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CROSSINGS OF BROWNIAN MOTION : A SEMI-MARTINGALE APPROACH

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SUMMARY. In this paper we study the crossings of an interval [a, b] by a Brownian motion. We do this by constructing an associated continuous semi-martingale which tracks the crossings of [a, b] by the Brownian motion. An analysis of the associated semi-martingale leads to asymptotic relations between crossings and local time, a probabilistic Taylor's formula and a new proof of Levy's crossing theorem.

0. Introduction

The origins of the current work goes back to an earlier paper (Rajeev, 1989) where we studied the sojourn times of margingales in an interval [a, b] in time t. The main result in Rajeev (1989) was a relationship between these sojourn times and the number of crossings of $[a, b]$ by the martinglale in time t (see Thm. I of Rajeev, 1989). More specifically we proved that if (X_t) is a continuous square integrable martingale and $X_0 \notin [a, b]$ almost surely, then,

$$E \int_0^t I_{[a,b]}(X_s) d < X >_s = (b-a)^2 \, E \, C^{X,t}_{[a,b]} + E(X_t - X_{\sigma_t})^2 \qquad \dots \quad (1)$$

where $C^{X,t}_{[a,b]}$ denotes the total number of crossings of $[a, b]$ in time t by X and σ_t is the last exit time before t, of X, from $[a, b]^c$. The arguments used to prove (1) are elementary, but two limiting cases of (1) are interesting. We consider the specific case of Brownian motion. Firstly, dividing (1) throughout by $(b-a)$ and noting that $(X_t - X_{\sigma_t})^2 \leqslant (b-a)^2$ we are led to

$$\operatorname*{Lt}_{b \to a} (b-a) \, E \, C^{X,t}_{[a,b]} = \operatorname*{Lt}_{b \to a} E \left(\frac{1}{b-a} \int_0^t I_{[a,b]}(X_s) \right) ds.$$

It is well known that the limit in the RHS is nothing but $E \, \Phi(t, a)$, where $\Phi(t, a)$ is the local time at a of the Brownian motion (X). We recognise this as an expected version of Levy's crossing theorem (see Sec. 2 for relevant references).

Secondly, dividing (1) throughout by $EC^{X,t}_{[a,b]}$ and noting that this increases to ∞ as $t \to \infty$, we get

$$\operatorname*{Lt}_{t \to \infty} \frac{E \int_0^t I_{[a,b]}(X_s) \, ds}{EC^{X,t}_{[a,b]}} = (b-a)^2.$$

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This says that the expected time spent in $[a, b]$ per crossing is $(b-a)^2$.

Thus eqn. (1) though simple, is revealing. The two observations we have just made certainly call for a closer scrutiny of (1). We proceed as follows :

Let $$Y_t = (b-a)^2 C^{X,t}_{[a,b]} + (X_t - X_{\sigma_t})^2$$

and $$Y'_t = Y_t - \int_0^t I_{[a,b]}(X_s)\, ds.$$

It is easily seen that Y_t is a continuous $\mathscr{F}_t$—adapted process and so is Y'_t. Eqn. (1) then says that $EY'_t = 0$ for all t. It can be shown that $EY'_\tau = 0$ for all bounded stop times τ. In other words the process (Y'_t) is a martingale or equivalently (Y_t) is a semi-martingale. This is a semi-martingale associated with the crossings of $[a, b]$ by X in time t. The term $(X_t - X_{\sigma_t})^2$ corresponds to the unfinished crossing at time t.

In this article, we study using techniques developed in Rajeev (1989), the process

$$Z_t = (b-a)\, C^{X,t}_{[a,b]} + |X_t - X_{\sigma_t}|.$$

We show in Section 1 that this is a (non-negative) submartingale and determine its martingale and bounded variation parts (Thm. 1.1). This fact leads to several interesting consequences which we develop in Section 2. These include an asymptotic relationship between local times and crossings (Thm. 2.4.1) and also a new proof of Levy's Crossing theorem (Th. 2.6.1). An application of Ito's formula shows that the process (Y_t) mentioned above is a semi-martingale (Corollary 2.3.1). We thus get back the results of Rajeev (1989). We also generalize the second of the limit theorems mentioned above, which leads to a probabilistic Taylor's formula. Finally we mention that in Rajeev (1989b) we have proved Thm. 1.1 for a continuous semi-martingale using the more recent Meyer-Yor theory of local times for semi-martingales (see Meyer, 1976, 361-371 and Yor (1976).

1. A SEMI-MARTINGALE ASSOCIATED WITH CROSSINGS

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_t)_{t \geqslant 0}$ a filtration on it satisfying usual conditions. Let (X_t) be a continuous square integrable martingale. The process $\{ < X >_t,\ t \geqslant 0\}$ will as usual denote the quadratic variation of the process (X_t). Let $\Phi(t, x)$ denote a jointly continuous version of the local time of (X_t). For a detailed account of the properties of $\Phi(t, x)$ see Meyer (1976) and Yor (1976). In particular for any bounded Borel function f on the line $\Phi(t, x)$ satisfies

$$\int_0^t f(X_s)\, d < X >_s = \int_R f(a)\Phi(t, a)\, da \qquad \ldots \quad (2)$$

Further $\Phi(t, x)$ is explicitly given by Tanaka's formula as follows :

$$\frac{1}{2}\Phi(t, x) = (X_t - x)^+ - (X_0 - x)^+ - \int_0^t I_{(x, \infty)}(X_s) dX_s. \qquad \dots \quad (3)$$

For any $a < b$ and $t > 0$, let

$U^{X, t}_{[a, b]}$ = number of upcrossings of the interval $[a, b]$ upto time t by the process (X_t),

= the largest integer k such that there are pairs $(t_i, s_i)_{i=1}^k$ with $X_{t_i} < a$ and $X_{s_i} > b$ and $0 \leqslant t_1 < s_1 < t_2 \dots < t_k < s_k \leqslant t$.

Similarly let $D^{X, t}_{[a, b]}$ be the number of down-crossings of $[a, b]$ upto time t by (X_t), and let

$$C^{X, t}_{[a, b]} = U^{X, t}_{[a, b]} + D^{X, t}_{[a, b]}$$

= total number of crossings upto time t.

For $t > 0$, we define

$$\tau_t = \inf\{u > 0 : X_u \notin [a, b]\} \wedge t$$

= first exit time from $[a, b]$ before t.

$$\sigma_t = \max\ \{u : \tau_t \leqslant u < t, X_u \notin [a, b]\}$$

= the first time after τ_t beyond which the path remains in $[a, b]$ upto t, if such a time exists, otherwise it is t.

