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CHARACTERIZATION OF UNIFORM DISTRIBUTIONS BY INEQUALITIES OF CHERNOFF-TYPE

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SUMMARY. A Chernoff-type inequality is obtained for uniform distribution on $[-1,1]$. Subsequently, uniform distribution is characterized among all distributions on $[-1,1]$ having symmetric unimodal densities via this inequality.

1. Introduction

Chernoff (1981) proved that if $X \sim N(0, 1)$, then for any absolutely continuous function g with $E[g^2(X)]$ finite,

$$V[g(X)] \leqslant E\{[g'(X)]^2\} \qquad \ldots \quad (1.1)$$

with equality iff $g(x)$ is linear. Borovkov and Utev (1983) have characterized standard normal distribution via the inequality (1.1). Several other authors have established inequalities analogous to (1.1), which subsequently we refer to as Chernoff-type inequalities, in connection with distributions other than normal or in more general setting and studied the related characterization problems. (Cacoullos (1982), Cacoullos and Papathanasiou (1985, 1986), Chen (1982, 1985), Chen and Lou (1987), Hwang and Shen (1987), Klaassen (1985), Prakasa Rao and Sreehari (1986a, 1986b), Srivastava and Sreehari (1987)).

Our aim in this note is to study Chernoff-type inequalities for distributions on $[-1, 1]$ having symmetric unimodal densities. All our results are in this context and are proved in Section 2. Our work essentially comprises of obtaining a Chernoff-type inequality for $U[-1, 1]$ distribution (Theorem 2.1), which is subsequently made use of in order to study this kind of inequalities for distributions on $[-1, 1]$ having symmetire unimodal densities (Theorem 2.2). Finally $U[-1, 1]$ distribution is characterized among all distributions on $[-1, 1]$ having symmetric unimodal densities through this kind of inequalities (Theorem 2.3)

2. Main results

At first we obtain a Chernoff-type inequality for $U[-1, 1]$ distribution.

Theorem 2.1 : *Let* $X \sim U[-1, 1]$. *Then*

$$\sup_g \frac{E\{[g(X)]^2\}}{E\{[g'(X)]^2\}} = \frac{4}{\pi^2} \qquad \ldots \quad (2.1)$$

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where the supremum is taken over all absolutely continuous functions $g : [-1, 1] \to \mathcal{R}$ for which $g(0) = 0$ and $E\{[g(X)]^2\}/E\{[g'(X)]^2\}$ is well-defined.

Proof : Note that in order to obtain the supremum in (2.1) it suffices to restrict our attention only to the even functions. We are, therefore, required to establish the following :

$$\sup_g \frac{\int_0^1 [g(x)]^2 dx}{\int_0^1 [g'(x)]^2 dx} = \frac{4}{\pi^2}. \qquad \dots \quad (2.2)$$

Observe, however, that in order to establish (2.2) it suffices to consider only those functions g such that g' is constant on each of $\left(\frac{i-1}{n}, \frac{i}{n}\right)$, $1 \leqslant i \leqslant n$, for some $n \geqslant 1$. But, it is easy to see that for such a function g,

$$\frac{\int_0^1 [g(x)]^2 dx}{\int_0^1 [g'(x)]^2 dx} = \frac{1}{n^2}\left[\frac{\boldsymbol{x}'\boldsymbol{A}_n\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}} - \frac{2}{3}\right] \qquad \dots \quad (2.3)$$

where $\boldsymbol{x}' = (x_1, \dots, x_n)$ with $x_i = g'(x)$ for $x \in \left(\frac{n-i}{n}, \frac{n-i+1}{n}\right)$, $1 \leqslant i \leqslant n$ and $\boldsymbol{A}_n = ((a_{ij}^{(n)}))_{1 \leqslant i, j \leqslant n}$ is defined by

$$a_{ij}^{(n)} = \begin{cases} i & \text{if } \; i = j \\ \min(i, j) - \frac{1}{2} & \text{if } \; i \neq j. \end{cases}$$

Therefore, it is enough to show that

$$\sup_{n \geqslant 1} \frac{\lambda_n}{n^2} = \frac{4}{\pi^2}, \qquad \dots \quad (2.4)$$

where λ_n = maximum eigenvalue of $\boldsymbol{A}_n$.

Let us now observe that if we define $\boldsymbol{B}_n = ((b_{ij}^{(n)}))_{1 \leqslant i, j \leqslant n}$ by $b_{ij}^{(n)} = \min(i, j)$, then for any $\boldsymbol{x} \in \mathcal{R}^n$ with $\boldsymbol{x} \neq \boldsymbol{0}$, we get

$$\frac{\boldsymbol{x}'\boldsymbol{B}_n\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}} - \frac{n}{2} \leqslant \frac{\boldsymbol{x}'\boldsymbol{A}_n\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}} \leqslant \frac{\boldsymbol{x}'\boldsymbol{B}_n\boldsymbol{x}}{\boldsymbol{x}'\boldsymbol{x}} + \frac{n}{2} \qquad \dots \quad (2.5)$$

so that

$$\frac{\alpha_n}{n^2} - \frac{1}{2n} \leqslant \frac{\lambda_n}{n^2} \leqslant \frac{\alpha_n}{n^2} + \frac{1}{2n} \qquad \dots \quad (2.6)$$

where α_n = maximum eigenvalue of $\boldsymbol{B}_n$.

