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CESÁRO UNIFORM INTEGRABILITY AND THE STRONG LAW OF LARGE NUMBERS

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SUMMARY. This paper studies some consequences of conditions like the uniform integrability and Cesàro uniform integrability (introduced by Chandra (1989)) in the context of strong laws of large numbers (SLLNs). Extending certain arguments of Etemadi (1981), Csörgo, Tandori and Totik (1983), we relax the condition of 'identical distribution and/or independence' in Etemadi's extension of Kolmogorov's SLLN and in the classical SLLNs of Markov and Cantelli. We also extend the recent SLLN of Landers and Rogge (1986) and a result of Calderon (1983) related to the classical SLLN of Marcinkiewicz and Zygmund.

1. Introduction

Although conditions like uniform integrability are well-known, these are not yet widely studied in the context of strong laws of large numbers (SLLNs). This paper attempts to fill up this gap to some extent. *Let* $\{X_n\}_{n \geqslant 1}$ *be a sequence of integrable random variables defined on the same probability space, and put* $S(n) = X_1 + \ldots\ldots + X_n$ *and* $\overline{X}_n = n^{-1} S(n)$ $(n \geqslant 1)$. Etemadi (1981, 1983a) has shown in an elementary way that in the classical SLLN of Kolmogorov (see Theorem 5.4.2 of Chung (1974)), the condition 'independent and identically distributed (i.i.d) random variables' can be relaxed to the condition 'pairwise independent and identically distributed random variables'; he has also been able to prove other SLLNs, with nice applications, for nonnegative pairwise independent random variables satisfying certain moment conditions. Csörgo, Tandori and Totik (1983) proved, by a novel extension of arguments of Etemadi (1981), an analogue of the other SLLN of Kolmogorov (see page 125 of Chung (1974)) for pairwise independent random variables. On the other hand, Landers and Rogge (1986) have obtained the SLLN for a certain class of uniformly integrable random variables which are also pairwise independent; they have also shown that *the SLLN need not hold for independent random variables which are merely uniformly integrable*. It may be noted that the SLLN of Landers and Rogge (1986) does not imply the Kolmogorov SLLN for iid random variables.

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Chandra (1989) has attempted to give a simple and straightforward proof of the classical weak law of large numbers (WLLN) of Khinchin (see Theorem 5.2.2 of Chung (1974)) and has pointed out that it is better to prove the stronger fact that the L_1-convergence of $(\overline{X}_n - E(\overline{X}_n))$ holds. The method of his proof leads naturally to a condition called 'Cesàro uniform integrability of $\{X_n\}$' which is weaker than the usual uniform integrability of $\{X_n\}$.

In this paper we modify the ideas of the above papers to get new SLLNs for pairwise independent random variables which are not necessarily identically distributed and satisfy certain moment conditions. *We are thus able to relax the condition of 'identical distribution and/or independence' in Etemadi's extension of Kolmogorov's* SLLN (*see Theorem* 2 *below*) *and in the classical SLLNs of Markov and Cantelli as well* (*see Corollary* 4 *and Theorem* 5 *below*). We have obtained an extension of SLLN of Csörgo, Tandori and Totik (1983) along the lines of Chung (1947). We also prove an extension of the SLLN of Landers and Rogge (1986) by replacing the uniform integrability by the Cesàro uniform integrability ; this situation is quite natural, since the laws of large numbers are, after all, properties of the *averages* $\overline{X}_n$. Finally, we establish in the Appendix an analogue of a classical result of La Vallée Poussin (on the necessary and sufficient condition of the uniform integrability) for a sequence of Cesàro uniformly integrable random variables.

Definition. A sequence $\{X_n\}_{n \geq 1}$ of random variables is said to be *Cesàro uniformly integrable* if

$$\lim_{N \to \infty} \sup_n \left[n^{-1} \sum_{i=1}^{n} E(|X_i| I(|X_i| > N))\right] = 0.$$

(Here $I(A)$ denotes the indicator function of the set A). Note that the above notion depends only on the marginal distributions of the X_n.

Definition. A sequence $\{a_n\}$ of nonnegative reals is said to be *Cesàro bounded* if the sequence $\{n^{-1}(a_1 + \ldots\ldots + a_n)\}$ is bounded.

For a better understanding of the significance of the Cesàro uniform integrability, we state a result of Chandra (1989).

Theorem. *A sequence $\{X_n\}$ of random variables is Cesàro uniformly integrable if and only if the following conditions are satisfied* :

(a) *$\{E(|X_n|)\}$ is Cesàro bounded* ; *and*

(b) *for each $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\{A_k\}_{k \geq 1}$ is a sequence of events satisfying the condition that* $\sup_n \left[n^{-1} \sum_{k=1}^{n} P(A_k)\right] < \delta$, *we have* $\sup_n \left[n^{-1} \sum_{k=1}^{n} E(|X_k| I(A))\right] < \epsilon$.

In this paper, C *stands for a generic constant, not necessarily the same at each appearance. Also* $\{f(n)\}$ *will stand for an increasing sequence such that* $f(n) > 0$ *for each* n *and* $f(n) \to \infty$.

For the convenience of the reader and the sake of the readability of the paper, we have made it self-contained by repeating some of arguments of the above papers.

2. Main results

We begin with a very simple and useful lemma. Recall the definition of the $S(n)$.