For a few properties of σ_t, see Rajeev (1989). Note that X_{σ_t} is $\mathcal{F}_t$ measurable and further

$$X_{\sigma_t} = a \text{ or } b \text{ if } X_t \in [a, b] \text{ and } \tau_t < t$$

$$= X_t \text{ if } X_t \notin [a, b] \text{ or } \tau_t = t.$$

In particular $|X_t - X_{\sigma_t}| \leqslant b - a$. From these observations it is not difficult to see that the process $(b-a)\, C^{X, t}_{[a, b]} + |X_t - X_{\sigma_t}|$ is a continuous $\mathcal{F}_t$-adapted process.

From now on we fix a continuous square integrable martingale (X_t) with local time $\Phi(t, x)$. We fix $a < b$ and $t > 0$. We now define two sequences of stop times (η^u_k) and (η^d_k) as follows :

$$\eta^d_0 = \inf\{s > 0 : X_s > b\} \wedge t$$

$$\eta^d_1 = \inf\{s > \eta^d_0, X_s < a\} \wedge t$$

$$\vdots$$

$$\eta^d_{2k} = \inf\{s > \eta^d_{2k-1}, X_s > b\} \wedge t$$

$$\eta^d_{2k+1} = \inf\{s > \eta^d_{2k}, X_s < a\} \wedge t$$

$$\vdots$$

Let $k^d(\omega) = \min\{k : \eta^d_{2k+1} = t\}$ Then for all $n < k^d$, $\eta^d_{2n+1} < t$. Similarly we define (η^u_k) as

$$\eta^u_0 = \inf\{s > 0 : X_s < a\} \wedge t$$
$$\eta^u_1 = \inf\{s > \eta^u_0, X_s > b\} \wedge t$$
$$\vdots$$
$$\eta^u_{2k} = \inf\{s > \eta^u_{2k-1}, X_s < a\} \wedge t$$
$$\eta^u_{2k+1} = \inf\{s > \eta^u_{2k}, X_s > b\} \wedge t$$
$$\vdots$$

Let $k^u(\omega) = \inf\{k : \eta^u_{2k+1} = t\}$. Then for all $n < k^u$, $\eta^u_{2n+1} < t$. Let

$$\psi^d(s, \omega) = \sum_{k=0}^{\infty} I_{(\eta^d_{2k}, \eta^d_{2k+1}]}(s)$$
$$\psi^u(s, \omega) = \sum_{k=0}^{\infty} I_{(\eta^u_{2k}, \eta^u_{2k+1}]}(s)$$
$$\psi(s, \omega) = \psi^u(s, \omega) - \psi^d(s, \omega).$$

We note that ψ is a bounded predictable function and that $\psi^u(s)$. $\psi^d(s) = 0$.

Let $\varphi(s, \omega) = I_{(a,b]}(X_s)$. We can now state our main result.

Theorem 1.1 : *For any $a < b$, the process $(b-a)\, C^{X; t}_{[a, b]} + |X_t - X_{\sigma_t}|$ is an $\mathcal{F}_t$-semi-martingale and we have*

$$(b-a) C^{X; t}_{[a, b]} + |X_t - X_{\sigma_t}| = \int_0^t \varphi(s)\psi(s) dX_s + \frac{1}{2}(\Phi(t, a) + \Phi(t, b)). \quad \ldots \quad (4)$$

To prove the theorem we need the following lemma which can be considered an L_2 version of Levy's crossing theorem.

Lemma 1.1 : *For any $b \in R$ and $t > 0$,*

$$\underset{\varepsilon \to 0}{\text{Lt}}\ \varepsilon\, D^{X; t}_{[b, b+\varepsilon]} \overset{L_2}{=} \frac{1}{2}\Phi(t, b).$$

Proof : For $\varepsilon > 0$, we define the following sequence of stop times :

$$\sigma^\varepsilon_1 = \inf\{s > 0 : X_s > b+\varepsilon\}$$
$$\tau^\varepsilon_1 = \inf\{s > \sigma^\varepsilon_1 : X_s < b\}$$
$$\vdots$$
$$\sigma^\varepsilon_k = \inf\{s > \tau^\varepsilon_{k-1}, X_s > b+\varepsilon\}$$
$$\tau^\varepsilon_k = \inf\{s > \sigma^\varepsilon_k, X_s < b\}.$$
$$\vdots$$

$$\text{Let } f^\varepsilon(t, \omega) = \sum_1^\infty I_{(\sigma_k^\varepsilon, \tau_k^\varepsilon]}(t, \omega)$$

Then f^ε is a bounded $\mathcal{F}_t$-predictable function and we have for any $t > 0$

$$\int_0^t f^\varepsilon(s) dX_s = \sum_{k=1}^\infty (X_{\tau_k^\varepsilon \wedge t} - X_{\sigma_k^\varepsilon \wedge t})$$

$$= -\varepsilon D^{X; t}_{[b, b+\varepsilon]} + (X_t - b - \varepsilon) f^\varepsilon(t, \omega) - (X_0 - (b+\varepsilon))^+ . \quad \ldots \quad (5)$$

As $\varepsilon \to 0$, almost surely, $f^\varepsilon(t, \omega) \to I_{\{x > b\}}(X_t)$ for almost every t, $d < X >$. This can be seen as follows : Fix any (t, ω). If $X_t < b$, then $f^\varepsilon(t, \omega) = 0$ for all ε. If $X_t > b$, then for sufficiently small ε, there exists $k = k(\varepsilon)$ such that $t \in [\sigma_k^\varepsilon, \tau_k^\varepsilon]$. In other words $f^\varepsilon(t, \omega) = 1$ for sufficiently small ε. Since by equation (2) we have almost surely

$$d < X > \{t : X_t = b\} = 0$$

the proof of the claim is complete. Hence

$$\int_0^t f^\varepsilon(s)\, dX_s \to \int_0^t I_{\{x > b\}}(X_s) dX_s \text{ in } L_2.$$

Also it is easy to see that $(X_t - b - \varepsilon) f^\varepsilon(t, \omega) \to (X_t - b)^+$ almost surely and hence in L_2 by dominated convergence. Similarly $(X_0 - (b+\varepsilon))^+ \to (X_0 - b)^+$ in L_2. Now taking $\varepsilon \to 0$ in (5) and using eqn. (3) we have the stated result.