We now obtain in the following lemma a set of numbers (for each n) which contains α_n. It will be used later to establish (2.4).

Lemma : *Define* $c_{m,n} = \dfrac{1}{2\left(1-\cos\dfrac{2m+1}{2n+1}\pi\right)}$, $\begin{array}{l} m=-n, \ldots, 0, \ldots, n \\ n=1, 2, \ldots \end{array}$

Then $\alpha_n \in \{c_{m,n} : -n \leqslant m \leqslant n\}$.

Proof (of the lemma) : It is easy to see that $\boldsymbol{B}_n$ is a positive definite matrix so that

$$\alpha_n = \max. \left\{\lambda : p_n\left(\frac{1}{\lambda}\right) = 0\right\}$$

where $p_n(x) = |\boldsymbol{B}_n^{-1} - x\boldsymbol{I}_n|$.

Now $\boldsymbol{B}_n^{-1} = \boldsymbol{H}_n = ((h_{ij}^{(n)}))_{1 \leqslant i, j \leqslant n}$ is given by

$$h_{ii}^{(n)} = 2 \quad \text{for } 1 \leqslant i \leqslant n-1,\ h_{nn}^{(n)} = 1$$

$$h_{ij}^{(n)} = -1 \text{ for } |i-j| = 1, \qquad h_{ij}^{(n)} = 0 \text{ for } |i-j| > 1.$$

From this we get

$$p_n(x) = (2-x)\,p_{n-1}(x) - p_{n-2}(x),\ n \geqslant 3 \qquad \ldots \quad (2.7)$$

$$p_2(x) = x^2-3x+1,\ p_1(x) = 1-x. \qquad \ldots \quad (2.8)$$

On solving the difference equation (2.7), subject to the condition (2.8), it can be seen immediately that

$$p_n\left(\frac{1}{x}\right) = 0 \Longrightarrow \left(\frac{1+\sqrt{1-4x}}{1+\sqrt{1-4x}}\right)^{2n+1} = 1. \qquad \ldots \quad (2.9)$$

The rest of the proof consists of routine algebraic computation and so we omit it.

With the previous steps in mind, we are now prepared to establish (2.4).

Write $\sup\limits_{n \geqslant 1} \dfrac{\lambda_n}{n^2} = a$ and obtain a subsequence $\{\lambda_{n_j}/n_j^2\}$ of $\{\lambda_n/n^2\}$ converging to a. In view of (2.6) this implies that $\alpha_{n_j}/n_j^2 \to a$. We prove that

$$a = \frac{4}{\pi^2}. \qquad \ldots \quad (2.10)$$

Observe that $\alpha_{n_j} = c_{m_j, n_j}$ for some m_j with $-n_j \leqslant m_j \leqslant n_j$. But the constants $c_{m,n}$ defined in the lemma satisfies $c_{-m,n} = c_{m-1,n}$ for $m = 1, \ldots, n$ and for every n, $c_{0,n} > c_{1,n} > \ldots > c_{n,n}$. Moreover, $\lim\limits_{n\to\infty} \dfrac{c_{0,n}}{n^2} = \dfrac{4}{\pi^2}$ and $\lim\limits_{n\to\infty} \dfrac{c_{1,n}}{n^2} = \dfrac{4}{9\pi^2}$. So, if we show that $a > \dfrac{4}{9\pi^2}$, (2.10) follows trivially.

To see this, note that with $g(x) = x$, the ratio appearing in the left-hand side of (2.3) becomes $\frac{1}{3}$ which is larger than $\frac{4}{9\pi^2}$, so that a must be greater than $\frac{4}{9\pi^2}$.

Therefore, $a = \frac{4}{\pi^2}$ and in view of our early discussions this completes the proof of our theorem.

Remark 2.1 : Klaassen (1988) has given an alternative proof of the above theorem.

Remark 2.2 : The supremum in (2.1) is attained for $g(x) = C. \sin. \frac{\pi x}{2}$ for any constant C. This choice of g is suggested by Corollary 4.3 of Chen and Lou (1987).

Theorem 2.2 : *Suppose X is an absolutely continuous random variable on* $[-1, 1]$ *with a symmetric unimodal density $f(x)$ having mode at* 0. *Then,*

$$E(X^2) \leqslant \sup_g \frac{E\{[g(X)]^2\}}{E\{[g'(X)]^2\}} \leqslant \frac{8}{\pi^2} . E|X| \qquad \dots \quad (2.11)$$

where the supremum is taken over all absolutely continuous functions $g : [-1, 1] \to \mathcal{R}$ *such that g is even, concave on* $[0, 1]$, $g(0) = 0$ *and* $E\{[g(X)]^2\}/E\{[g'(X)]^2\}$ *is well-defined.*

Proof : Note that the lower bound can be obtained by taking $g(x) = |x|$.