Lemma 1. *Let* $\{X_n\}_{n \geq 1}$ *be a sequence of random variables with finite* $E(X_n^2)$. *Suppose that*

(i) *there is a constant* $C > 0$ *such that* $E(S(n))^2 \leqslant C \sum_{i=1}^{n} E(X_i^2)$, $\forall\, n \geqslant 1$;

and

(ii) $\sum_{n=1}^{\infty} n^{-2} E(X_n^2) < \infty$.

Then for every subsequence $\{k_n\}$ *of positive integers such that* $\liminf_{n \to \infty} (k_n/k_{n-1}) > 1$, *one has* $S(k_n)/k_n \to 0$ *almost surely (a.s) as* $n \to \infty$.

Proof. Let $\delta > 0$. The Chebyshev inequality, Condition (i) and a change of order of summation imply that

$$\delta^2 \sum_{n=1}^{\infty} P(|S(k_n)| \geqslant k_n\, \delta) \leqslant C \sum_{j=1}^{\infty} E(X_j^2) \sum_{n : k_n \geqslant j} (k_n)^{-2} \qquad \ldots (1)$$

Since $\liminf_{n \to \infty} (k_n/k_{n-1}) > 1$, there exist $b > 1$ and an integer $n_0 \geqslant 2$ such that $k_n > b\, k_{n-1}$ for each $n \geqslant n_0$. Put $n(j) = \min\,\{n \geqslant 1 : k_n \geqslant j\}$, $j \geqslant 1$. Then $n(j) \uparrow \infty$ as $j \uparrow \infty$, so that the set $\{j \geqslant 1 : n(j) < n_0\}$ is finite, say $\{1, 2, \ldots, j_0 - 1\}$. Let $j \geqslant j_0$; then $n(j) \geqslant n_0$ and so $k_n > b^{n-n(j)}\, k_{n(j)}$ for each $n \geqslant n(j)$; therefore

$$\sum_{n : k_n \geqslant j} (k_n)^{-2} = \sum_{n=n(j)}^{\infty} (k_n)^{-2} \geqslant (k_{n(j)})^{-2} \sum_{n=n(j)}^{\infty} b^{-2(n-n(j))}$$
$$= (k_{n(j)})^{-2}\,(1-b^{-2})^{-1} \leqslant j^{-2}\,(1-b^{-2})^{-1}$$

(by definition of $n(j)$). Condition (ii) now implies

$$\sum_{j=j_0}^{\infty} E(X_j^2) \sum_{n : k_n \geqslant j} (k_n)^{-2} < \infty.$$

Next, note that for each $j \geqslant 1$, the series $\sum_{n : k_n \geqslant j} (k_n)^{-2} \leqslant \sum_{n=1}^{\infty} n^{-2} < \infty$.

Thus the left side of (1) is finite. By a standard result (see, e.g., Loève, 1977, page 18) we have the desired result. □

The next theorem and Corollary 1 can be obtained from the arguments of Csörgő *et al.* (1983). We therefore indicate only the main steps of the proofs.

Theorem 1. *Let $\{X_n\}_{n\geq 1}$ be a sequence of nonnegative random variables with finite $var(X_n)$. Assume that*

(i) $\sup_{n\geq 1}\left[\sum_{k=1}^{n} E(X_k)/f(n)\right] = A(say) < \infty$;

(ii) *there is a double sequence $\{\rho_{ij}\}$ of non-negative reals such that*

$$var\,(S_n) \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} \text{ for each } n \geqslant 1;$$

(iii) $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij}/(f(i \vee j))^2 < \infty, \quad i \vee j = max\,(i, j).$

Then $[S(n)-E(S(n))]/f(n) \to 0$ almost surely as $n \to \infty$.

Proof. Let $\alpha > 1$, $\epsilon > 0$ and put $L = [A/\epsilon]$, the integer part of A/ϵ, and $Z(n) = [S(n)-E(S(n))]/f(n)$. For each integer $n \geqslant 1$, there exist integers $m(n)$ and $s(n)$ such that $m(n) \to \infty$, $0 \leqslant s(n) \leqslant L$, $\alpha^{m(n)} \leqslant f(n) < \alpha^{m(n)+1}$ and $s(n)\,\epsilon \leqslant E(S(n))/f(n) < (s(n)+1)\,\epsilon$. Let T_n be the set of all integers $k \geqslant 1$ such that $\alpha^{m(n)} \leqslant f(k) < \alpha^{m(n)+1}$ and $s(n)\,\epsilon \leqslant E(S(k))/f(k) < (s(n)+1)\,\epsilon$; let $k_n^+ = \sup T_n$ and $k_n^- = \inf T_n$. Then $k_n^- \to \infty$ and $\{m(n)\}$ is increasing. Note that

$$\sum_{n=1}^{\infty} (f(k_n^{\pm}))^{-2} \text{ var } (S(k_n^{\pm})) \leqslant \sum_{n=1}^{\infty} (f(k_n^{\pm}))^{-2} \sum_{i=1}^{k_n^{\pm}} \sum_{j=1}^{k_n^{\pm}} \rho_{ij}$$

$$\leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{n=p}^{\infty} \alpha^{-2m(n)} \leqslant \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{m=m(p)}^{\infty} \alpha^{-2m}$$

where $p = \inf\,\{n \geqslant 1 : \alpha^{m(n)+1} \geqslant f(i \vee j)\}$. Thus the last sum is

$$\leqslant C \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij}\, \alpha^{-2m(p)} \leqslant C \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij}\,(f(i \vee j))^{-2} < \infty.$$

