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ON SOJOURN TIMES OF MARTINGALES

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SUMMARY. For continuous square integrable martingales we derive a formula for the expected time spent in an interval $[a, b]$ by the martingale in terms of the expected number of crossings of that interval. From this we deduce some new limit theorems for martingales. Some applications to stochastic integrals and local times are also given.

0. Introduction

Given a continuous martingale, an interval $[a, b]$ on the real line, and a time point $t > 0$, we look at the following two random variables viz. the time spent by the process in the interval $[a, b]$ upto time t, as measured by the quadratic variation process, and the number of crossings of $[a, b]$ made by the process upto time t. It is natural to ask what is the relationship between these two random variables. The answer to this question is our main result (Theorem 1). The rest of the paper is devoted to applications of our main result.

The paper is divided into two sections. Section 1 contains the main theorem. The proof of the theorem depends on a stochastic decomposition of the set $\{s \leqslant t : X_s \in [a, b]\}$. As a corollary we derive a new type of limit theorem for martingales. In Section 2, we develop further applications of our main theorem, which are also of independent interest. The main result is Theorem 2 where we calculate explicitly the expected value of the local time of the process at x, at the first time it hits 'c' viz. $E\,\varphi(\tau_c, x)$. The main applications here is the existence of certain stochastic integrals.

Let $(\Omega, \mathcal{F}, P)$ be a probability triple and let $(X_t)_{t \geqslant 0}$ be a continuous square integrable martingale. Let $(A_t)_{t \geqslant 0}$ be its quadratic variation process. For all w, let $\mu_A(w)$ be the measure induced on $[0, \infty)$ by the increasing function $t \to A_t(w)$. We fix $a < b$.

Let $U^{X, t}_{[a, b]}$ be the number of upcrossings of $[a, b]$ upto time t i.e. the largest integer k such that there are pairs $(t_i, s_i)_{i=1}^{k}$ with $X_{t_i} < a$ and $X_{s_i} > b$ and $0 \leqslant t_1 < s_1 < t_2 < s_2 < \ldots < t_k < s_k \leqslant t$. Similarly let $D^{X, t}_{[a, b]}$ be the number of down-crossings of $[a, b]$ upto time t. Let $C^{X, t}_{[a, b]} = U^{X, t}_{[a, b]} + D^{X, t}_{[a, b]}$ be the total number of crossings of $[a, b]$ upto time t.

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1. The main theorem

Let

$$\tau_t = \inf\{u > 0\ X_u \notin [a, b]\} \wedge t$$

$$= \text{first hitting time of } [a, b]^c \text{ before } t.$$

and $$\sigma_t = \inf\{u : \tau_t \leqslant u \leqslant t \text{ and } X_s \in [a, b]\ \forall\ s \in [u, t)\}$$

= the first time after τ_t beyond which the path remains in $[a, b]$ upto t, if such a time exists, otherwise it is t.

In other words τ_t is the hitting time of $[a, b]^c$ before t, whereas σ_t is the last exit time for $[a, b]^c$ after τ_t and before t. Thus the process is in $[a, b]$ during $[0, \tau_t)$ as well as $(\sigma_t, t]$. τ_t is an $(\mathcal{F}_u)$ stop time, but σ_t need not in general be a stop time. However it is easy to see that σ_t is $\mathcal{F}_t$ measurable. With this notation, we have the following theorem.

Theorem 1 :

$$E\mu_A\{s \leqslant t.\ x_s \in [a, b]\} = E(X_{\tau_t} - X_0)^2 + (b-a)^2 EC^{X;\,t}_{[a,\,b]} + E(X_t - X_{\sigma t})^2,$$

$$\left(\text{The LHS above can also be written as } E\int_0^t 1_{[a,b]}(X_s)\, dA_s\right).$$

