ is given by

$$
\frac{1}{k(1+q)}((\mathbf{e}-q\mathbf{z})-q\mathbf{e}_k) - \frac{1}{1+q}(-q\mathbf{e}_k) = \frac{1}{k(1+q)}(\mathbf{e}-q\mathbf{z}+q(k-1)\mathbf{e}_k). \tag{2.16}
$$

The expressions (2.11) and (2.12) are respectively, equal to the expressions (2.15) and (2.16). Hence, if the two sides of (2.3) are partitioned conformally as in (2.1), then the $(1, 1)$ and $(1, 2)$ blocks on both sides are equal. By symmetry, the $(2, 1)$ block on both sides are also equal. Since a tree has at least two pendant vertices, we can repeat the argument using the second pendant vertex and thus conclude that the $(2, 2)$ block on both sides of (2.3) are equal. Thus, by the induction hypothesis, we obtain the required result. □

For $q = 1$, the Theorem 2.1 reduces to the inverse of the distance matrix D, obtained by Graham and Lovasz [4].

Corollary 2.2. *Let T be a tree on n vertices and let D be its distance matrix. Then*

$$D^{-1} = \frac{1}{2(n-1)}(\mathbf{e}-\mathbf{z})(\mathbf{e}-\mathbf{z})^{\mathrm{t}} - \frac{1}{2}L = \frac{1}{2(n-1)}\delta\delta^{\mathrm{t}} - \frac{1}{2}L.$$

3. Exponential distance matrix of a tree

We now define another matrix using the distance matrix of a tree. Let T be a tree on n vertices and let $D = (D_{ij})$ be its distance matrix. We now consider an $n \times n$ matrix $F = (F_{ij})$, called the exponential distance matrix, with

$$F_{ij} = \begin{cases} 1 & \text{if } i = j, \\ q^{D_{ij}} & \text{if } i \neq j. \end{cases}$$

Proposition 3.3. *Let T be a tree on n vertices and F be the corresponding exponential distance matrix. If $q \neq \pm 1$ then*

$$F^{-1} = I - \frac{q}{1-q^2}A + \frac{q^2}{1-q^2}\operatorname{diag}(\mathbf{d}).$$

Proof. We will prove the result by induction on n. The result can be easily verified for $n = 2$. Let the result be true for $n = k$, and let $\bar{T}$ be a tree on $k+1$ vertices with $k+1$ being a pendant vertex and the vertex k being adjacent to $k+1$. As before, let the tree $T = \bar{T} \setminus \{k+1\}$. Suppose $\bar{F}$, F respectively, represent the matrices corresponding to the trees $\bar{T}$ and T. Then

$$\bar{F} = \begin{bmatrix} F & \underline{\mathbf{q}} \\ \underline{\mathbf{q}}^{\mathrm{t}} & 1 \end{bmatrix},$$

where for any $q \in \mathbb{R}$,

$$\underline{\mathbf{q}}^{\mathrm{t}} = (q^{D_{1,k+1}}, q^{D_{2,k+1}}, \ldots, q^{D_{k,k+1}}, q^{D_{k+1,k+1}} = q^0 = 1). \tag{3.17}$$

We are now ready to prove the formula for $\bar{F}^{-1}$. Note that by induction hypothesis, for $q \neq \pm 1$, F is already invertible. So, if $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is the inverse of $\bar{F}$, then we need to show that

$$A_{11} = F^{-1} + F^{-1}\underline{\mathbf{q}}W^{-1}(F^{-1}\underline{\mathbf{q}})^{\mathrm{t}}, \quad \text{and} \quad A_{12} = -F^{-1}\underline{\mathbf{q}}W^{-1}, \tag{3.18}$$

where $W = 1 - \underline{\mathbf{q}}^{\mathrm{t}}F^{-1}\underline{\mathbf{q}}$. As $F\mathbf{e}_k = \frac{1}{q}\underline{\mathbf{q}}$, $W = 1 - q^2$. Thus,

$$A_{11} = F^{-1} + \frac{1}{1-q^2}(q\mathbf{e}_k)(q\mathbf{e}_k)^{\mathrm{t}} = F^{-1} + \frac{q^2}{1-q^2}\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}} \quad \text{and}$$
$$A_{12} = -\frac{q}{1-q^2}\mathbf{e}_k. \tag{3.19}$$

Also, from the statement of the proposition and (3.19),

$$\bar{F}_{11}^{-1} = I - \frac{q}{1-q^2}A + \frac{q^2}{1-q^2}\operatorname{diag}(\mathbf{d} + \mathbf{e}_k)$$

$$
\begin{aligned}
&= I - \frac{q}{1-q^2}A + \frac{q^2}{1-q^2}\text{diag}(\mathbf{d}) + \frac{q^2}{1-q^2}\mathbf{e}_k\mathbf{e}_k^t = F^{-1} + \frac{q^2}{1-q^2}\mathbf{e}_k\mathbf{e}_k^t \\
&= A_{11}
\end{aligned} \tag{3.20}
$$

and

$$\tilde{F}_{12}^{-1} = \mathbf{0} - \frac{q}{1-q^2}\mathbf{e}_k = A_{12}. \tag{3.21}$$