Remark 1.1 : Similarly we can prove that

$$\operatorname*{Lt}_{\varepsilon \to 0} \varepsilon \; U^{X; t}_{[b-\varepsilon, b]} \overset{L_2}{=} \frac{1}{2} \Phi(t, b).$$

Proof of *Theorem* 1.1 : We first note that if $a_n \uparrow a$ then $\Phi(t, a_n) \to \Phi(t, a)$ in L_2. This is an easy consequence of the Tanaka formula for the local time given by eqn. (3). Fix a sequence (a_n) strictly increasing to a. By Lemma 1 we choose an a_n' such that $a_n < a_n' < a_{n+1}$ and such that

$$\| (a_n' - a_n) D^{X, t}_{[a_n, a_n']} - \Phi(t, a_n) \|_{L_2} < \frac{1}{2^n}$$

$$\text{So } \operatorname*{Lt}_{n \to \infty} (a_n' - a_n)\, D^{X, t}_{[a_n, a_n']} \overset{L_2}{=} \Phi(t, a).$$

Similarly fix a sequence (b_n) strictly decreasing to b, and for each n fix b_n' such that $b_{n+1} < b_n' < b_n$ and

$$\operatorname*{Lt}_{n \to \infty} (b_n - b_n')\, U^{X, t}_{[b_n', b_n]} \overset{L_2}{=} \Phi(t, b).$$

Now we define a sequence of stop times (σ_k^n) and (τ_k^n) as follows :

$$\sigma_0^n = 0$$
$$\tau_0^n = \inf\{0 < s \leqslant t : X_s < a_n \text{ or } X_s > b_n\} \wedge t$$
$$\sigma_1^n = \inf\{\tau_0^n < s \leqslant t : |X_s - a| < |a - a_n'| \text{ or } |X_s - b| < |b - b_n'|\} \wedge t$$
$$\vdots$$
$$\sigma_k^n = \inf\{\tau_{k-1}^n < s \leqslant t : |X_s - a| < |a - a_n'| \text{ or } |X_s - b| < |b - b_n'|\} \wedge t$$
$$\tau_k^n = \inf\{\sigma_k^n < s \leqslant t : X_s < a_n \text{ or } X_s > b_n\} \wedge t$$
$$\vdots$$

Let $k_n(\omega) = \min \; \{k : \tau_k^n = t\}$. Then $k_n(\omega) < \infty$ almost surely, $\tau_k^n = t$ for all $k \geqslant k_n$ and $\sigma_k^n = t$ for all $k > k_n$.

Let $E_n = \overset{\infty}{\underset{k=0}{U}} (\sigma_k^n, \tau_k^n)$. Then it is easy to show that (see Rajeev, 1989a)

(1) $E_{n+1} \subset E_n$

(2) $\{0 < s < t : X_s \in [a, b]\} = \bigcap_n E_n$. ... (6)

Let $\varphi_n(s, \omega) = \sum_{k=0}^{\infty} I_{(\sigma_k^n, \tau_k^n]}(s)$. Then φ_n are a sequence of bounded predictable functions and almost surely $\varphi_n(s) \to \varphi(s)$ for almost every $s, d < X >$.

Hence $$\int_0^t \varphi_n(s)\psi(s)dX_s \xrightarrow{L_2} \int_0^t \varphi(s)\psi(s)dX_s$$

But $$\int_0^t \varphi_n(s)\psi(s)dX_s = \int_0^t \varphi_n(s)\psi^u(s)dX_s - \int_0^t \varphi_n(s)\psi^d(s)dX_s$$
$$= I_1 - I_2.$$

We now look at the integral I_1 more carefully. The integrand in I_1 is

$$\varphi_n(s)\psi^u(s) = \Big(\sum_{k=0}^{\infty} I_{(\sigma_k^n, \tau_k^n]}(s)\Big)\Big(\sum_{j=0}^{\infty} I_{(\eta_{2j}^u, \eta_{2j+1}^u]}(s)\Big)$$

$$= \sum_{j,k} I_{(\sigma_k^n \vee \eta_{2j}^u, \tau_k^n \wedge \eta_{2j+1}^u]}(s)$$

We note that all the sums involved are finite amost surely. Hence

$$\int_0^t \varphi_n(s)\psi^u(s)dX_s = \sum_{j,k} \big(X_{(\tau_k^n \wedge \eta_{2j+1}^u)\wedge t} - X_{(\sigma_k^n \vee \eta_{2j}^u)\wedge t}\big)$$

Each summand on the right side takes the values $(b-a)$, $(a_n - a_n')$, $(b - a_n')$ or $(a_n - a)$, except when $(j, k) = (0, 0)$ or $(k_n, 2k^u)$. Accordingly we can write

$$\int_0^t \varphi_n(s)\, \psi^u(s)\, dX_s = \sum_1^6 S_i$$

To explain S_i, let us denote by $U([a, b])$ etc. the number of upcrossings of $[a, b]$ etc. by (X_t) in the time interval $[(\tau_0^n \wedge \eta_1^u) \wedge t, (\sigma_{k_n}^n \vee \eta_{2k^u}^u) \wedge t]$. We denote by $U([c, d], [a, b])$ the number of upcrossings of $[c, d]$ during the time interval $[(\tau_0^n \wedge \eta_1^u) \wedge t, (\sigma_{k_n}^n \vee \eta_{2k^u}^u) \wedge t]$ which contain at least one crossing (up or down) of $[a, b]$. A similar notation is used for the downcrossings. We also note that for all ω and for sufficiently large n, there are no upcrossings of $[a, b]$ or downcrossings of $[a_n, a_n']$ in either of the time intervals $[(\sigma_0^n \vee \eta_0^u) \wedge t, (\tau_0^n \wedge \eta_1^u) \wedge t]$ or $[(\sigma_{k_n}^n \vee \eta_{2k^u}^u, t]$. With the above notation we have,

$$S_1 = X_{(\tau_0^n \wedge \eta_1^u) \wedge t} - X_{(\sigma_0^n \vee \eta_0^u) \wedge t}$$

$$S_2 = (b-a) \{U([a, b]) - U([a_n', b]) - D([a_n, a_n'], [a, b])\}$$

$$S_3 = (a_n - a_n') \{D([a_n, a_n']) - D([a_n, b_n']) - D([a_n, a_n'], [a, b])\}$$

$$S_4 = (b - a_n') \{U([a_n', b_n]) + D([a_n, a_n'], [a, b])$$

$$S_5 = (a_n - a) \{D([a_n, b_n'] + D([a_n, a_n'], [a, b])\}$$

$$S_6 = X_{(\tau_{k_n}^n \wedge \eta_{2k^u+1}^u) \wedge t} - X_{(\sigma_{k_n}^n \vee \eta_{2k^u}^u) \wedge t}$$

Since $\tau_0^n \downarrow \tau_t$, the first exit time from $[a, b]$ before t, S_1 tends to zero almost surely. Since (see Rajeev, 1989), $\sigma_{k_n}^n \uparrow \sigma_t$ and $\tau_{k_n}^n = t$, $\eta_{2k^u+1}^u = t$, S_6 converges to $\left(X_t - X_{\sigma_t \vee \eta_{2k^u}^u}\right)$ almost surely and in L_2. It is easy to see that $S_2 + S_4$ increases to $(b-a)\ U_t([a, b])$. Regarding the first term in S_3, we note that

$$(a_n - a_n')\, D\, ([a_n, a_n']) = (a_n - a_n').\ D_t\, ([a_n, a_n']) \xrightarrow{L_2} -\frac{1}{2}\, \Phi\, (t, a)$$

as noted in the beginning of the proof. The other two terms in S_3 are bounded by $(a_n - a_n')\ D_t\ ([a, b])$ and since $a - a_n' \to 0$, these two terms tend to zero. Similarly S_5 converges to zero almost surely and in L_2.