Before proving the theorem, we define for each $n \geqslant 1$ and $k \geqslant 0$, the stop times σ_k^n, τ_k^n, as follows :

$$\sigma_0^n \equiv 0$$

$$\tau_0^n \equiv \inf\left\{\sigma_0^n < s \leqslant t :\ X_s < a - \frac{1}{n} \text{ or } X_s > b + \frac{1}{n}\right\} \wedge t$$

$$\sigma_1^n \equiv \inf\left\{\tau_0^n < s \leqslant t :\ |X_s - b| < \frac{1}{n+1} \text{ or } |X_s - a| < \frac{1}{n+1}\right\} \wedge t$$

$$\tau_1^n \equiv \inf\left\{\sigma_1^n < s \leqslant t :\ X_s > b + \frac{1}{n} \text{ or } X_s < a - \frac{1}{n}\right\} \wedge t$$

$$\vdots$$

$$\sigma_k^n \equiv \inf\left\{\tau_{k-1}^n < s \leqslant t :\ |X_s - b| < \frac{1}{n+1} \text{ or } |X_s - a| < \frac{1}{n+1}\right\} \wedge t$$

$$\tau_k^n \equiv \inf\left\{\sigma_k^n < s \leqslant t :\ X_s > b + \frac{1}{n} \text{ or } X_s < a - \frac{1}{n}\right\} \wedge t$$

$$\vdots$$

Let $k_n(w) = \min\{k \geqslant 1 :\ \tau_k^n(w) = t\}$. Then $k_n(w) < \infty$ almost surely because (X_u) is continuous. σ_k^n and τ_k^n are all $(\mathcal{F}_u)$ stop times bound by t and

$$\tau_k^n = t\ \forall\ k \geqslant k_n,\ \sigma_k^n = t\ \forall\ k > k_n.$$

Let $$E_n = \bigcup_{k=0}^{\infty} (\sigma_k^n, \tau_k^n).$$

We start with a series of observations.

*F*1. $E_{n+1} \subset E_n$: Indeed if $s \in E_{n+1}$ then for some k, $s \in (\sigma_k^{n+1}, \tau_k^{n+1})$. Hence $a - \frac{1}{n+1} < X_s < b + \frac{1}{n+1}$. But if $s \notin E_n$, then it is easy to see that either $X_s \geqslant b + \frac{1}{n+1}$ or $X_s \leqslant a - \frac{1}{n+1}$, a contradiction.

*F*2. $\{0 < s < t : X_s \in [a, b]\} = \bigcap_{n=1}^{\infty} E_n$. As noted above if $s \notin E_n$ then either $X_s \geqslant b + \frac{1}{n+1}$ or $X_s \leqslant a - \frac{1}{n+1}$ and hence $X_s \notin [a, b]$. If $s \in E_n \; \forall \; n$, then $a - \frac{1}{n} < X_s < b + \frac{1}{n} \; \forall \; n$ and hence $X_s \in [a, b]$.

*F*3. $\tau_0^n \downarrow \tau_t$: By *F*1, τ_0^n's are decreasing. If the path remains in $[a, b]$ upto t, then $\tau_0^n = t = \tau_t$ for large n. If $\tau_t < t$, then $X_s \in [a, b] \; \forall \; s < \tau_t$. Hence $\tau_0^n \geqslant \tau_t \; \forall \; n$. Hence $\tau_t \leqslant \operatorname{Lt}_{n \to \infty} \tau_0^n$. Also $\forall \; u < \operatorname{Lt}_{n \to \infty} \tau_0^n$, $X_u \in [a, b]$. So $\tau_t = \operatorname{Lt}_{n \to \infty} \tau_0^n$.

*F*4. $\sigma_{k_n}^n \uparrow \sigma_t$: By *F*1, $\sigma_{k_n}^n$'s are increasing. If $\tau_t = t$ then $\sigma_t = \sigma_{k_n}^n \; \forall \; n$. Let now $\tau_t < t$. By definition of σ_t, $\sigma_{k_n}^n \leqslant \sigma_t \; \forall \; n$. Moreover for any $n \geqslant 1$ and $u > \sigma_{k_n}^n$, $a - \frac{1}{n} \leqslant X_u < b + \frac{1}{n}$, so that $\sigma_t \leqslant \operatorname{Lt}_{n \to \infty} \sigma_{k_n}^n$, which completes the proof.