Therefore, from (3.20) and (3.21), if the two sides of F^{-1} are partitioned conformally as in (2.1), then the $(1, 1)$ and $(1, 2)$ blocks on both sides are equal. By symmetry, the $(2, 1)$ block on both sides are also equal. Since a tree has at least two pendant vertices, we can repeat the argument using the second pendant vertex and thus conclude that the $(2, 2)$ block on both sides of (2.3) are equal. So, by the induction hypothesis, the required result follows. □

Comparing the expression for $\mathscr{D}^{-1}$ given in (2.3) with the one given by Bapat et al. (see (2.1) in [1]), we introduce the q-Laplacian matrix, $\mathscr{L}$, of a tree T by

$$\mathscr{L} = qL - (q-1)I + q(q-1)\text{diag}(\mathbf{z}). \tag{3.22}$$

That is, if v_i, v_j are any two vertices of the tree T, then

$$\mathscr{L}_{ij} = \begin{cases} 1 + (\deg(v_i) - 1)q^2 & \text{if } i = j, \\ -q & \text{if } i \neq j, \quad (v_i, v_j) \in E(T), \\ 0 & \text{if } i \neq j, \quad (v_i, v_j) \notin E(T). \end{cases}$$

Remark 1. The q-Laplacian matrix $\mathscr{L}$ reduces to

1. $\mathscr{L} = \text{diag}(d) - A = L$, the Laplacian matrix of a tree whenever $q = 1$.
2. $\mathscr{L} = \text{diag}(d) + A$, the signless Laplacian matrix of a tree (see [7]), whenever $q = -1$.

We now state a few properties of the q-Laplacian matrix $\mathscr{L}$.

Proposition 3.4. *Let T be a tree on n vertices and let $\mathscr{L}$ be the q-Laplacian matrix. Then*

1. $\det(\mathscr{L}) = 1 - q^2$.
2. *The matrix $\mathscr{L}$ is positive definite if and only if $q \in (-1, 1)$.*

Proof. We use induction to prove both parts of the proposition. The result is clearly true for $n = 2$ as the corresponding matrix is given by $\mathscr{L} = \begin{bmatrix} 1 & -q \\ -q & 1 \end{bmatrix}$ and

$$\det(\mathscr{L}) = 1 - q^2 > 0 \text{ if and only if } q \in (-1, 1).$$

Let us assume the result to be true for $n = k$. We now prove the result for $n = k + 1$. As before, let $\tilde{T}$ be a tree on $k + 1$ vertices. Let $k + 1$ be a pendant vertex adjacent to vertex k. Then in the block form, $\tilde{\mathscr{L}}$ is given by

$$\tilde{\mathscr{L}} = \begin{bmatrix} q^2\mathbf{e}_k\mathbf{e}_k^t + \mathscr{L} & -q\mathbf{e}_k \\ -q\mathbf{e}_k^t & 1 \end{bmatrix}$$

Thus, by the induction hypothesis

$$\det(\bar{\mathscr{L}}) = 1 \cdot \det(q^2 \mathbf{e}_k \mathbf{e}_k^{\mathrm{t}} + \mathscr{L} - (-q\mathbf{e}_k) \cdot 1^{-1} \cdot (-q\mathbf{e}_k^{\mathrm{t}})) = \det(\mathscr{L}) = 1 - q^2. \tag{3.23}$$

Hence by the induction argument the proof of the first part is complete.

For the proof of the second part, observe that, by the induction hypothesis, $\mathscr{L}$ is a positive definite matrix. So, the matrix $q^2\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}} + \mathscr{L}$ is also a positive definite matrix.

We now suppose that $\bar{\mathscr{L}}$ is a positive definite matrix. Then $\det(\bar{\mathscr{L}}) = 1 - q^2$ must be positive. That is, we need $q \in (-1, 1)$.

If $q \in (-1, 1)$, then $\det(\bar{\mathscr{L}}) = 1 - q^2 > 0$. Also, by the induction argument, the matrix $q^2\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}} + \mathscr{L}$, which corresponds to the first block of the matrix $\bar{\mathscr{L}}$, is positive definite. Hence, the matrix $\bar{\mathscr{L}}$ is itself a positive definite matrix.

Therefore, by the induction argument the proof of the second part is also complete. □

The proof of the following corollary is omitted as it is an immediate consequence of Proposition 3.4 and Remark 1.