Hence $$\operatorname*{Lt}_{n \to \infty} \int_0^t \varphi_n(s) \psi^u(s) dX_s \overset{L_2}{=} (b-a)\, U^{X,t}_{[a,b]} + \left(X_t - X_{\sigma_t \vee \eta_{2k^u}^u}\right) - \frac{1}{2}\, \Phi(t, a).$$

In an analogous fashion we get,

$$\operatorname*{Lt}_{n \to \infty} \int_0^t \varphi_n(s) \psi^d(s) dX_s \overset{L_2}{=} (a-b) D^{X,t}_{[a,b]} + \left(X_t - X_{\sigma_t \vee \eta_{2k^d}^d}\right) + \frac{1}{2}\, \Phi(t; b).$$

Subtracting the above two equations, the proof is complete.

Remark 1.2 : Since $\psi(s) = 0$ for $0 \leq s < \tau_t$, where τ_t is the first exit time of the process (X_t) from $[a, b]$, we see that for $s \in [0, \tau_t]$ both sides of eqn. (4) are identically zero.

In a similar manner, by taking $\psi(s) = 1$ in Theorem 1.1 we can prove the following :

Theorem 1.2 : $\left(X_{\tau_t} - X_0\right) + (b-a) [U^{X,t}_{[a,b]} - D^{X,t}_{[a,b]} + \left(X_t - X_{\sigma_t}\right)$

$$= \int_0^t I_{(a,b]}(X_s) dX_s + \frac{1}{2}(\Phi(t, a) - \Phi(t, b)). \qquad \dots (7)$$

But it is interesting to note that eqn. (7) can be quickly derived from Tanaka's formula (3) as follows : Using (3) at $X = a$ and $X = b$ and subtracting we get

$$(X_t - a)^+ - (X_0 - a)^+ - (X_t - b)^+ + (X_0 - b)^+$$
$$= \int_0^t I_{(a,b]}(X_s)\, dX_s + \frac{1}{2}(\Phi(t, a) - \Phi(t, b)).$$

It is now a matter of verification to see that the LHS above is precisely the LHS of eqn. (7).

2. Some consequences of theorem 1.1

In this section we detail some of the consequences of Theorem 1.1.

2.1 Theorem 1.1 says that the process $(b-a)C^{X,t}_{[a,b]} + |X_t - X_{\sigma_t}|$ is a non-negative submartingale whose increasing part in the Doob-Meyer decomposition is $\frac{1}{2}(\Phi(t, a) + \Phi(t, b))$. The asymptotic behaviour of the process (X_t) gets reflected in the convergence of this submartingale. For example, if (X_t) is Brownian motion then $\overline{\text{Lt}}_{t \to \infty} X_t = \pm \infty$ implies that this sub-martingale converges to ∞ almost surely as $t \to \infty$ for all $a < b$.

2.2 We have used Tanaka's formula in deriving Theorem 1.1. Conversely by letting $b \to \infty$ in Theorem 1.1 we can get back Tanaka's formula. Note that for fixed t, ω, $C^{X,t}_{[a,b]} = 0$ for large b. $\Phi(t, b)$ is also zero for large b, since it is supported on the set $\{s : X_s = b\}$. Strictly speaking, letting $b \to \infty$ gives Tanaka formula only in the interval $[\tau, \infty)$ where τ is the first hit of $(-\infty, a)$ by (X_t). But on $[0, \tau)$, $X_{\tau \wedge t} - X_0$ itself does the job.

2.3 In addition to the remarks in 2.1 above, Theorem 1.1 says that $|X_t - X_{\sigma_t}|$ is in fact a (special) semi-martingale. This follows immediately if we note that $(b-a)C^{X,t}_{[a,b]}$ is a left continuous (and hence predictable) increasing process. In particular it follows that $(b-a)C^{X,t}_{[a,b]}$ is the sum of the jumps upto time t of the semi-martingale $|X_t-X_{\sigma_t}|$. Now for a smooth function f, we can apply Ito's formula to $|X_t-X_{\sigma_t}|$ to get the following lemma, when (X_t) is a Brownian motion.

Lemma 2.3.1 : *Let (X_t) be a Brownian motion, and f a smooth function. Then,*

$$\begin{aligned}(f(b-a)-f(0))C^{X,t}_{[a,b]}+f(|X_t-X_{\sigma_t}|)-f(0)\\ =\int_0^t f'(|X_s-X_{\sigma_s}|_-)\psi(s)\varphi(s)dX_s\\ +\frac{1}{2}\int_0^t f''(|X_s-X_{\sigma_s}|_-)\varphi(s)\,ds\\ +\frac{1}{2}f'(0)\,(\Phi(t,a)+\Phi(t,b)).\end{aligned} \quad \ldots \quad (8)$$

Proof : From Theorem 1.1 we have,

$$|X_t-X_{\sigma_t}| = \int_0^t \psi(s)\varphi(s)dX_s+\frac{1}{2}(\Phi(t,a)+\Phi(t,b))-(b-a)C^{X,t}_{[a,b]}$$