Hence $Z(k_n^{\pm}) \to 0$ almost surely as $n \to \infty$. Following now the arguments of Csörgő et al. (1983), we get $-\epsilon-(\alpha-1)A/\alpha \leqslant \liminf Z(n) \leqslant \limsup Z(n) \leqslant (\alpha-1)\,A+\epsilon$. Letting $\epsilon \to 0+$ and $\alpha \to 1+$ along sequences, we get the desired result. □

Remark 1. In Theorem 1, the condition 'the X_n are non-negative and $\{E(X_n)\}$ satisfies Condition (i) above' can be replaced by the condition '$X_n \geqslant c_n \ \forall\ n \geqslant 1$ and $\{E(X_n - c_n)\}$ satisfies Condition (i) above'. This observation is occasionally useful. Finally, *Conditions* (ii) *and* (iii) *will usually be satisfied with the particular choice :* $\rho_{ij} = C \max(\mathrm{cov}(X_i, X_j), 0)$; it is not known whether this is the 'optimal' choice.

We denote by X^+ *and* X^- *the positive and negative parts respectively of a random variable* X.

Corollary 1. *Let* $\{X_n\}_{n \geqslant 1}$ *be a sequence of pairwise independent random variables with finite* $var(X_n)$. *Assume that*

$$\text{(i)} \quad \sup_{n>1} \left[\sum_{n=1}^{n} E(|X_k - E(X_k)|)/f(n) \right] < \infty ;$$

and

$$\text{(ii)} \quad \sum_{n=1}^{\infty} (f(n))^{-2} \ var(X_n) < \infty,$$

Then $[S(n) - E(S(n))]/f(n) \to 0$ *almost surely as* $n \to \infty$.

Proof. Put $Y_n = (X_n - E(X_n))^+$ and $Z_n = (X_n - E(X_n))^-$ $(n \geqslant 1)$. It suffices to show that as $n \to \infty$,

$$(f(n))^{-1} \sum_{i=1}^{n} (Y_i - E(Y_i)) \to 0 \text{ a.s.}, \ (f(n))^{-1} \sum_{i=1}^{n} (Z_i - E(Z_i)) \to 0 \text{ a.s.}, \quad \ldots \quad (2)$$

since $(f(n))^{-1} \sum_{i=1}^{n} E(Y_i) - (f(n))^{-1} \sum_{i=1}^{n} E(Z_i) = (f(n))^{-1} \sum_{i=1}^{n} E(X_i - E(X_i)) = 0.$

Since var $(Y_n) \leqslant E(Y_n^2) \leqslant$ var (X_n) and $E(Y_n) \leqslant E(|X_n - E(X_n)|)$ $(n \geqslant 1)$, it follows from Theorem 1 that the first part of (2) holds ; replacing X_n by $-X_n$, one gets the second part of (2). □

Corollary 2. *Let* $\{X_n\}_{n \geqslant 1}$ *be a sequence of pairwise independent integrable random variables such that there is a sequence* $\{B_n\}$ *of Borel subsets of* $\mathbf{R}^1$ *satisfying the following conditions* (a)—(d) :

$$\text{(a)} \quad \sum_{n=1}^{\infty} P(X_n \in B_n^c) < \infty ;$$

$$\text{(b)} \quad \sum_{i=1}^{n} E(X_i I(X_i \in B_i^c)) = o(f(n)) ;$$

$$\text{(c)} \quad \sum_{n=1}^{\infty} (f(n))^{-2} \ var\,(X_n I(X_n \in B_n)) < \infty ;$$

and

(d) $\sup_{n \geqslant 1} \left[\sum_{k=1}^{n} E(|X_k| I(X_k \in B_n))/f(n) \right] < \infty$;

here B_n^c is the complement of B_n. Then $(f(n))^{-1}$ $[S(n) - E(S(n))] \to 0$ almost surely as $n \to \infty$.

Proof. Let $Y_n = X_n I(X_n \in B_n)$, $n \geqslant 1$. By (c) and (d), Corollary 1 applied to $\{Y_n\}$ yields $(f(n))^{-1} \sum_{i=1}^{n} (Y_i - E(Y_i)) \to 0$ almost surely as $n \to \infty$. By (b), we get $(f(n))^{-1} \sum_{i=1}^{n} (Y_i - E(X_i)) \to 0$ almost surely as $n \to \infty$. By (a) and the first Borel-Cantelli lemma, the desired result follows. □

The next theorem, our first main result, is an extension of the classical Kolmogorov SLLN for independent and identically distributed random variables (see Stout (1974) and Etemadi (1981)). *Our intention is to replace the condition of 'identical distribution' by suitable weaker conditions of simple nature.* It is known that the SLLN need not hold for uniformly integrable sequence of independent random variables. *We show instead that the SLLN holds under the stronger assumption of 'domination in distribution by an integrable random variable'. or, the assumption of 'L_p-boundedness of X_n for some $p > 1$'* (see, in this connection, page 32 of Billingsley (1968)).

Theorem 2. *Let $\{X_n\}_{n \geqslant 1}$ be a sequence of pairwise independent random variables and put $G(x) = \sup_{n \geqslant 1} P(|X_n| \geqslant x)$ for $x \geqslant 0$. If*

$$\int_0^\infty G(x)dx < \infty, \tag{3}$$

then $n^{-1} \sum_{i=1}^{n} c_i(X_i - E(X_i)) \to 0$ almost surely as $n \to \infty$ for each bounded sequence $\{c_n\}$.