*F*5. From the definition of τ_t it follows that

$$
\begin{aligned}
X_{\tau_t} &= X_0 \quad \text{if } X_0 \notin [a, b] \\
&= X_t \quad \text{if } X_s \in [a, b] \; \forall \; s \leqslant t \\
&= a \text{ or } b \text{ otherwise.}
\end{aligned}
$$

In particular it follows that $|X_{\tau_t} - X_0| \leqslant b - a$.

*F*6. X_{σ_t} is $\mathcal{F}_t$ measurable : This follows from the fact that σ_t is $\mathcal{F}_t$ measurable and $X(t, w)$ is jointly $\mathcal{B}_{[0,t]} \times \mathcal{F}_t$ measurable. Further,

$$
\begin{aligned}
X_{\sigma_t} &= a \text{ or } b \text{ if } X_t \in [a, b], \; \tau_t < t, \\
&= X_t \qquad \text{if } X_t \notin [a, b] \text{ or } \tau_t = t.
\end{aligned}
$$

In particular $|X_t - X_{\sigma_t}| \leqslant b - a$.

Proof of Theorem 1 : By $F1$, $F2$, $\mu_A\{s \leqslant t : X_s \in [a, b]\} = \underset{n\to\infty}{\text{Lt}}\ \mu_A(E_n)$ and also since $\mu_A(E_n) \leqslant A_t \ \forall\ w$, the Dominated Convergence Theorem implies that

$$E\mu_A\{s \leqslant t : X_s \in [a, b]\} = \underset{n\to\infty}{\text{Lt}}\ E\mu_A(E_n) \qquad \ldots \quad (1)$$

and since $\mu_A(E_n) = \sum_{k=0}^{\infty} (A_{\tau_k^n} - A_{\sigma_k^n})$, we have

$$E\mu_A(E_n) = E\left[\sum_{k=0}^{\infty} (X_{\tau_k^n} - X_{\sigma_k^n})^2\right] \qquad \ldots \quad (2)$$

Now we analyse the sum inside the expectation in (2). Firstly, since σ_k^n and τ_k^n are eventually equal to t, the sum is in fact a finite sum almost surely. Secondly, in case $\tau_k^n < t$, $X_{\sigma_k^n} = a - \frac{1}{n+1}$ or $b + \frac{1}{n+1}$ and $X_{\tau_k^n} = a - \frac{1}{n}$ or $b + \frac{1}{n}$. Hence during the interval $[\sigma_k^n, \tau_k^n]$, either there is an upcrossing of one of the intervals $\left[a - \frac{1}{n+1}, b + \frac{1}{n}\right]$, $\left[b + \frac{1}{n+1}, b + \frac{1}{n}\right]$ or a down-crossing of one of the intervals $\left[a - \frac{1}{n}, a - \frac{1}{n+1}\right]$, $\left[a - \frac{1}{n}, b + \frac{1}{n+1}\right]$. Hence the sum inside the expectation can be broken up into six parts as follows :

An initial term and a final term, a term corresponding to upcrossings of $\left[a - \frac{1}{n+1}, b + \frac{1}{n}\right]$, a term corresponding to down-crossings of $\left[a - \frac{1}{n}, b + \frac{1}{n+1}\right]$ a term corresponding to downcrossings of $\left[a - \frac{1}{n}, a - \frac{1}{n+1}\right]$ and a term corresponding to upcrossings of $\left[b + \frac{1}{n+1}, b + \frac{1}{n}\right]$. This gives us

$$\sum_{k=0}^{\infty} \left(X_{\tau_k^n} - X_{\sigma_k^n}\right)^2 = \left(X_0 - X_{\tau_0^n}\right)^2 + \left(b - a + \frac{1}{n} + \frac{1}{n+1}\right)^2 U^{X,t}_{\left[a - \frac{1}{n+1},\ b + \frac{1}{n}\right]}$$