For a smooth function f, we recall that the Ito's formula applied to a semi-martingale (Y_t) gives

$$\begin{aligned}f(Y_t) = f(Y_0) + \int_0^t f'(Y_{s-})dY_s+\frac{1}{2}\int_0^t f''(Y_{s-})d<Y^c>_s\\ +\sum_{0<s\leq t}(f(Y_s)-f(Y_{s-})-f'(Y_{s-})\Delta Y_s)\end{aligned}$$

where $<Y^c>$ denotes the quadratic variation of the continuous martingale part of Y and ΔY_s is the jump of Y at time s. With $Y_t = |X_t-X_{\sigma_t}|$, one has the following :

$$\int_0^t f''(Y_{s-})d<Y^c>_s = \int_0^t f''(|X_s-X_{\sigma_s}|_-)\varphi(s)ds$$

$$\begin{aligned}\sum_{0<s\leq t}(f(Y_s)-f(Y_{s-})-f'(Y_{s-})\Delta Y_s)\\ = -(f(b-a)-f(0))C^{X,t}_{[a,b]}+f'(b-a)(b-a)C^{X,t}_{[a,b]}\end{aligned}$$

$$\begin{aligned}\int_0^t f'(Y_{s-})dY_s = \int_0^t f'(|X_s-X_{\sigma_s}|_-)\psi(s)\varphi(s)dX_s\\ +\frac{1}{2}\int_0^t f'(|X_s-X_{\sigma_s}|_-)\,(\Phi(ds,a)+\Phi(ds,b))\\ -f'(b-a)\,(b-a)C^{X,t}_{[a,b]}.\end{aligned}$$

Hence,

$$f(|X_t - X_{\sigma_t}|) - f(0) = \int_0^t f'(|X_s - X_{\sigma_s}|_-)\psi(s)\varphi(s)dX_s$$

$$+\frac{1}{2}\int_0^t f'(|X_s - X_{\sigma_s}|_-)(\Phi(ds, a)+\Phi(ds, b))$$

$$-(f(b-a)-f(0))C^{X,t}_{[a,b]}+\frac{1}{2}\int_0^t f''(|X_s - X_{\sigma_s}|_-)\varphi(s)ds.$$

Since for the Brownian motion, the hitting time of $[b, \infty)$, (b, ∞) are almost surely equal and since $\Phi(ds, a)$, $\Phi(ds, b)$ are supported on the sets $\{s : X_s = a\}$, $\{s : X_s = b\}$ respectively, we have

$$\int_0^t f'(|X_s - X_{\sigma_s}|_-)(\Phi(ds, a)+\Phi(ds, b) = f'(0)(\Phi(t, a)+\Phi(t, b)).$$

Substituting this into the expression derived above for $f(|X_t - X_{\sigma_t}|)$, the proof is complete.

Taking expectations in eqn. (8) with $f(x) = x^2$, we get back the relationship between sojourn time in $[a, b]$ and the number of crossings of $[a, b]$.

Corollary 2.3.1 :

$$E\left(\int_0^t I_{[a,b]}(X_s)ds\right) = E(X_{\tau_t} - X_0)^2 + (b-a)^2\, EC^{X,t}_{[a,b]} + E(X_t - X_{\sigma_t})^2$$

where τ_t is the first exit time of (X_t) from $[a, b]$ before t.

2.4 We shall show in Section 2.6 that Levy's crossing theorem follows from Theorem 1.1 on letting $b \to a$. We now derive, as a consequence of Theorem 1.1, a closely related result. The crossing theorem says that $(b-a)C^{X,t}_{[a,b]} \sim \Phi(t, a)$ as $b \to a$. The parameter t is of course fixed. We will show that the above relationship holds even if we let $t \to \infty$ i.e. $(b-a)C^{X,t}_{[a,b]} \sim \Phi(t, a)$ as $t \to \infty$. More precisely we have the following theorem.

Theorem 2.4.1 : *Let (X_t) be a Brownian motion and $a < b$. Then almost surely,*

$$\operatorname*{Lt}_{t\to\infty} \frac{\Phi(t, a)}{C^{X,t}_{[a,b]}} = (b-a)$$

Proof : We will show that almost surely,

$$\operatorname*{Lt}_{t\to\infty} \frac{\Phi(t, a)+\Phi(t, b)}{2C^{X,t}_{[a,b]}} = (b-a) \qquad \dots \quad (9)$$

and $$\underset{t\to\infty}{\text{Lt}} \frac{\Phi(t, a)-\Phi(t, b)}{2C^{X,t}_{[a,b]}} = 0 \qquad \dots \quad (10)$$

Adding eqns. (9) and (10) the proof is complete. We will prove only eqn. (9) using Theorem 1.1. The proof of eqn. (10) is similar, using Theorem 1.2.

We assume without loss that $X_0 = a$. Let (τ_n) be the successive crossing times of the interval $[a, b]$. Observe that

$$\int_0^{\tau_{n+1}} \psi(s)\,\varphi(s)dX_s = \sum_{k=1}^{n+1} (-1)^{k-1} \int_{\tau_{k-1}}^{\tau_k} \varphi(s)dX_s$$

$$= \sum_k \int_0^\infty I_{(0,b-a]}(Y_k(s))dY_k(s)$$

$$- \sum_k \int_0^\infty I_{(-(b-a),0]}(Z_k(s))dZ_k(s)$$

where $$Y_k(s) = X_{\tau_{2k+1}\wedge s}-X_{\tau_{2k}\wedge s} \quad k = 0, 1, 2, \dots$$

$$Z_k(s) = X_{\tau_{2k}\wedge s}-X_{\tau_{2k-1}\wedge s} \quad k = 1, 2, \dots,$$

The strong renewal property of the Brownian motion and the strong law of large numbers imply that almost surely,

$$\frac{1}{n}\int_0^{\tau_{n+1}} \psi(s)\varphi(s)dX_s \to 0 \text{ as } n\to \infty.$$

Hence by Theorem 1.1 we conclude that

$$\underset{n\to\infty}{\text{Lt}} \frac{\Phi(\tau_{n+1}, a)+\Phi(\tau_{n+1}, b)}{2n} = (b-a).$$

If now $\tau_n < t \leqslant \tau_{n+1}$, then $n-1 < C^{X,t}_{[a,b]} \leqslant n$ and hence

$$\frac{1}{2}\left(\frac{n-1}{n}\right)\left(\frac{\Phi(\tau_n, a)+\Phi(\tau_n, b)}{n-1}\right) < \frac{1}{2}\,\frac{(\Phi(t, a)+\Phi(t, b))}{C^{X,t}_{[a,b]}}$$

$$\leqslant \frac{1}{2}\left(\frac{n}{n-1}\right)\left(\frac{\Phi(\tau_{n+1},a)+\Phi(\tau_{n+1},b)}{n}\right)$$

eqn. (9) follows on letting $n \to \infty$.

An easy consequence of Theorem 2.4.1 is the following intuitively obvious result.

Corrollary 2.4.1 : *For any a, b*

$$\underset{t\to\infty}{\text{Lt}} \frac{\Phi(t, a)}{\Phi(t, b)} = 1 \textit{ almost surely.}$$

Proof : The proof is straightforward using Theorems 1.2 and 2.4.1.