Corollary 3.5. *Let T be a tree on n vertices. Then the q-Laplacian matrix $\mathscr{L}$ of T is positive semidefinite for $q = -1, 1$.*

The next proposition gives a bound on the smallest eigenvalue of the q-Laplacian matrix $\mathscr{L}$.

Proposition 3.6. *Let T be a tree and let $\mathscr{L}$ be the q-Laplacian matrix. If $\tau(\mathscr{L})$ denotes the smallest eigenvalue of $\mathscr{L}$, then $\tau(\mathscr{L}) \leqslant 1$ for all $q \in \mathbb{R}$. Also, $\tau(\mathscr{L}) = 1$ if and only if $q = 0$.*

Proof. If $q = 0$ then $\mathscr{L} = I$ and hence $\tau(\mathscr{L}) = 1$. For $q \neq 0$, consider a tree T with 1 as a pendant vertex. Suppose the vertex 2 is adjacent to 1 and has degree d. Then the 2×2 matrix $M = \begin{bmatrix} 1 & -q \\ -q & 1 + (d-1)q^2 \end{bmatrix}$ is a submatrix of $\mathscr{L}$. Note that, the characteristic polynomial of this submatrix is $p(\lambda) = (\lambda - 1)^2 + q^2(d - 2 - (d-1)\lambda)$. Note that $p(\frac{d-2}{d-1}) > 0$ and $p(1) < 0$ $(q \neq 0)$. As $p(\lambda)$ is a continuous function of λ, by the intermediate value theorem, there exists a real number $x_0 \in \left(\frac{d-2}{d-1}, 1\right)$ such that $p(x_0) = 0$. So, by the interlacing eigenvalue theorem (for example, see [8]) $\tau(\mathscr{L}) < 1$. Therefore, the required result follows. □

We now show that for $|q| > 1$, $\mathscr{L}$ has exactly one negative eigenvalue.

Proposition 3.7. *Let T be a tree and let $\mathscr{L}$ be the q-Laplacian matrix. Then for $|q| > 1$, $\mathscr{L}$ has exactly one negative eigenvalue.*

Proof. The result is clearly true for $n = 2$, as $\det(\mathscr{L}) = 1 - q^2 < 0$ for $|q| > 1$. So, let us assume the result to be true for $n = k$. We now prove the result for $n = k + 1$. As before, let $\bar{T}$ be a tree on $k + 1$ vertices. Let $k + 1$ be a pendant vertex adjacent to vertex k. Then in the block form, $\bar{\mathscr{L}}$ is given by $\bar{\mathscr{L}} = \begin{bmatrix} q^2\mathbf{e}_k\mathbf{e}_k^{\mathrm{t}} + \mathscr{L} & -q\mathbf{e}_k \\ -q\mathbf{e}_k^{\mathrm{t}} & 1 \end{bmatrix}$. Let $Q = \begin{bmatrix} I & q\mathbf{e}_k \\ \mathbf{0}^{\mathrm{t}} & 1 \end{bmatrix}$. Then it is easy to verify that $Q\bar{\mathscr{L}}Q^{\mathrm{t}} = \begin{bmatrix} \mathscr{L} & \mathbf{0} \\ \mathbf{0}^{\mathrm{t}} & 1 \end{bmatrix} \equiv B$ (say). Then by Sylvester's inertia theorem, the matrices $\bar{\mathscr{L}}$ and B have the same inertia. Therefore, the conclusion follows by appealing to the induction hypothesis. □

We now relate the two matrices $\mathscr{L}$ and F. By definition,

$$\begin{aligned} F^{-1} &= I - \frac{q}{1-q^2}A + \frac{q^2}{1-q^2}\mathrm{diag}(\mathbf{d}) \\ &= \frac{1}{1-q^2}((1-q^2)I - qA + q^2\mathrm{diag}(\mathbf{d})). \end{aligned} \tag{3.24}$$

Also,

$$\begin{aligned} \mathscr{L} &= qL - (q-1)I + q(q-1)\mathrm{diag}(\mathbf{z}) \\ &= q(\mathrm{diag}(\mathbf{d}) - A) - (q-1)I + q(q-1)\mathrm{diag}(\mathbf{d} - e) \\ &= (1-q^2)I - qA + q^2\mathrm{diag}(\mathbf{d}) \end{aligned} \tag{3.25}$$

Thus, from (3.25) and (3.24), we see that $(1-q^2)F^{-1} = \mathscr{L}$. Hence, we arrive at the following lemma.

Lemma 3.8. *Let T be a tree on n vertices and let F be the corresponding exponential matrix. If $\mathscr{L}$ is the q-Laplacian matrix and $q \neq \pm 1$, then*

$$(1-q^2)F^{-1} = \mathscr{L}.$$

Using the above lemma, we get the following corollary to Proposition 3.4.