Proof. First note that $\sup_{n \geqslant 1} E(|X_n|) < \infty$. It suffices to prove the result for $c_n \equiv 1$. To this end, we use Corollary 2 with $B_n = [-n, n]$ for $n \geqslant 1$ (it is also possible to apply Corollary 2 of Theorem 1 of Etemadi (1983a)). Condition (a) follows since $\sum_{n=1}^{n} P(|X_n| > n) \leqslant \sum_{n=1}^{\infty} G(n) < \infty$. To verify Condition (b), note that for any nonnegative random variable Z and $\alpha \geqslant 0$,

$$E(ZI(Z \geqslant \alpha)) = \alpha P(Z \geqslant \alpha) + \int_\alpha^\infty P(Z \geqslant x)dx\,;$$

see, e.g., Equation (3), page 223 of Billingsley (1968). Hence

$$E(|X_n|I(|X_n| > n)) \leqslant nP(|X_n| \geqslant n) + \int_n^\infty G(x)dx \to 0,$$

so that Condition (b) holds. Obviously Condition (d) holds. Thus it remains to verify Condition (c). Observe that for a nonnegative random variable Z and $\alpha > 0$,

$$E(ZI(Z \leqslant \alpha)) = \int_0^\alpha P(ZI(Z \leqslant \alpha) > x)dx$$

$$= \int_0^\alpha P(x < Z \leqslant \alpha)dx \leqslant \int_0^\alpha P(Z \geqslant x)dx.$$

Hence

$$\Sigma n^{-2}E(X_n^2 I(|X_n| \leqslant n)) \leqslant \Sigma n^{-2} \int_0^{n^2} P(|X_n| \geqslant x^{1/2})dx$$

$$\leqslant 2\Sigma n^{-2} \int_0^n yG(y)dy = 2\Sigma n^{-2} \sum_{j=1}^n \int_{j-1}^j yG(y)dy$$

$$= 2 \sum_{j=1}^\infty \int_{j-1}^j yG(y)dy \sum_{n=j}^\infty n^{-2} \leqslant 4 \sum_{j=1}^\infty j^{-1} \int_{j-1}^j yG(y)dy$$

$$\leqslant 4 \sum_{j=1}^\infty \int_{j-1}^j G(y)dy < \infty. \quad \square$$

Remark 2. Our Condition (3), though stronger than uniform integrability, is by no means any stronger than the conditions imposed by Landers and Rogge (1986).

The next theorem, our second main result, is an analogue of the SLLN of Chung (1947); for other related results, the interesting paper of Chung (1947) may be consulted.

Theorem 3. *Let* $\{X_n\}$ *be pairwise independent and*

$$\sup_{n \geqslant 1} \Big[\sum_{k=1}^n E(|X_k| I(|X_k| \leqslant a_k))/f(n) \Big] < \infty.$$

Let $g_n : (0, \infty) \to (0, \infty)$ *be increasing in* x *for each* $n \geqslant 1$, $g_n(0)$ *being defined arbitrarily. Assume that*

$$x/g_n(x) \textit{ and } \frac{g_n(x)}{x^2} \textit{ decrease in } x, \qquad \ldots \quad (4)$$

$\Sigma E(g_n(|X_n|))/g_n(a_n) < \infty$ *and* $\{a_n/f(n)\}$ *bounded ; then* $(f(n))^{-1}[S(n) - E(S(n))] \to 0$ *almost surely as* $n \to \infty$.

Proof. We use Corollary 2 with $B_n = [-a_n, a_n]$. To verify Condition (a), note that

$$\Sigma P(|X_n| > a_n) \leqslant \Sigma P(g_n(|X_n|) \geqslant g_n(a_n)) < \infty.$$

Next note that

$$\Sigma(f(n))^{-1} E(|X_n| I(|X_n| > a_n) \leqslant \Sigma a_n E(g_n(|X_n|)/(f(n)g_n(a_n)) < \infty$$

so that Condition (b) follows by the Kronecker lemma. Finally, Condition (c) follows, since

$$\Sigma(f(n))^{-2} E(X_n^2 I(|X_n| \leqslant a_n)) \leqslant \Sigma a_n^2 E(g_n(|X_n|))/(g_n(a_n)(f(n)^2)) < \infty. \square$$

Theorem 4. *Let $\{X_n\}$, $\{a_n\}$ etc. be as in Theorem 3, except that Condition (4) is replaced by $x/g_n(x)$ is increasing in x and $E(X_n) \equiv 0$. Then the conclusion of Theorem 4 holds.*

Proof. The proof of Theorem 4 goes through except that we have to verify Condition (b) of Corollary 2. To this end, we use the fact that $E(X_n) \equiv 0$ and

$$\Sigma(f(n))^{-1} E(|X_n| I(|X_n| \leqslant a_n)) \leqslant \Sigma a_n E(g_n(|X_n|))/(f(n)g_n(a_n)) < \infty. \quad \square$$

Corollary 3. *Let $\{X_n\}$ be as in Theorem 3, and assume that $E(X_n) \equiv 0$ if $0 < p_n < 1$. If $0 < p_n \leqslant 2$ for each $n \geqslant 1$ and $\Sigma n^{-p_n} E(|X_n|^{p_n}) < \infty$, then $n^{-1}[S(n) - E(S(n))] \to 0$ almost surely as $n \to \infty$.*

Remark 3. It may be noted that the last condition with $p_n \equiv 1$ implies that '$\{E(|X_n|)\}$ is Cesàro bounded'. Clearly, Corollary 3 generalises the SLLN of Csörgő *et al.* (1983).