Remark 2.4.1 : Theorem 2.4.1, and its Corollary together imply that for any x,

$$\operatorname{Lt}_{t\to\infty} \frac{C^{X,t}_{[a,b]}}{\Phi(t,x)} = \frac{1}{(b-a)} \text{ almost surely.}$$

Corollary : *Let $a < b$ and $c < d$. Then,*

$$\operatorname{Lt}_{t\to\infty} \frac{C^{X,t}_{[a,b]}}{C^{X,t}_{[c,d]}} = \frac{d-c}{b-a} \textit{ almost surely.}$$

2.5 We now derive a limit theorem which is a generalization of the following result :

Theorem 2.5.1 : *If (X_t) is a Brownian motion and $a < b$, then almost surely*

$$\frac{\int_0^t I_{(a,b)}(X_s)ds}{C^{X,t}_{[a,b]}} \to (b-a)^2 \textit{ as } t \to \infty.$$

For a proof of this result see Rajeev and Rao (1988). Prof. Meyer has drawn our attention to results of Burdzy *et al* (1987) where a more general result in the context of Hunt processes is obtained.

To start with, take expectations in eqn. (8) and divide through out by $EC^{X,t}_{[a,b]}$ and let $t\to\infty$ to get

$$\operatorname{Lt}_{t\to\infty} \frac{E\int_0^t f''(|X_s-X_{\sigma_s}|_-)\varphi(s)ds}{EC^{X,t}_{[a,b]}} = f(b-a)-f(0)-\tfrac{1}{2}f'(0)\operatorname{Lt}_{t\to\infty}\frac{E(\Phi(t,a)+\Phi(t,b))}{EC^{X,t}_{[a,b]}}$$

By Theorem 1.1 the limit in the RHS is just $2(b-a)$. Hence we have the following lemma.

Lemma 2.5.1 : *Let (X_t) be a Brownian motion, $a < b$, and f a smooth function. Then*

$$f(b-a)-f(0)-f'(0)(b-a) = \operatorname{Lt}_{t\to\infty}\frac{E\left(\int_0^t f''(|X_s-X_{\sigma_s}|_-)\varphi(s)ds\right)}{EC^{X,t}_{[a,b]}}.$$

We now look at the almost sure version of the above result. More precisely we will prove the following theorem.

Theorem 2.5.2 : *Let (X_t) be a Brownian motion, $a < b$, and f a smooth function. Then almost surely*

$$f(b-a)-f(0) = f'(0)(b-a) + \operatorname{Lt}_{t\to\infty}\frac{\int_0^t f''(|X_s-X_{\sigma_s}|_-)\varphi(s)ds}{2C^{X,t}_{[a,b]}} \qquad \dots \quad (11)$$

Remark 2.5.1 : It is interesting to compare the above result with the classical Taylor's formula of differential calculas. The remainder term is now probabilistic ! Also note that Theorem 2.5.1 follows by taking $f(x) = x^2$ in the above theorem.

Proof of Theorem 2.5.2 : We assume without loss of generality that $X_0 = a$ almost surely. Let $\tau_k, k = 1, 2, \ldots$ be the successive crossing times.

Let
$$Y_k(s) = X_{\tau_{2k+1}\wedge s} - X_{\tau_{2k}\wedge s} \quad k = 0, 1, 2, \ldots$$
$$Z_k(s) = X_{\tau_{2k}\wedge s} - X_{\tau_{2k-1}\wedge s} \quad k = 1, 2, 3, \ldots .$$

We note that
$$f''(|X_s - X_{\sigma_s}|_-)\varphi(s) = f''(Y_k(s))I_{[0,b-a]}(Y_k(s)),\ \tau_{2k} \leqslant s < \tau_{2k+1}$$
$$= f''(-Z_k(s))I_{[-(b-a),0]}(Z_k(s)),\ \tau_{2k-1} \leqslant s < \tau_{2k}.$$

Hence
$$\int_0^{\tau_{s+1}} f''(|X_s - X_{\sigma_s}|_-)\varphi(s)ds = \sum_k \int_0^\infty f''(Y_k(s))I_{[0,b-a]}(Y_k(s))I_{[\tau_{2k},\tau_{2k+1})}(s)ds$$
$$+ \sum_k \int_0^\infty f''(-Z_k(s))I_{[-(b-a),0]}(Z_k(s))I_{[\tau_{2k-1},\tau_{2k})}(s)ds.$$

Now by the strong renewal property of the Brownian motion the two sums in the RHS are in fact sums of i.i.d. random variables. Further during an upcrossing (respy. downcrossing) Y_k (respy. Z_k) behaves like a Brownian motion started at zero and stopped at $b-a$ (respy. $a-b$). Hence (see Rajeev, 1989, Cor. 2.1),
$$E\int_0^\infty f''(Y_k(s))I_{[0,b-a]}(Y_k(s))I_{[\tau_{2k},\tau_{2k+1})}(s)ds$$
$$= E\int_0^\infty f''(-Z_k(s))I_{[-(b-a),0]}(Z_k(s))I_{[\tau_{2k-1},\tau_{2k})}(s)ds$$
$$= 2[f(b-a) - f(0) - f'(0)(b-a)].$$

The strong law of large numbers now implies that almost surely
$$\underset{n\to\infty}{\mathrm{Lt}} \frac{1}{2n}\int_0^{\tau_n+1} f''(|X_s - X_{\sigma_s}|_-)\varphi(s)ds = f(b-a) - f(0) - f'(0)(b-a). \quad \ldots \quad (12)$$

Now we observe that for fixed ω and $\tau_n \leqslant t < \tau_{n+1}$
$$\frac{1}{C^X_{[a,b]}}\int_0^t f''(|X_s - X_{\sigma_s}|_-)\varphi(s)ds = \left(1-\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\int_0^{\tau_n} f''(|X_s - X_{\sigma_s}|_-)\varphi(s)ds$$
$$+\frac{1}{C^X_{[a,b]}}\left(\int_{\tau_n}^t f''(|X_s - X_{\sigma_s}|_-)\varphi(s)ds\right)$$

The second term in the RHS can be dominated by $C_0\left(\frac{1}{C^{X,t}_{[a,b]}}\int\limits_{\tau_n}^{t}\varphi(s)ds\right)$, where C_0 is a constant. By Theorem 2.5.1, this tends to zero almost surely as $t\to\infty$. From eqn. (12) it follows that the first term converges to the required limit. This completes the proof of the theorem.