Corollary 3.9. *Let T be a tree on n vertices and let F be the corresponding exponential matrix. Then F is a positive definite matrix for $q \in (-1, 1)$.*

Proof. Note that a matrix A is positive definite if and only if A^{-1} is positive definite. By Proposition 3.4, we know that $\mathscr{L}$ is positive definite for all $q \in (-1, 1)$. Also, $1 - q^2 > 0$ for all $q \in (-1, 1)$. So, by Lemma 3.8, F^{-1} is a positive definite matrix and hence F itself is a positive definite matrix. □

4. Invariant factors of the q-distance matrix

We first prove a preliminary result.

Lemma 4.10. *Let T be a tree on $n \geqslant 3$ vertices. Then one of the following holds:*

1. *T has a pendant vertex adjacent to a vertex of degree 2.*
2. *T has 2 pendant vertices adjacent to the same vertex.*

Proof. Let $P = [u_1, u_2, \dots, u_{k-2}, u_{k-1}, u_k]$ be a path corresponding to the diameter of T. Note that as $n \geqslant 3$, $k \geqslant 3$. If $\deg_T(u_{k-1}) = 2$ then the first condition holds.

If $\deg_T(u_{k-1}) > 2$, let v be another vertex adjacent to u_{k-1}, other than u_{k-2} and u_k. Since the diameter of T is the same as the length of P, it follows that $\deg_T(v) = 1$. Thus Case 2 holds. □

Recall that a square matrix A with polynomial entries over $\mathbb{R}$ is called *unimodular* if $\det(A)$ is a nonzero real number. For our purpose, we use the word "unimodular" to describe a matrix which satisfies the stronger condition that its determinant is ± 1.

Theorem 4.11. *Let T be a tree on $n \geqslant 3$ vertices and $\mathscr{D}$ be the q-distance matrix of T. Also, let n be a pendant vertex. Then there exists a unimodular matrix U_n such that*

$$U_n\mathscr{D}U_n^{\mathrm{t}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigoplus_{i=1}^{n-2} \left[-\frac{i+1}{i}(1+q) \right]$$

and $U_n\mathbf{e}_n = \mathbf{e}_n$.

Proof. We will prove the result by induction on n. For $n = 3$, $\mathscr{D} = \begin{bmatrix} 0 & 1 & 1+q \\ 1 & 0 & 1 \\ 1+q & 1 & 0 \end{bmatrix}$.

Let

$$P_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(1+q) & 1 \end{bmatrix}.$$

Then

$$P_n\mathscr{D}P_n^{\mathrm{t}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2(1+q) \end{bmatrix} \quad \text{and} \quad P_n\mathbf{e}_n = \mathbf{e}_n.$$

So, the statement is true for $n = 3$. Let the statement be true for $n = k$ and $\tilde{T}$ be a tree on $k + 1$ vertices with $k + 1$ as a pendant vertex. We will prove the result by considering two cases.

Case 1: Suppose that the vertex $k + 1$ is adjacent to the vertex k of degree 2 (Fig. 2).

Let the vertex k be adjacent to the vertex $k - 1$. Then the matrix $\mathscr{D}_{k+1}$ has the form

$$\mathscr{D}_{k+1} = \begin{bmatrix} \mathscr{D}_{k-1} & \mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1} & \mathbf{e} + q\mathbf{e} + q^2\mathscr{D}_{k-1}\mathbf{e}_{k-1} \\ (\mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1})^{\mathrm{t}} & 0 & 1 \\ (\mathbf{e} + q\mathbf{e} + q^2\mathscr{D}_{k-1}\mathbf{e}_{k-1})^{\mathrm{t}} & 1 & 0 \end{bmatrix},$$

where $\mathscr{D}_{k-1}$ is the polynomial matrix corresponding to the tree $\tilde{T} \setminus \{k, k+1\}$. Let $E_{ij} = \mathbf{e}_i\mathbf{e}_j^{\mathrm{t}}$ and define $P_1 = I - (1+q)E_{k+1,k}$. Then

$$P_1\mathscr{D}_{k+1}P_1^{\mathrm{t}} = \begin{bmatrix} \mathscr{D}_{k-1} & \mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1} & -q\mathscr{D}_{k-1}\mathbf{e}_{k-1} \\ (\mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1})^{\mathrm{t}} & 0 & 1 \\ (-q\mathscr{D}_{k-1}\mathbf{e}_{k-1})^{\mathrm{t}} & 1 & -2(1+q) \end{bmatrix}.$$

Now taking $P_2 = I + qE_{k+1,k-1}$, we get

$$P_2P_1\mathscr{D}_{k+1}P_1^{\mathrm{t}}P_2^{\mathrm{t}} = \begin{bmatrix} \mathscr{D}_{k-1} & \mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1} & \mathbf{0} \\ (\mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1})^{\mathrm{t}} & 0 & 1+q \\ \mathbf{0} & 1+q & -2(1+q) \end{bmatrix}.$$

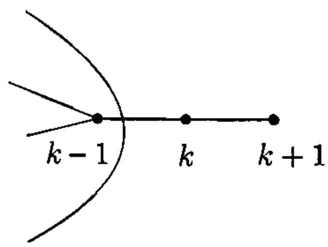

Fig. 2. Figure for Case 1.