$$+ \left(b - a + \frac{1}{n} + \frac{1}{n+1}\right)^2 D^{X,t}_{\left[a - \frac{1}{n},\ b + \frac{1}{n+1}\right]}$$

$$+ I_1^n + I_2^n + \left(X_{\tau^n_{k_n}} - X_{\sigma^n_{k_n}}\right)^2$$

where

$$I_1^n \leqslant \left(\frac{1}{n} - \frac{1}{n+1}\right)^2 D^{X,t}_{\left[a - \frac{1}{n},\ a - \frac{1}{n+1}\right]}$$

$$I_2^n \leqslant \left(\frac{1}{n} - \frac{1}{n+1}\right)^2 U^{X,t}_{\left[b + \frac{1}{n+1},\ b + \frac{1}{n}\right]}$$

By Doob's lemma on crossings $E(I_1^n) \to 0$ and $E(I_2^n) \to 0$ as $n \to \infty$. Regarding the last term note that $\tau^n_{k_n} = t$ and if $\sigma^n_{k_n} < t$ then $X_t \in \left[a-\frac{1}{n},\ b+\frac{1}{n}\right]$ since, $(X_{\tau^n_{k_n}} - X_{\sigma^n_{k_n}})^2 \to (X_t - X_{\sigma_t})^2$, by Bounded Convergence Theorem we have,

$$E(X_{\tau^n_{k_n}} - X_{\sigma^n_{k_n}})^2 \to E(X_t - X_{\sigma_t})^2,$$

similarly

$$E(X_0 - X_{\tau^n_0})^2 \to E(X_0 - X_{\tau_t})^2.$$

Also we note that

$$U^{X,t}_{\left[a-\frac{1}{n+1},\ b+\frac{1}{n}\right]} \uparrow U^{X,t}_{[a,b]}.$$

Hence

$$E\, U^{X,t}_{\left[a-\frac{1}{n+1},\ b+\frac{1}{n}\right]} \to E\, U^{X,t}_{[a,b]},$$

similarly

$$E\, D^{X,t}_{\left[a-\frac{1}{n},\ b+\frac{1}{n+1}\right]} \to E\, D^{X,t}_{[a,b]}.$$

Hence

$$\underset{n\to\infty}{\mathrm{Lt}}\ E\left(\sum_{k=0}^{\infty}(X_{\tau^n_k} - X_{\sigma^n_k})^2\right)$$

$$= E(X_0 - X_{\tau_t})^2 + (b-a)^2\, E(U^{X,t}_{[a,b]} + D^{X,t}_{[a,b]}) + E(X_t - X_{\sigma_t})^2. \quad \ldots \quad (3)$$

The proof is now completed using (1), (2) and (3).

We will now derive some limit theorems as corollaries of our main result. But first we note the following lemma.

Lemma 1 : *For a continuous martingale* (X_t) *the following statements are equivalent* :

(a) $A_t \uparrow \infty$ *as* $t \to \infty$ *almost surely.*

(b) $\overline{\underset{t\to\infty}{\mathrm{Lt}}}\ X_t = +\infty$ *and* $\underset{t\to\infty}{\underline{\mathrm{Lt}}}\ X_t = -\infty$ *almost surely.*

(c) $\underset{t\to\infty}{\underline{\mathrm{Lt}}}\ X_t < \overline{\underset{t\to\infty}{\mathrm{Lt}}}\ X_t$ *almost surely.*

Proof : (a) $\Rightarrow$ (b) because X_{τ_t} is a Brownian motion where τ_t is the inverse map of A_t given by $\tau_t = \inf\{s : A_s > t\}$.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a) because $X_t = B_{A_t}$ for some Brownian motion (B_t) on an extension space (see Ikeda and Watanabe, 1981, page 91, Theorem 1.2).