Remark 2.5.2 : The above theorem can also be proved directly using Theorem 1.1. We give only a sketch of the proof : Let τ_n be the successive crossing times. Lemma 2.3.1. together with Theorem 2.4.1 gives

$$\underset{t\to\infty}{\text{Lt}}\ \frac{1}{2C^{X,t}_{[a,b]}}\int\limits_0^t f''(|X_s-X_{\sigma_s}|_-)\varphi(s)ds$$

$$=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{2n}\int\limits_0^{\tau_{n+1}} f''(|X_s-X_{\sigma_s}|_-)\varphi(s)ds$$

$$=f(b-a)-f(0)-f'(0)(b-a)+\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\int\limits_0^{\tau_{n+1}} f'(|X_s-X_{\sigma_s}|_-)\psi(s)\varphi(s)dX_s.$$

By the strong renewal property of the Brownian motion and the strong law of large numbers

$$\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\int\limits_0^{\tau_{n+1}} f'(|X_s-X_{\sigma_s}|_-)\psi(s)\varphi(s)dX_s\to 0 \text{ almost surely.}$$

This completes (the sketch) of the proof.

Remark 2.5.3 : It is worth noting that the last term on the right side of eqn. (11) is almost surely constant (as is also clear from the equation) because $\bigcap\limits_{t>0}\sigma(X_s, s\geqslant t)$ is trivial. Thus in particular, lemma 2.5.1 and Theorem 2.5.2 together imply the following ratio ergodic theorem

$$\underset{t\to\infty}{\text{Lt}}\ \frac{\int\limits_0^t f''(|X_s-X_{\sigma_s}|_-)\varphi(s)ds}{C^{X,t}_{[a,b]}}=\underset{t\to\infty}{\text{Lt}}\ \frac{E\left(\int\limits_0^t f''(|X_s-X_{\sigma_s}|_-)\varphi(s)ds\right)}{EC^{X,t}_{[a,b]}}.$$

2.6 We now use Theorem 1.1 to prove Levy's crossing theorem. Levy conjectured the result in the following form : If (X_t) is a Brownian motion, then $\underset{b\to a}{\text{Lt}}\ (b-a)D^{[X],t}_{[a,b]}=\Phi(t,a)$ almost surely. The first proof of this striking result, using excursions of the Brownian motion seems to be in Ito-Mckean (1965). Subsequently several proofs have been given in the literature. Chung and Durret (1976) use theta functions, Williams (1977) uses elementary arguments involving poisson processes, Ito (see Ikeda and Watanble (1981)) uses excursion theory, Maisonneuve (1976) uses regenerative systems. A different proof is outlined in Stroock (1982). In this context, Meyer (1976) discusses the uniform integrability of the random variables $(b-a)\ U^{X,t}_{[a,b]}$ when X is a martingale. On the other hand, Karoui (1976) has shown that this result is true for general semi-martingales, both continuous and discontinuous. Here we treat essentially the Brownian motion case, using a different technique, which is suggested quite naturally by Theorem 1.1.

The idea is to exploit the structure of the semi-martingale $|X_t - X_{\sigma_t}|$ and show that the martingale part tends to zero as $b \to a$. To do this we use some estimates for moments of stochastic integrals viz. $E\left(\int\limits_0^t I_{[a,b]}(X_s)dX_s\right)^k$. Such estimates have been obtained by Yor (see Yor (1976)) to establish the joint continuity properties of local times for semi-martingales. These also play a crucial role in the proof given in Karoui (1976).

Let (X_t) be a continuous square integrable martingale. Then we have the following lemma.

Lemma 2.6.1 : *There exists a constant $C_{k,t}$ such that for all $k \geqslant 1$,*

$$E\left(\int\limits_0^t I_{[a,b]}(X_s)dX_s\right)^k \leqslant C_{k,t}(b-a)^k\, E < X >_t^{k/2}.$$

Proof : With $f = I_{[a,b]}$, eqn. (2) gives

$$\int\limits_0^t I_{[a,b]}(X_s)\, d < X >_s = \int\limits_a^b \Phi(t, x)dx.$$

Raising to power k on both sides and taking expectations, the RHS can be dominated by $(b-a)^k \sup\limits_{x \in [a,b]} E[\Phi(t, x)]^k$, using Jensen's inequality. The required estimate is then obtained by using the Tanaka formula (2) for $\Phi(t, x)$, together with the $B-D-G$ inequalities. The details can be found in Yor (1976).

Using the above lemma and the $B-D-G$ inequalities it is easy to see that

$$E\left(\int\limits_0^t \varphi(s)\psi(s)dX_s\right)^{2k} \leqslant C_{k,t}(b-a)^k\, E < X >_t^{k/2} \qquad \dots \quad (13)$$

We now state and prove the crossing theorem.

Theorem 2.6.1 : *Let (X_t) be a continuous square integrable martingale and $\Phi(t, x)$ a jointly continuous version of its local time. Then for each x,*

$$\operatorname*{Lt}_{\epsilon \to 0}\ \epsilon\, C^{X,t}_{\left[x-\frac{\epsilon}{2},\, x+\frac{\epsilon}{2}\right]} = \Phi(t, x) \textit{ almost surely.}$$

Proof: With $a = x - \frac{\varepsilon}{2}$, $b = x + \frac{\varepsilon}{2}$, Theorem 1.1 gives

$$\int_0^t \varphi_\varepsilon(s)\psi_\varepsilon(s)dX_s = \varepsilon C^{X,t}_{\left[x-\frac{\varepsilon}{2},\, x+\frac{\varepsilon}{2}\right]} + |X_{\sigma_t} - X_t|$$

$$-\frac{1}{2}\left(\Phi\left(t, x-\frac{\varepsilon}{2}\right) + \Phi\left(t, x+\frac{\varepsilon}{2}\right)\right)$$

Applying (13) to $\int_0^t \varphi_\varepsilon(s)dX_s$, we get for all $k \geqslant 1$,

$$E\left(\int_0^t \varphi_\varepsilon(s)\psi_\varepsilon(s)X_s\right)^{2k} \leqslant C_{k,t}E < X >_t^{k/2} \cdot \varepsilon^k.$$

A standard application of the Borel-Cantelli lemma shows that

$$\underset{n\to\infty}{\mathrm{Lt}} \int_0^t \varphi_{1/n^2}(s)\psi_{1/n^2}(s)dX_s = 0 \text{ almost surely.}$$

Further we note that $|X_{\sigma_t^n} - X_t| < \frac{1}{n^2} \to 0$ almost surely, where $\sigma_t^n = \sigma_t$ for the interval $\left[x - \frac{1}{2n^2}, x + \frac{1}{2n^2}\right]$.