Note that the upper left 2×2 block matrix is nothing but the q-distance matrix $\mathscr{D}_k$ of the tree $\bar{T} \setminus \{k+1\}$. Observe that for this tree, the vertex k is a pendant vertex. So, by the induction hypothesis, there exists a unimodular matrix U_1 such that

$$U_1 \mathscr{D}_k U_1^{\mathrm{t}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigoplus_{i=1}^{k-2} \left[-\frac{i+1}{i}(1+q) \right] \quad \text{and} \quad U_1 \mathbf{e}_k = \mathbf{e}_k.$$

Thus,

$$\begin{aligned}
&\begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} P_2 P_1 \mathscr{D}_{k+1} P_1^{\mathrm{t}} P_2^{\mathrm{t}} \begin{bmatrix} U_1^{\mathrm{t}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \\
&= \begin{bmatrix} U_1 \mathscr{D}_k U_1^{\mathrm{t}} & (1+q) U_1 \mathbf{e}_k \\ (1+q)\mathbf{e}_k^{\mathrm{t}} U_1^{\mathrm{t}} & -2(1+q) \end{bmatrix} \\
&= \left[\begin{array}{cccccc|c}
0 & 1 & 0 & & & 0 & 0 \\
1 & 0 & 0 & & & 0 & 0 \\
0 & 0 & -2(1+q) & \cdots & & 0 & 0 \\
\vdots & \vdots & & \ddots & & & \vdots \\
0 & 0 & 0 & \cdots & & -\frac{k-1}{k-2}(1+q) & 1+q \\
\hline
0 & 0 & 0 & \cdots & & 1+q & -2(1+q)
\end{array}\right]
\end{aligned}$$

So, taking $P_3 = I + \frac{k-2}{k-1} E_{k+1,k}$, we have

$$P_3 \begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} P_2 P_1 \mathscr{D}_{k+1} P_1^{\mathrm{t}} P_2^{\mathrm{t}} \begin{bmatrix} U_1^{\mathrm{t}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} P_3^{\mathrm{t}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigoplus_{i=1}^{k-1} \left[-\frac{i+1}{i}(1+q) \right].$$

It can be easily verified that

$$\det\left(P_3 \begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} P_2 P_1 \right) = 1 \quad \text{and} \quad P_3 \begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} P_2 P_1 \mathbf{e}_{k+1} = \mathbf{e}_{k+1}.$$

Case 2: Suppose that the vertices $k+1$ and k are both pendant and are adjacent to the vertex $k-1$ (see Fig. 3).

In this case, the matrix $\mathscr{D}_{k+1}$ has the form

$$\mathscr{D}_{k+1} = \begin{bmatrix} \mathscr{D}_{k-1} & \mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1} & \mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1} \\ (\mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1})^{\mathrm{t}} & 0 & 1+q \\ (\mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1})^{\mathrm{t}} & 1+q & 0 \end{bmatrix}.$$

Let us take $P_1 = I - E_{k+1,k}$. Then

$$P_1 \mathscr{D}_{k+1} P_1^{\mathrm{t}} = \begin{bmatrix} \mathscr{D}_{k-1} & \mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1} & \mathbf{0} \\ (\mathbf{e} + q\mathscr{D}_{k-1}\mathbf{e}_{k-1})^{\mathrm{t}} & 0 & 1+q \\ \mathbf{0} & 1+q & -2(1+q) \end{bmatrix}.$$

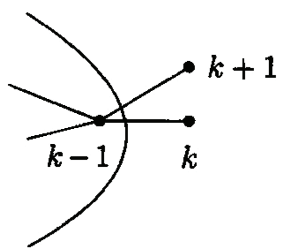

Fig. 3. Figure for Case 2.

Note again that the upper left 2×2 block matrix is nothing but the q-distance matrix $\mathscr{D}_k$. Observe again that for this tree, the vertex k is a pendant vertex. So, by the induction hypothesis, there exists a unimodular matrix U_1 such that

$$U_1 \mathscr{D}_k U_1^{\mathrm{t}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigoplus_{i=1}^{k-2} \left[-\frac{i+1}{i}(1+q) \right] \quad \text{and} \quad U_1 \mathbf{e}_k = \mathbf{e}_k.$$

So, taking $P_3 = I + \frac{k-2}{k-1} E_{k+1,k}$, we have

$$P_3 \begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} P_1 \mathscr{D}_{k+1} P_1^{\mathrm{t}} \begin{bmatrix} U_1^{\mathrm{t}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} P_3^{\mathrm{t}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigoplus_{i=1}^{k-1} \left[-\frac{i+1}{i}(1+q) \right].$$

It can be easily verified that

$$\det \left(P_3 \begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} P_1 \right) = 1 \quad \text{and} \quad P_3 \begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} P_1 \mathbf{e}_{k+1} = \mathbf{e}_{k+1}.$$

Hence, by the induction hypothesis, the statement holds for all $n \geqslant 3$. □

As a corollary to Theorem 4.11, we get the following result about the *inertia* of the matrix $\mathscr{D}$. Recall that inertia of a Hermitian matrix A is defined as the triplet (p, n, z), where p, n, z are the number of positive, negative and zero eigenvalues of A, respectively.