Corollary 4. *Let $\{X_n\}_{n \geqslant 1}$ be a sequence of pairwise independent random variables with $E(X_n) \equiv 0$. Assume that $\sup_{n \geqslant 1} E(|X_n|^p) < \infty$ for some $p > 1$. Then $n^{-1}S(n) \to 0$ almost surely as $n \to \infty$.*

Following an argument of Chandra (1991), we now strengthen Corollary 4.

Theorem 5. *Let $\{X_n\}$ be pairwise independent, $\{E(|X_n|)\}$ Cesaro bounded, $E(|X_n|^p) < \infty$ for some $1 < p \leqslant 2$ and $E(X_n) \equiv 0$. With $b_n = n^{-1} \sum_{k=1}^{n} E(|X_k|^p)$, assume that*

$$\Sigma b_n n^{-p} < \infty \text{ and } b_n = o(n^{p-1}).$$

Then $n^{-1} S(n) \to 0$ almost surely as $n \to \infty$.

Proof. We use the formula of summation by parts (see page 194 of Apostol (1974)). Note that if $N \geqslant 2$

$$\sum_{n=1}^{N} n^{-p} E(|X_n|^p) = \sum_{n=1}^{N} n^{-p} (nb_n-(n-1)b_{n-1})$$

$$= Nb_N/N^p + \sum_{n=1}^{N-1} nb_n(n^{-p}-(n+1)^{-p})$$

$$\leqslant Nb_N/N^p + p\sum_{n=1}^{N-1} nb_n/n^{p+1}$$

so that Corollary 3 applies. □

Remark 4. *Corollary 4 generalizes Markov's SLLN* to the case of pairwise independent sequences of random variables ; see pages 125–126 of Chung (1974). *It also generalizes Cantelli's SLLN* to the case of pairwise independent sequences of random variables which need not be identically distributed; see pages 106–107 of Chung (1974) and page 436 of Rényi (1970) (see also page 377 of Shiryaev (1983)).

We next generalize the SLLN of Landers and Rogge (1986). The reader should note the naturality of Cesàro uniform integrability in the context of laws of large numbers.

Theorem 6. *Let $\{X_n\}_{n \geqslant 1}$ be a sequence of pairwise independent random variables. Assume that there is a function $\phi : (0, \infty) \to (0, \infty)$ such that*

(i) *$t^{-1}\phi(t)$ is increasing to ∞ as $t \uparrow \infty$;*

(ii) *$\sup_n \left[n^{-1} \sum_{i=1}^{n} E(\phi(|X_n|))\right] = c$ (say) $< \infty$;*

and

(iii) $\sum_{n=1}^{\infty} (\phi(n))^{-1} < \infty.$

Then $n^{-1}[S(n)-E(S(n))] \to 0$ almost surely as $n \to \infty$.

Proof. We use Corollary 2 with $B_n = [-n, n]$ for $n \geqslant 1$. Also we use the following lemma which can be proved using the formula of summation by parts.

Lemma 2. *If $\Sigma b_n < \infty$ and b_n is decreasing, then for any bounded $\{\alpha_n\}$ such that $\{n\alpha_n\}$ is increasing, $\Sigma[n\alpha_n-(n-1)\ \alpha_{n-1}]\, b_n < \infty$.*

Put $\alpha_n = n^{-1} \sum_{i=1}^{n} E(\phi(|X_i|))$ for $n \geqslant 1$. We first verify Condition (a) (of the above-mentioned corollary) ;

$$\Sigma P(|X_n| > n) \leqslant \Sigma P(\phi(|X_n|) \geqslant \phi(n))$$
$$\leqslant \Sigma E(\phi(|X_n|))/\phi(n) = \Sigma[n\alpha_n-(n-1)\alpha_{n-1}]/\phi(n) < \infty$$

(by (iii). To prove Condition (b), let $\epsilon < 0$. There is an integer $N_1 > 1$ such that for each $n \geqslant 1$,

$$n^{-1} \sum_{i=1}^{n} E(|X_i|\ I(|X_i| > N_1) < \epsilon/2\ ;$$

this is possible since $\{X_n\}$ is Cesàro uniformly integrable (see Remark 5 below). Next there is an integer $N > N_1$ such that for each $n \geqslant N$, $n^{-1} \sum_{i=1}^{N_1} E(|X_i|) < \epsilon/2$. Then for $n \geqslant N$,

$$\sum_{i=1}^{n} E(|X_i| I(|X_i| > i))$$

$$\leqslant \sum_{i=1}^{N_1} E(|X_i|) + \sum_{i=1}^{n} E(|X_i| > N_1)) < n\epsilon.$$

To prove Condition (c), it suffices to show that

$$\Sigma n^{-2}\, E(X_n^2\, I(A_n)) < \infty \qquad \ldots \quad (5)$$

where $A_n = [n^{1/4} \leqslant |X_n| \leqslant n]$, $n \geqslant 1$. For each $n \geqslant 1$, there is a z_n in the interval $[n^{1/4}, n]$ such that

$$\phi(z_n)/z_n^2 \leqslant 2 \inf \{\phi(x)/x^2 : n^{1/4} \leqslant x \leqslant n\}\ ;$$

note that the right side of the above inequality is positive. Then for $x \epsilon [n^{1/4}, n]$, we have

$$x^2 \leqslant 2nz_n\, \phi(x)/\phi(z_n) \qquad (\text{as } z_n \leqslant n)$$

$$\leqslant 2n^2\, \phi(x)/t_n \qquad (\text{by (i) and as } z_n \geqslant n^{1/4})$$