Corollary 1.1 : *Let* (X_t) *be a continuous square integrable martingale with* $<X>_t = A_t \uparrow \infty$ *almost surely as* $t \to \infty$. *Then, for any* $a < b$,

$$\operatorname*{Lt}_{t \to \infty} \frac{E\,\mu_A\{s \leqslant t : X_s \in [a, b]\}}{E\, C^{X;\,t}_{[a,\,b]}} = (b-a)^2.$$

Proof : By the hypothesis on (A_t) and Lemma 1(b) it follows that $C^{X;\,t}_{[a,\,b]} \uparrow \infty$ as $t \to \infty$, almost surely. The result now follows from Theorem 1, together with $F5$ and $F6$.

The next corollary is in the same spirit as Corollary 1.1. However instead of letting $t \to \infty$, we choose a sequence of stop times $\tau_n \uparrow \infty$.

Let (X_t) and (A_t) be as in Corollary 1.1. For $n \geqslant 1$, we define

$$\tau_n(w) = \inf\{t > 0 : C^{X;\,t}_{[a,\,b]} = n\}$$

since $C^{X;\,t}_{[a,\,b]}$ is a left continuous $\mathcal{F}_t$ adapted process, τ_n is an $\mathcal{F}_t$ stop time $\forall\, n$. Further note that $\tau_n < \infty$ almost surely because $A_t \uparrow \infty$ almost surely.

Corollary 1.2 : $\forall\, n \geqslant 1$,

$$E\,\mu_A\{s \leqslant \tau_n : X_s \in [a, b]\} = E(X_0 - X_\tau)^2 + n(b-a)^2$$

where τ *is the first exit time of* X *from the interval* $[a, b]$. *In particular,*

$$\operatorname*{Lt}_{n \to \infty} \frac{E\,\mu_A\{s \leqslant \tau_n : X_s \in [a, b]\}}{n} = (b-a)^2.$$

Proof : Theorem 1 applied to the process $Y_t = X_{t \wedge \tau_n}$ gives

$$E\mu_A\{s \leqslant t \wedge \tau_n : X_s \in [a, b]\} = E(Y_0 - Y_{\tau_t})^2 + (b-a)^2 E C^{X,\, t \wedge \tau_n}_{[a,\,b]} + E(Y_t - Y_{\sigma_t})^2 \qquad \ldots (4)$$

Regarding the first term in the RHS of Eqn. (4) observe that since $\tau_t = \tau \wedge t$, where τ is the first exit time of X from $[a, b]$, we have by DCT,

$$\operatorname*{Lt}_{t \to \infty} E(Y_0 - Y_{\tau_t})^2 = E(Y_0 - Y_\tau)^2 = E(X_0 - X_\tau)^2.$$

Regarding the 2nd term in the RHS of (4) observe that

$$\operatorname*{Lt}_{t \to \infty} E\, C^{X,\, t \wedge \tau_n}_{[a,\,b]} = E\, C^{X,\, \tau_n}_{[a,\,b]} = n - 1.$$

Regarding the last term on the RHS of (4) we observe that for $t > \tau_n$, $Y_t = X_{\tau_n}$ $= b$ or a according as there is an upcrossing or downcrossing at time τ_n.

Since $\sigma_t = \min \{u < t : Y_s \in [a, b] \ \forall \ s \in [u, t)\}$ it is clear that for $t > \tau_n$

$$Y_{\sigma_t} = a \text{ if } Y_{\tau_n} = b$$
$$= b \text{ if } Y_{\tau_n} = a.$$

Hence
$$\operatorname*{Lt}_{t \to \infty} (Y_t - Y_{\sigma_t})^2 = (b-a)^2.$$

Hence by the Dominated Convergence Theorem,

$$\operatorname*{Lt}_{t \to \infty} E(Y_t - Y_{\sigma_t})^2 = (b-a)^2.$$

These observations complete the proof.