By the joint continuity of $\Phi(t, x)$,

$$\frac{1}{2}\left(\Phi\left(t, x-\frac{1}{2n^2}\right) + \Phi\left(t, x+\frac{1}{2n^2}\right)\right) \to \Phi(t, x) \text{ almost surely,}$$

so that,

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{1}{n^2} C^{X,t}_{\left[x-\frac{1}{2n^2},\, x+\frac{1}{2n^2}\right]} = \Phi(t, x) \text{ almost surely.}$$

Finally we note that if $\frac{1}{(n+1)^2} \leqslant \varepsilon < \frac{1}{n^2}$, then

$$\frac{1}{(n+1)^2} C^{X,t}_{\left[x-\frac{1}{2n^2},\, x+\frac{1}{2n^2}\right]} \leqslant \varepsilon C^{X,t}_{\left[x-\frac{\varepsilon}{2},\, x+\frac{\varepsilon}{2}\right]}$$

$$\leqslant \frac{1}{n^2} C^{X,t}_{\left[x-\frac{1}{2(n+1)^2},\, x+\frac{1}{2(n+1)^2}\right]}$$

Hence $$\underset{\varepsilon\to 0}{\mathrm{Lt}}\, \varepsilon C^{X,t}_{\left[x-\frac{\varepsilon}{2},\, x+\frac{\varepsilon}{2}\right]} = \Phi(t, x) \text{ almost surely.}$$

Using eqns. (4) and (7) of Section 1 and Lemma 2.6.1 we can prove in a similar manner.

Theorem 2.6.2 : $\underset{\epsilon \to 0}{\text{Lt}} \; \epsilon \, U^{X,t}_{\left[x-\frac{\epsilon}{2},\, x+\frac{\epsilon}{2}\right]} = \frac{1}{2} \Phi(t, x)$

and $$\underset{\epsilon \to 0}{\text{Lt}} \; \epsilon \, D^{X,t}_{\left[x-\frac{\epsilon}{2},\, x+\frac{\epsilon}{2}\right]} = \frac{1}{2} \Phi(t, x) \textit{ almost surely.}$$

Remark 2.6.1 : We conclude these investigations with an indication of the relationship between Theorem 2.6.1 and the 'classical' form of Levy's crossing theorem. Suppose we define

$U^{X;t}_{(a,b)}$ = the largest integers k, such that there are pairs $(t_i, s_i)_{i=1}^{k}$ with $X_{t_i} \leqslant a$ and $X_{s_i} \geqslant b$ and $0 \leqslant t_1 < s_1 < t_2 \ldots < t_k < s_k \leqslant t$.

We note that in the definition of $U^{X;t}_{[a,b]}$ in Sec. 1, strict inequality was used. Then of course for general semi-martingales $U^{X;t}_{(a,b)}$ and $U^{X;t}_{[a,b]}$ are different. For instance, if (X_t) is reflected Brownian motion then $D^{X;t}_{[0,a]} = 0$ always. On the other hand if (X_t) is Brownian motion, then the law of Iterated Logarithm (or a weaker version thereof) along with the strong markov property yields that for every fixed t, a, b, $D^{X;t}_{[a,b]} = D^{X;t}_{(a,b)}$ almost surely. As a consequence using arguments as in proof of Theorem 2.6.1 it is not difficult to see that in case (X_t) is Brownian motion,

$$\underset{\epsilon \to 0}{\text{Lt}} \; \epsilon \, D^{X;t}_{(0,\epsilon)} = \frac{1}{2} \Phi(t, 0) \text{ and}$$

and

$$\underset{\epsilon \to 0}{\text{Lt}} \; \epsilon \; U^{X,t}_{(-\epsilon,0)} = \frac{1}{2} \Phi(t, 0),$$

so that $\underset{\epsilon \to 0}{\text{Lt}} \; \epsilon \; D^{|X|;t}_{(0,\epsilon)} = \Phi(t, 0)$. This is the classical' version.

3. Conclusion

We conclude with a few comments on the proof of Theorem 1.1. We could have used techniques similar to that used in (8) to prove Theorem 1.1. But the proof we have given here is more elementary. We have essentially used only an L_2-version of Levy's crossing theorem and the fact that the limit in the crossing theorem has nice L_2 properties in the space variable. We have ued Tanaka's formula only to identify this limit as the local time, More importantly, the approximation (6) of the set $\{0 < s < t : X_s \in [a, b]\}$ by a decreasing limit of union of stochastic intervals, which plays a crucial role in our proof, works for any continuous $\mathcal{F}_t$ adapted process. This suggests a method for studying the process $|X_t - X_{\sigma_t}|$ for more general X—at least those for which the crossing theorem is true. Lastly, in the approximation (6) we have used crossings of closed intervals $[a_n, b_n] \supset [a, b]$. We could well have used open intervals $(a_n, b_n) \supset [a, b]$. In either case the result—Theorem 1.1—is the same. For the Brownian motion this can be expected—as we have noted earlier $C^{X,t}_{[a,b]} \equiv C^{X,t}_{(a,b)}$ almost surely. But for general semi-martingales the situation can be quite different.

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References

Burdzy, K., Pitman, J. N., Yor, M. (1987) : Some asymptotic laws for crossings and excursions (presonal communication).

Chung, K. L. and Durret, R. (1976) : Down crossings and local time. *Z. Wahr. Verw. Geb.*, **35**, 147-149.

Ito, K. and Mckean, J. P. (1965) : *Diffusion Processes and their Sample Paths*, Springer Verlag, Berlin.

Ikeda, N. and Watanabe, S. (1981) : *Stochastic Differential Equations and Diffusion Processes*, North Holland, 123-131.

Karoui, N. El, (1976) : Sur les montees des Semi-martingales. *Asterisque*, **52-53**, 63-88,

Meyer, P. A. (1976) : Un *Course* les Integrales Stochastiques, *Sem. de Prob.* X, 511.

Maisonneuve, B. (1976) : On Lévy's down crossing theorem and various extensions, Sem. de Prob. XV.

Rajeev, B. (1989a) : On sojourn times of martingales, *Sankhyā*, **51**, A, 1-12.

——— (1990) : On semi-martingales associated with crossings (to appear in *Sem. de Prob.* XXIV).

Rajeev, B. and Rao, B. V. (1988) : On a ratio limit theorem for martingales. (Pre-print).

Stroock, D. W. (1982) : Topics in stochastic differential equations, *TIFR*, Bombay.

Williams, David (1977) : Lévy's down crossing theorem. *Z. Wahr. Verw. Geb.*, **40**, 157-158.

Yor, Marc (1976) : Sur la Continuite des temps locaux associeise a certain Semi-martingales. *Asterisque*, **52-53**, 23-36.

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