Corollary 4.12. *Let T be a tree on n vertices, $n \geqslant 3$. Also, let $\mathscr{D}$ be the corresponding q-distance matrix. Then the following hold:*

1. *If $q > -1$, then the inertia of $\mathscr{D}$ is $(1, n-1, 0)$.*
2. *If $q < -1$, then the inertia of $\mathscr{D}$ is $(n-1, 1, 0)$.*
3. *If $q = -1$, then the inertia of $\mathscr{D}$ is $(1, 1, n-2)$.*

Proof. Since the matrices $\mathscr{D}$ and $U\mathscr{D}U^{\mathrm{t}}$ are congruent, the result follows from *Sylvester's law of inertia*. □

It may be remarked that when $q > -1$, $\mathscr{D}$ is an elliptic matrix with a zero diagonal in the sense of Fiedler [3]. Also, for $q = 1$, the q-distance matrix is the distance matrix, and one gets the well known result (see Theorem 3, [5]) that the distance matrix has exactly one positive eigenvalue and $n - 1$ negative eigenvalues.

As another application of Theorem 4.11, we obtain the Smith normal form of the q-distance matrix.

Corollary 4.13. *Let T be a tree on $n \geqslant 3$ vertices and $\mathscr{D}$ be the q-distance matrix of T. Then there exist unimodular matrices U, V such that*

$$U \mathscr{D} V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bigoplus_{i=1}^{n-2} \left[-\frac{i+1}{i}(1+q) \right],$$

a diagonal matrix.

Proof. Let n be a pendant vertex of T. By Theorem 4.11 there exists a unimodular matrix U_n such that $U_n \mathscr{D} U_n^{\mathrm{t}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigoplus_{i=1}^{n-2} \left[-\frac{i+1}{i}(1+q) \right]$. Note that the matrix $U_n \mathscr{D}^{\mathrm{t}} U_n$ is not a diagonal matrix. This matrix differs from the diagonal matrix only in the first block. Therefore, if we take $U = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I\right) U_n$ and $V = U_n^{\mathrm{t}}$, then the new matrix

$$U\mathscr{D}V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bigoplus_{i=1}^{n-2} \left[-\frac{i+1}{i}(1+q)\right]$$

is a diagonal matrix. Also, the matrices U and V are unimodular as the matrix U_n was unimodular. □

Remark 2. Observe that the matrix $U_n\mathscr{D}U_n^{\mathrm{t}}$ in Theorem 4.11 is not a diagonal matrix, whereas the matrix $U\mathscr{D}V$ in Corollary 4.13 is a diagonal matrix.

5. q-distance matrix of a weighted tree

We now define the q-distance matrix of a weighted tree T on n vertices. Let $D = (d_{ij})$ be its distance matrix. Suppose the weights on the $n-1$ edges of the tree T are any real numbers $w_1, w_2, \ldots, w_{n-1}$. Let $i = i_0, (i_0, i_1), i_1, (i_1, i_2), i_2, \ldots, i_{k-1}, (i_{k-1}, i_k), i_k = j$ be a path of length k from a vertex i to a vertex j of T. If the edge (i_t, i_{t+1}) has weight w_t, then the (i, j)th entry of the q-distance matrix $\mathscr{D}$ is set to be $w_0 + qw_1 + q^2w_2 + \cdots + w_{k-1}q^{k-1}$. Note that the diagonal entries of the matrix $\mathscr{D}$ are zero and $\mathscr{D}$ is not a symmetric matrix in general. Also, let $\sigma_n = \sum_{i=1}^{n} w_i$.

For example, the distance matrix $\mathscr{D}$, for the tree T shown in Fig. 4 is given by

$$\begin{bmatrix} 0 & w_1 & w_1+qw_2 & w_1+qw_2+w_3q^2 & w_1+qw_2+w_4q^2 & w_1+w_5q \\ w_1 & 0 & w_2 & w_2+w_3q & w_2+w_4q & w_5 \\ w_2+qw_1 & w_2 & 0 & w_3 & w_4 & w_2+w_5q \\ w_3+qw_2+q^2w_1 & w_3+qw_2 & w_3 & 0 & w_3+w_4q & w_3+qw_2+w_5q^2 \\ w_4+qw_2+q^2w_1 & w_4+qw_2 & w_4 & w_4+w_3q & 0 & w_4+qw_2+w_5q^2 \\ w_5+qw_1 & w_5 & w_5+qw_2 & w_5+qw_2+w_3q^2 & w_5+qw_2+w_4q^2 & 0 \end{bmatrix}$$

In the next result we obtain a formula for the determinant of $\mathscr{D}$.