Remarks : (1) Implicit in Theorem 1 is a decomposition of the set $\{s \leqslant t : X_s \in [a, b]\}$ into three parts viz. $[0, \tau_t)$, $[\tau_t, \sigma_t)$ and $(\sigma_t, t]$. Between the times 0 to τ_t and σ_t to t the path lies entirely in $[a, b]$. It is between τ_t and σ_t that the crossings are made. τ_t is a stop time and hence $E(X_{\tau_t} - X_0)^2 =$ $E\, A_{\tau_t}$. σ_t as noted earlier, need not be a stop time. However this leaves open the question : $E(X_{\sigma_t} - X_t)^2 = E(A_{\sigma_t} - A_t)$?

(2) In the case of Brownian motion, Lévy's theorem states that $\operatorname*{Lt}_{\epsilon \to 0} \epsilon\, D_{[0,\epsilon]}^{|X|,t} = \varphi\ (t, 0)$ almost surely where $\varphi(t, 0)$ is the local time at zero. Theorem 1 does in some sense provide a motivation for Lévy's theorem. In fact it is easy to show that Theorem 1 implies the following :

$$\operatorname*{Lt}_{\epsilon \to 0} \ \epsilon\, E\, C^{X,t}_{\left[x - \frac{\epsilon}{2},\ x + \frac{\epsilon}{2}\right]} = E\varphi(t, x)$$

for every x, outside a Lebesgue null set.

2. Applications

In this section we give two applications of our formula (Theorem 1). These are derived as corollaries to Theorem 2, which is the main result in this section.

Let (X_t) be a continuous square integrable martingale and let $< X >_t$ $= A_t \uparrow \infty$ almost surely as $t \to \infty$. For any real number c, let

$$\tau_c = \inf \{s \geqslant 0,\ X_s = c\}$$

Let $\varphi(t, x)$ be the local time at x of the martingale (X_t). The existence of such times are well known (see for example Meyer (1976) and Jacod (1979)). Since $A_t \uparrow \infty$ almost surely by Lemma 1 it follows that $\tau_c < \infty$ almost surely. Then we have,

Theorem 2 : *Let $X_0 = y$ almost surely. Then*

(i) *If $y \leqslant c$,*

$$E\varphi(\tau_c, a) = 2\,(c-a) \wedge (c-y) \qquad a \leqslant c$$
$$= 0 \qquad a > c.$$

(ii) *When $y > c$,*

$$E\varphi(\tau_c, a) = 2(a-c) \wedge (y-c) \qquad a \geqslant c$$
$$= 0 \qquad a < c.$$

To prove the theorem we need the following lemma.

Lemma 2 : *Let (X_t) be a continuous square integrable martingale with $< X >_t = A_t \uparrow \infty$ almost surely. For any interval $[a, b]$ and $c \notin [a, b]$ we have, $E\mu_A\{s \leqslant \tau_c : X_s \in (a, \infty)\} < \infty$ in case $c > b$ and $X_0 \leqslant c$ almost surely and $E\mu_A\{s \leqslant \tau_c : X_s \in (-\infty, b)\} < \infty$ in case $c < a$ and $X_0 \geqslant c$ almost surely.*

Proof : We will do the case $c > b$. The case $c < a$ is similar. Let $x_0 \leqslant c$ almost surely. We first compute $E\, D^{X, \tau_c}_{[a, b]}$ explicitly. First, we define the following sequence of stop times

$$\sigma_0 = \inf\{t > 0 \quad : X_t > b\}$$
$$\tau_0 = \inf\{t > \sigma_0 \quad : X_t < a\}$$
$$\vdots$$
$$\sigma_k = \inf\{t > \tau_{k-1} : X_t > b\}$$
$$\tau_k = \inf\{t > \sigma_k \quad : X_t < a\}.$$
$$\vdots$$

Since $A_t \uparrow \infty$ almost surely, it follows from our earlier observations that $\forall\, k$, $\sigma_k < \infty$, $\tau_k < \infty$ almost surely and that $\sigma_k < \tau_k < \sigma_{k+1}$ almost surely.