where $\quad t_n = n^{3/4}\, \phi(n^{1/4})$ for $n \geqslant 1$. Observe that

$$\Sigma\, n^{-2}\, E(X_n^2\, I(A_n)) \leqslant 2\ \Sigma\, E(\phi(|X_n|))/t_n$$

$$= 2\ \Sigma[n\alpha_n - (n-1)\alpha_n]/t_n.$$

So (5) will follow if we show that $\Sigma\ 1/t_n < \infty$ (use Lemma 2). For this purpose, we use Lemma 15 of Petrov (1975, 277–278) with $a_n = n^{1/4} - (n-1)^{1/4}$ for $n \geqslant 1$, $\psi(x) = \phi(|x|)/|x|$; here we are following the notation of Petrov (1975) and using Assumptions (i) and (iii). As $a_n \geqslant 1/(4n^{3/4})$ for each n and $t_n = n\, \psi(n^{1/4})$, we get $\Sigma\ 1/t_n < \infty$.

We finally prove Condition (d). There is a $t_0 > 0$ such that $\phi(t) \geqslant t$ for each $t \geqslant t_0$, and so $|x| \leqslant t_0 + \phi(|x|)$ which implies that for each $n \geqslant 1$, $n^{-1} \sum_{i=1}^{n} E(|X_i|) \leqslant t_0 + c$. □

Remark 5. The existence of a function ϕ with the properties (i) and (ii) of Theorem 4 is equivalent to the Cesàro uniform integrability of $\{X_n\}$. The proof is given in Appendix.

We next state and prove a result that arises in connection with the following interesting question. Now that Kolmogorov's SLLN is known to hold under more relaxed conditions, it is natural to ask whether the classical SLLN of Marcinkiewicz and Zygmund (see Theorem 2, page 122, of Chow and Teicher (1978)) also holds under such relaxed conditions. We come up with the following partial answer. Incidentally, *our result strengthens considerably a result of Calderon* (1983) *along similar lines.*

Theorem 7. *Let $\{X_n\}_{n \geqslant 1}$ be a sequence of pairwise independent random variables such that $E(X_n) \equiv 0$, and there is a random variable Y with $E(Y^p) < \infty$ for some $1 < p < 2$ and satisfying the condition that*

$$\sup_n P(|X_n| \geqslant \alpha) \leqslant CP(Y \geqslant \alpha) \ \forall \ \alpha \geqslant 0.$$

(Here C is a constant). Then

(a) $\sum_{n=1}^{\infty} P(|X_n|^p > n) < \infty$;

(b) $n^{-1/p} \sum_{i=1}^{n} |EY_i| \to 0$ *as* $n \to \infty$;

and

(c) *for every subsequence $\{k_n\}$ of positive integers such that $\liminf_{n\to\infty} (k_n/k_{n-1}) > 1$, one has*

$$(k_n)^{-1/p} \sum_{i=1}^{k_n} (Y_i - EY_i) \to 0 \text{ completely}$$

in the sense of Hsu and Robbins (1947) *(see also page* 225 *of Stout* (1974)), *where $Y_n = X_n I(|X_n|^p \leqslant n)$ for each $n \geqslant 1$.*

Proof. (a) $\sum_{n=1}^{\infty} P(|X_n|^p > n) \leqslant C \sum_{n=1}^{\infty} P(Y^p \geqslant n) \leqslant CE(Y^p) < \infty.$

(b) By Kronecker's lemma, it suffices to show that

$$\sum_{n=1}^{\infty} n^{-1/p} |EY_n| < \infty.$$

But the last series is

$$\leqslant \sum_{n=1}^{\infty} n^{-1/p} E(|X_n| I(|X_n|^p > n))$$

$$\leqslant C \sum_{n=1}^{\infty} n^{-1/p} E(YI(Y^p \geqslant n)).$$

Now put $A_n = [n^{1/p} \leqslant Y < (n+1)^{1/p}]$ for each $n \geqslant 1$, and note that

$$\sum_{p=1}^{j} i^{-1/p} \leqslant \int_{1}^{j+1} x^{-1/p}\, dx \leqslant d\, j^{(p-1)/p}$$

for a suitable constant d. Hence

$$\sum_{n=1}^{\infty} n^{-1/p} E(YI(Y^p \geqslant n)) = \sum_{n=1}^{\infty} n^{-1/p} \sum_{j=n}^{\infty} E(YI(Y \in A_j))$$

$$= \sum_{j=1}^{\infty} E(YI(Y \in A_j)) \sum_{n=1}^{j} n^{-1/p} \leqslant d \sum_{j=1}^{\infty} E(YI(Y \in A_j))\, j^{-(p-1)/p}$$

$$\leqslant d \sum_{j=1}^{\infty} E(Y^p I(Y \in A_j)) \leqslant d\, E(Y^p) < \infty,$$

since on A_j, we have $j^{-(p-1)/p}\, Y \leqslant Y^p$.