Observe now that the supremum in (2.11) will remain the same, even if we restrict our attention only to those functions g having non-negative derivative on $[0, 1]$. Now with any such g, we have

$$E\{[g(X)]^2\} = 2 \int_0^1 [g(x)]^2 f(x) \, dx$$

$$= -2 \int_0^1 [g(x)]^2 \left(\int_x^1 df(u) \right) dx + 2f(1) \int_0^1 [g(x)]^2 \, dx$$

$$= -2 \int_0^1 \left(\int_0^u [g(x)]^2 \, dx \right) df(u) + 2f(1) \int_0^1 [g(x)]^2 \, dx$$

$$\leqslant -2 \int_0^1 \frac{4u^2}{\pi^2} \left(\int_0^u [g'(x)]^2 \, dx \right) df(u) + 2f(1) \int_0^1 [g(x)]^2 \, dx$$

(using Theorem 2.1 and the fact that f is non-increasing on [0, 1])

$$\leqslant -2 \int_0^1 \frac{4u^2}{\pi^2} \left(\int_0^u [g'(x)]^2 \, dx \right) df(u) + 2.f(1).\frac{4}{\pi^2} \int_0^1 [g'(x)]^2 \, dx$$

$$= \frac{8}{\pi^2} \left(\int_0^1 [g'(x)]^2 dx \right) \left(-2 \int_0^1 u^2 \, df(u) + f(1) \right)$$

$$= \frac{8}{\pi^2} \left(\int_0^1 [g'(x)]^2 dx \right) . \left(2 \int_0^1 u \, f(u) du \right)$$

$$= \frac{8}{\pi^2} \left(\int_0^1 [g'(x)]^2 dx \right) . E|X| .$$

The proof will now be completed if we show that

$$\int_0^1 [g'(x)]^2 dx \leqslant 2 \int_0^1 [g'(x)]^2 f(x) dx. \qquad \dots \quad (2.12)$$

To see this, note that both the functions $2f(x)$ and $h(x) \equiv 1$ are densities on [0, 1] and moreover $f(x)$ and $[g'(x)]^2$ are decreasing functions, the later being a consequence of the facts that g is concave and has non-negative derivative on [0, 1]. It is then easy to establish (2.12).

This completes the proof of our theorem.

Remark 2.3 : The left-hand inequality in (2.11) is most stringent in the following sense :

Given $\epsilon > 0$, $\exists$ a random variable X on $[-1, 1]$ having a symmetric, unimodal density f with mode at 0 such that

$$1 \leqslant \sup_g \frac{E\{[g(X)]^2\}}{E\{[g'(X)]^2\} E(X^2)} \leqslant 1+\epsilon. \qquad \dots \quad (2.13)$$

This can be seen by truncating a $N\left(0, \frac{1}{t^2}\right)$ variable on $[-1, 1]$ for a suitable $t > 0$.

Remark 2.4 : If in Theorem 2.2, we take X to be a non-uniform random variable, then

$$\sup_g \frac{E\{[g(X)]^2\}}{E\{[g'(X)]^2\}} < \frac{4}{\pi^2}. \qquad \dots \quad (2.14)$$

This is an immediate consequence of (2.11) and the fact that for any non-uniform random variable X on $[-1, 1]$ having a symmetric unimodal density f with mode at 0 we have $E|X| < \frac{1}{2}$.

Remark 2.5 : If in Theorem 2.2 we take $X \sim U[-1, 1]$, then the supremum considered in (2.11) is $\frac{4}{\pi^2}$.

To see this, consider $g(x) = \left|\sin \frac{\pi x}{2}\right|$ and note that this g satisfies the conditions of Theorem 2.2. Then, with this g,

$$\frac{E\{[g(X)]^2\}}{E\{[g'(X)]^2\}} = \frac{4}{\pi^2}.$$

Hence, in view of Theorem 2.1 the supremum in (2.11) must be $\frac{4}{\pi^2}$.

Observe now that from Theorem 2.2 and Remarks 2.4, 2.5 we have the following characterization of uniform distribution on $[-1, 1]$.

Theorem 2.3 : *Suppose X is an absolutely continuous random variable on $[-1, 1]$ with a symmetric unimodal density $f(x)$ having mode at 0. Then*

$$\sup_g \frac{E\{[g(X)]^2\}}{E\{[g'(X)]^2\}} = \frac{4}{\pi^2} \Longleftrightarrow X \sim U[-1, 1],$$

where the supremum is taken over all absolutely continuous functions $g : [-1, 1] \to \mathcal{R}$, such that g is even, concave on $[0, 1]$, $g(0) = 0$ and $E\{[g(X)]^2\}/E\{[g'(X)]^2\}$ is well-defined.

Remark 2.6 : In an earlier draft of this paper we put it as an open question whether in the above theorem we can drop the restriction of concavity of g on $[0, 1]$, other conditions remaining unchanged.

Later, Klaassen (1988) has observed that the answer is "yes".

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