Theorem 5.1. *Let T be a weighted tree on n vertices with edge weights $w_1, w_2, \ldots, w_{n-1}$. If $q \neq -1$, then*

$$\det(\mathscr{D}) = (-1)^{n-1}(1+q)^{n-2}\sigma_n \prod_{i=1}^{n} w_i.$$

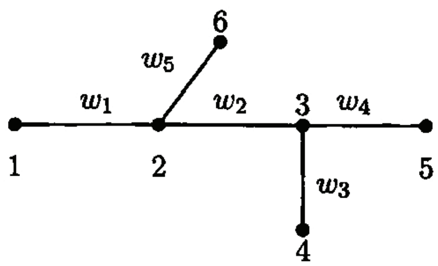

Fig. 4. A weighted tree on six vertices.

Proof. We will prove the result by induction on n. For $n=2$, we have $\mathscr{D}=\begin{bmatrix}0 & w_1\\ w_1 & 0\end{bmatrix}$. So, $\det(\mathscr{D})=-w_1^2=(-1)^1\sigma_1 w_1$. That is, the result is true for $n=2$.

Let the result be true for $n=k$. We now prove the result for $n=k+1$. Suppose $\bar{T}$ is a tree with edge weights $w_1,w_2,\ldots,w_k$. Suppose further that $\bar{T}$ has a pendant vertex $k+1$ and is adjacent to the vertex k with edge weight w_k. We assume that $q\neq -1$ and that $\sum_{i=1}^{k-1} w_i\neq 0$. This results in no loss of generality since the restrictions can be removed by a continuity argument. Then

$$\bar{\mathscr{D}}=\begin{bmatrix}\mathscr{D} & \mathscr{D}\mathbf{e}_k+w_k\underline{\mathbf{q}}\\ w_k\mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D} & 0\end{bmatrix},$$

where for any $q\in\mathbb{R}$, $\underline{\mathbf{q}}$ is defined in (3.17).

The proof of the induction part will be done in four steps.

Step 1: $(\mathbf{e}-q\mathbf{z})^{\mathrm{t}}\underline{\mathbf{q}}=\mathbf{e}^{\mathrm{t}}\underline{\mathbf{q}}-q\mathbf{z}^{\mathrm{t}}\underline{\mathbf{q}}=1+q$.

To prove this, suppose that there is a vertex i_0 adjacent to t vertices, say, $i_1,i_2,\ldots,i_t$. Also suppose i_0 is at a distance d from the vertex $k+1$. Then in the expression $\mathbf{e}^{\mathrm{t}}\underline{\mathbf{q}}$, the contribution due to the presence of t vertices being adjacent to i_0 is $q^{d-2}+q^{d-1}+(t-1)q^d$. But in the expression $q\mathbf{z}^{\mathrm{t}}\underline{\mathbf{q}}$, the information that the degree of the vertex i_0 is t, gives $q\cdot(t-1)q^{d-1}=(t-1)q^d$. Thus, in $(\mathbf{e}-q\mathbf{z})^{\mathrm{t}}\underline{\mathbf{q}}$, the contribution at vertex i_0 is $q^{d-2}+q^{d-1}$. That is, there is no contribution from the vertices that are at a distance $d+1$ from the vertex $k+1$. But then this will be true for all vertices that are at a distance 1 or more. Hence, the only term left out in the expression $(\mathbf{e}-q\mathbf{z})^{\mathrm{t}}\underline{\mathbf{q}}$, is $1+q$.

Step 2: In this step, we show that $\mathbf{e}^{\mathrm{t}}\mathscr{D}^{-1}=\frac{1}{\sigma_{k-1}}(\mathbf{e}^{\mathrm{t}}-q\mathbf{z}^{\mathrm{t}})$. That is, we show that $\sigma_{k-1}\mathbf{e}^{\mathrm{t}}=(\mathbf{e}^{\mathrm{t}}-q\mathbf{z}^{\mathrm{t}})\mathscr{D}$.