(c) We proceed as in the proof of Lemma 1, and define $b > 1$, the $n(j)$ and j_0 as before. Let $\delta > 0$. Now note that

$$\sum_{n=1}^{\infty} \delta^2 P\left(\left|\sum_{n=1}^{k_n} (Y_j - EY_j)\right| \geqslant k_n^{1/p}\delta \right)$$

$$\leqslant \sum_{n=1}^{\infty} (k_n)^{-2/p} \sum_{j=1}^{k_n} E(X_j^2 I(|X_j|^p \leqslant j))$$

$$\leqslant \sum_{n=1}^{\infty} (k_n)^{-2/p} \sum_{k=1}^{k_n} \int_{0}^{j^{2/p}} P(|X_j|^2 > x)dx$$

$$\leqslant C \sum_{n=1}^{\infty} (k_n)^{-2/p} \sum_{j=1}^{k_n} \int_{0}^{j^{2/p}} P(Y^2 > x)dx$$

$$\leqslant C \sum_{j=1}^{\infty} (k_n)^{-2/p} \sum_{j=1}^{k_n} [E(Y^2 I(Y^p < j)) + j^{2/p} P(Y^p \geqslant j))].$$

Next observe that

$$\sum_{n=1}^{\infty} (k_n)^{-2/p} \sum_{j=1}^{k_n} j^{2/p} P(Y^p \geqslant j) = \sum_{j=1}^{\infty} P(Y^p \geqslant j) j^{2/p} \sum_{n : k_n \geqslant j} (k_n)^{-2/p}$$

$$= \sum_{j=1}^{\infty} P(Y^p \geqslant j) j^{2/p} \sum_{n=n(j)}^{\infty} (k_n)^{-2/p} \leqslant (1-b^{-2/p})^{-1} \sum_{j=1}^{\infty} P(Y^p \geqslant j) < \infty.$$

Finally, with the set A_n defined in the proof of (b),

$$\sum_{n=1}^{\infty}(k_n)^{-2/p}\sum_{j=1}^{\infty}E(Y^2I(Y^p<j))=\sum_{j=1}^{\infty}E(Y^2I(Y^p<j))\sum_{n=n(j)}^{\infty}(k_n)^{-2/p}$$

$$\leqslant(1-b^{-2/p})^{-1}\sum_{j=1}^{\infty}j^{-2/p}E(YI(Y^p<j))$$

$$=(1-b^{-2/p})^{-1}\sum_{j=0}^{\infty}j^{-2/p}\sum_{k=0}^{j-1}E(Y^2I(Y\in A_k))$$

$$=(1-b^{-2/p})^{-1}\sum_{k=0}^{\infty}E(Y^2I(Y\in A_k))\sum_{j=k+1}^{\infty}j^{-2/p}$$

$$\leqslant d\sum_{k=0}^{\infty}(k+1)^{(p-2)/p}E(Y^2I(Y\in A_k))\leqslant d\sum_{k=0}^{\infty}E(Y^2Y^{p-2}I(Y\in A_k))\ (\text{as } p<2)$$

$$\leqslant d\,E(Y^p)<\infty,$$

where d is a suitable constant. □

Modifying the proof of Theorem 2 of Etemadi (1981) along the line of arguments used to prove Theorem 2, one can establish the following result.

Theorem 8. *Let $\{X_{m,n}\}_{m\geqslant 1,\,n\geqslant 1}$ be a double sequence of pairwise independent random variables such that there exist a random variable Y and a constant $C\geqslant 0$ satisfying the conditions*

$$P(|X_{m,n}|\geqslant\alpha)\geqslant CP(Y\geqslant\alpha)\ \forall\ \alpha\geqslant 0,\ \ \forall\ m,n$$

and

$$E(Y\log^+ Y)<\infty.$$

Then $(mn)^{-1}(S_{m,n})\to 0$ almost surely as $(m,n)\to\infty$ where $S_{m,n}=\sum_{i=1}^{m}\sum_{j=1}^{n}X_{i,j}$.

Csörgő *et al.* (1983) prove Corollary 1 with $f(n)=n$; we shall now show that a *general* SLLN like Corollary 1 automatically yields a corresponding SLLN for *weighted converges* (see in this connection Etemadi (1983b) who used *separate* arguments for such an extension) ; to this end, one needs only replace X and $f(n)$ by respectively $w_i X_i$ and $\sum_{i=1}^{n} w_i$ where $\{w_n\}$ stands for the weights. We are thus able to strengthen Theorem 1 and Corollary 2 of Etemadi (1983b) ; as observed in Etemadi (1983b), Theorems 2–4 of Jamison *et al.* (1965) remain true for *pairwise* independent random variables.

Appendix

Consider the statement (A) below for a sequence $\{X_n\}_{n\geqslant 1}$ of integrable random variables :

(A) there is a measurable function $\phi:(0,\infty)\to(0,\infty)$ such that $\phi(t)/t \to\infty$ as $t\to\infty$ *and* $\sup_{n\geqslant 1}\left[n^{-1}\sum_{i=1}^{n}E(\phi(|X_i|))\right]=c$ (say) $<\infty$. *If* (A) *holds, then* $\{X_n\}$ *is Cesàro uniformly integrable.* For if $\epsilon>0$, there is an integer $N\geqslant 1$ such that $\phi(t)\geqslant t(c+1)/\epsilon$ for $t>N$, and so for each $n\geqslant 1$,

$$\sum_{i=1}^{n}E(|X_i|I(|X_i|>N))$$

$$\leqslant \epsilon\sum_{i=1}^{n}E(\phi(|X_i|)I(|X_i|>N))]/(c+1)<n\epsilon.$$

The converse implication is more interesting, and is analogous to a classical result of La Vallée Poussin (see, e.g., page 19 of Meyer (1966)) on uniform integrability.

Theorem. *Let $\{X_n\}_{n\geqslant 1}$ be Cesàro uniformly integrable. Then (A) holds; moreover, ϕ can be chosen so that $\phi(t)/t$ is increasing and ϕ is convex.*

Proof. 1°. For any sequence $\{u_n\}_{n\geqslant 1}$ of nonnegative reals with $u_n\uparrow\infty$ as $n\uparrow\infty$, if we put

$$\phi(t)=\int_0^t g(x)dx,\quad (t>0)$$

($\phi(0)=0$), where $g:(0,\infty)\to(0,\infty)$ is defined as

$$g(x)=u_n \text{ if } n-1\leqslant x<n \quad (n\geqslant 1),$$

then $\phi(t)/t\uparrow\infty$ as $t\uparrow\infty$; such a function ϕ is convex (see, e.g., Theorem A, page 9, of Roberts and Varberg (1973)).

Since $\beta(t):=\phi(t)/t$ is continuous on $(0,\infty)$ and is differentiable in each interval $(n-1,n)$, $n\geqslant 1$, to see that $\beta(t)$ is increasing, it suffices to prove that $\beta'(t)\geqslant 0$ for $t\in(n-1,n)$ for each $n\geqslant 1$. Fix an $n\geqslant 1$ and a $t\in(n-1,n)$. Note that $\phi(t)=\sum_{i=0}^{n-1}u_i+(t-n+1)u_n$ $(u_0=0)$ so that $t^2\beta'(t)=\sum_{i=1}^{n}(u_n-u_i)\geqslant 0$. Now that β is shown to be increasing, to show that $\beta(t)\to\infty$ as $t\to\infty$, it suffices to show that $\beta(n)\to\infty$ as $n\to\infty$. But $\beta(n)=n^{-1}\sum_{i=1}^{n}u_i\to\infty$ as $n\to\infty$, since $u_n\to\infty$ as $n\to\infty$.

2°. It remains to show that such a sequence $\{u_n\}_{n\geqslant 1}$ can be chosen so as to also satisfy

$$\sup_n \left[\quad n^{-1} \sum_{i=1}^{n} E(\phi(|X_i|)) \right] \leqslant 1. \qquad \ldots \quad \text{(A1)}$$

where ϕ is defined through $\{u_n\}$ as in 1°.

To this end, we first use the Cesàrc uniform integrability of $\{X_n\}$ to get hold of a sequence $\{N_j\}_{j\geqslant 1}$ of positive integers such that for each $j \geqslant 1$,

$$\sup_n \left\{ n^{-1} \sum_{n=1}^{n} E(|X_i| I(|X_i| \geqslant N_j)) \right\} \leqslant 2^{-j},$$

and $N_j \to \infty$ as $j \to \infty$. We now put for each $n \geqslant 1$,

$$u_n = \text{card}\,\{j \geqslant 1 : N_j < n\}.$$

(Note that since $N_j \to \infty$ as $j \to \infty$, the set $\{j \geqslant 1 : N_j < n\}$ is finite for each $n \geqslant 1$.) Clearly, $\{u_n\}$ is increasing. Finally, for any $M \geqslant 1$, there is an integer $n_0 \geqslant 1$ such that $N_j < n_0$ for each $j = 1, \ldots, M$ which implies that $u_n \geqslant M$ for each $n \geqslant n_0$; thus $u_n \to \infty$ as $n \to \infty$.

We now establish (A1). Note that

$$\phi(t) \leqslant \sum_{i=1}^{n} u_i \text{ for } t \in (n-1, n],\ n \geqslant 1. \qquad \ldots \quad \text{(A2)}$$

Next note that for any $k \geqslant 1$,

$$E(\phi(|X_k|)) \leqslant E\Big[\sum_{n=1}^{\infty} I(n-1 < |X_k| \leqslant n) \sum_{i=1}^{n} u_i \Big] \qquad \text{(by (A2))}$$

$$= E\Big[\sum_{i=1}^{\infty} u_i \Big(\sum_{n=i}^{\infty} I(n-1 < |X_k| \leqslant n) \Big) \Big] = \sum_{i=1}^{\infty} u_i P(|X_k| > i-1).$$

So for any $n \geqslant 1$,

$$\sum_{k=1}^{n} E(\phi(|X_k|)) \leqslant \sum_{k=1}^{n} \sum_{m=1}^{\infty} \Big(\sum_{j\,:\,N_j<m} 1 \Big) P(|X_k| > m-1)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{n} \sum_{m=N_j}^{\infty} P(|X_k| > m) \leqslant \sum_{j=1}^{\infty} \sum_{k=1}^{n} E(|X_k| I(|X_k| \geqslant N_j))$$

$$\leqslant n \sum_{j=1}^{\infty} 2^{-j} \leqslant n.$$

Here we have used the result of 3° below.

3°. For any integrable random variable X and any integer $N \geqslant 1$

$$\sum_{m=N}^{\infty} P(|X| \geqslant m) \leqslant E(|X| I(|X| \geqslant N)).$$

For, if $j \geqslant 1$,

$$\sum_{m=N}^{N+j} E(|X| I(m \leqslant |X| < m+1)) \geqslant \sum_{m=N}^{N+j} m\, P(m \leqslant |X| < m+1)$$

$$= \sum_{m=N}^{N+j} mP(|X| \geqslant m) - \sum_{m=N}^{N+j} mP(|X| \geqslant m+1)$$

$$\geqslant \sum_{m=N}^{N+j+1} P(|X| \geqslant m) - (N+j+1)P(|X| \geqslant N+j+1)$$

Letting $j \to \infty$, we get the desired inequality. □

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