The result will also be proved by induction. The initial step in the induction argument can be easily verified. Let the result be true for all trees with k vertices. We now prove the result for a tree with $k+1$ vertices. From (2.1), note that

$$\mathbf{e}^{\mathrm{t}}-q\bar{\mathbf{z}}^{\mathrm{t}}=[\mathbf{e}^{\mathrm{t}}|1]-q[(\mathbf{z}+\mathbf{e}_k)^{\mathrm{t}}|0]=[\mathbf{e}^{\mathrm{t}}-q(\mathbf{z}+\mathbf{e}_k)^{\mathrm{t}}|1].$$

Now, using step 1 and the induction hypothesis, we get

$$\begin{aligned}(\mathbf{e}^{\mathrm{t}}-q\bar{\mathbf{z}}^{\mathrm{t}})\bar{\mathscr{D}}&=(\mathbf{e}^{\mathrm{t}}-q(\mathbf{z}+\mathbf{e}_k)^{\mathrm{t}})\begin{bmatrix}\mathscr{D} & \mathscr{D}\mathbf{e}_k+w_k\underline{\mathbf{q}}\\ w_k\mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D} & 0\end{bmatrix}\\ &=[\mathbf{e}^{\mathrm{t}}-q(\mathbf{z}+\mathbf{e}_k)^{\mathrm{t}}|1]\begin{bmatrix}\mathscr{D} & \mathscr{D}\mathbf{e}_k+w_k\underline{\mathbf{q}}\\ w_k\mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D} & 0\end{bmatrix}\\ &=[\sigma_k\mathbf{e}^{\mathrm{t}}|\sigma_k]=\sigma_k\mathbf{e}^{\mathrm{t}}.\end{aligned}\tag{5.1}$$

Thus, by the induction hypothesis, the proof of step 2 is complete.

Step 3: We now show that $(w_k\mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D})\mathscr{D}^{-1}(\mathscr{D}\mathbf{e}_k+w_k\underline{\mathbf{q}})=\frac{(1+q)w_k\sigma_k}{\sigma_{k-1}}$. We use the results obtained in step 1 and step 2, to prove this step. We have

$$\begin{aligned}&(w_k\mathbf{e}^{\mathrm{t}}+q\mathbf{e}_k^{\mathrm{t}}\mathscr{D})\mathscr{D}^{-1}(\mathscr{D}\mathbf{e}_k+w_k\underline{\mathbf{q}})\\ &\quad=w_k\mathbf{e}^{\mathrm{t}}\mathbf{e}_k+w_k^2\mathbf{e}^{\mathrm{t}}\mathscr{D}^{-1}\underline{\mathbf{q}}+q\mathbf{e}_k\mathscr{D}\mathbf{e}_k+qw_k\mathbf{e}_k^{\mathrm{t}}\underline{\mathbf{q}}\\ &\quad=w_k+w_k^2\mathbf{e}^{\mathrm{t}}\mathscr{D}^{-1}\underline{\mathbf{q}}+q\cdot 0+qw_k=w_k(1+q)+\frac{w_k^2}{\sigma_{k-1}}(\mathbf{e}^{\mathrm{t}}-q\mathbf{z}^{\mathrm{t}})\underline{\mathbf{q}}\end{aligned}$$

$$= w_k(1+q) + (1+q)\frac{w_k^2}{\sigma_{k-1}} = \frac{(1+q)w_k\sigma_k}{\sigma_{k-1}}. \tag{5.2}$$

Step 4: We now use (5.2) to complete the induction step. By the induction hypothesis and (5.2), we have

$$\begin{aligned}
\det(\bar{\mathscr{D}}) &= \det(\mathscr{D}) \cdot (0 - (w_k \mathbf{e}^t + q\mathbf{e}_k^t\mathscr{D})\mathscr{D}^{-1}(\mathscr{D}\mathbf{e}_k + w_k\underline{\mathbf{q}}))\\
&= -\det(\mathscr{D})\frac{w_k\sigma_k(1+q)}{\sigma_{k-1}}\\
&= -(-1)^{k-2}\sigma_{k-1}\prod_{i=1}^{k-1} w_i(1+q)^{k-2} \times \frac{w_k\sigma_k(1+q)}{\sigma_{k-1}}\\
&= (-1)^{k-1}(1+q)^{k-1}\sigma_k\prod_{i=1}^{k} w_i.
\end{aligned}$$

Thus, by induction, the proof is over. □

It is tempting to obtain a formula for $\mathscr{D}^{-1}$, in the case of a weighted tree. However, it appears that such a formula will be very complicated and we leave it as an open problem. As a consequence of Theorem 5.1, we derive the determinant formula for an unweighted tree.

Corollary 5.2. *Let T be a tree on n vertices and let $\mathscr{D}$ be its q-distance matrix. Then* $\det(\mathscr{D}) = (-1)^{n-1}(n-1)(1+q)^{n-2}$.

Proof. In this case, the weight of each edge is 1. So, $\sigma_{n-1} = 1 + 1 + \cdots + 1 = n - 1$. Hence, the result follows. □

For $q = 1$, the above result reduces to the result of Graham and Pollak [5] on $\det(D)$.

Corollary 5.3. *Let T be a tree on n vertices and let D be its distance matrix. Then* $\det(D) = (-1)^{n-1}(n-1)2^{n-2}$.

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