Let $$f(t, w) = \sum_{k=0}^{\infty} I_{(\sigma_k, \tau_k]}(t).$$

Let $Y_t = X_{t\wedge\tau_c}$ and $Y'_t = Y_t - Y_0$. Then $f(t, w)$ is a bounded $\mathcal{F}_t$-predictable process and we have almost surely,

$$\int_0^t f(s, w)\, dY'_s = \sum_{k=0}^{\infty} Y_{t\wedge\tau_k} - Y_{t\wedge\sigma_k}$$
$$= -(X_0-b)^+ - (b-a)\, D^{Y, t}_{[a, b]} + (Y_t - b)\, f(t \wedge \tau_c).$$

Taking expectations we get,

$$(b-a)\, ED^{Y,t}_{[a,b]} = E(Y_t-b)f(t \wedge \tau_c)-E(X_0-b)^+.$$

Further as $t \to \infty$,

$$(Y_t-b)\, f(t \wedge \tau_c) \to (c-b)$$

and

$$(a-b) \leqslant (Y_t-b)\, f(t \wedge \tau_c) \leqslant c-b \;\forall\; t.$$

Hence

$$E(Y_t-b)\, f(t \wedge \tau_c) \to c-b.$$

But as $t \to \infty$, $$ED^{Y,t}_{[a,b]} \uparrow ED^{Y,\infty}_{[a,b]} = ED^{X,\tau_c}_{[a,b]}$$

Hence $$(b-a)\, E\, D^{X,\tau_c}_{[a,b]} = (c-b)-E(X_0-b)^+ \qquad \ldots \quad (5)$$

We recall that

$$\tau_t = \tau_t^{[a,b]} = \inf \;\{s \leqslant t : X_s \notin [a, b]\}$$

as $t \to \infty$, $\tau_t^{[a,b]} \uparrow \tau^{[a,b]}$ where,

$$\tau^{[a,b]} = \inf \{s > 0 : X_s \notin [a, b]\}.$$

With this notation we have,

$$U^{X,\tau_c}_{[a,b]} = D^{X,\tau_c}_{[a,b]} + I_{\{X_{\tau[a,b]} \leqslant a\}}$$

Hence taking expectations and using (5) we get

$$(b-a)^2\, E\, C^{X,\tau_c}_{[a,b]} = (b-a)(c-b)-E(X_0-b^+)+(b-a)^2\, P\{X_{\tau[a,b]} \leqslant a\} \quad \ldots \quad (6)$$

Applying Theorem 1 to the martingales $X_{t \wedge \tau_c}$ and letting $t \to \infty$ we get

$$E\mu_A\{s \leqslant \tau_c : X_s \in [a, b]\} = (b-a)^2\, E\, C^{X,\tau_c}_{[a,b]} + \operatorname*{Lt}_{t \to \infty} E(X_{\tau_t}-X_0)^2.$$

By DCT (Section 1, F5)

$$\operatorname*{Lt}_{t \to \infty} E(X_{\tau_t}-X_0)^2 = E(X_{\tau[a,b]}-X_0)^2$$

Hence by (b) we get,

$$E\mu_A\{s \leqslant \tau_c : X_s \in [a, b]\} = E(X_{\tau[a,b]}-X_0)^2+(b-a)(c-b)-E(X_0-b)^+$$

$$+(b-a)^2\, P(X_{\tau[a,b]} \leqslant a) \qquad \ldots \quad (7)$$

It is easy to see that the RHS in (7) as a function of b is bounded $\forall\; a \leqslant b \leqslant c$. Hence letting $b \to c$ in LHS of (7) we get by Monotone Convergence Theorem

$$E\mu_A\{s \leqslant \tau_c : X_s \in [a, c)\} < \infty.$$

$$E\mu_A\{s \leqslant \tau_c : X_s \in [a, \infty)\} < \infty.$$

This proves the lemma.

We now turn to the proof of Theorem 2, which is now an easy consequence of Tanaka's formula for the local time.

Proof of Theorem 2: By Tanaka's formula (see Meyer (1976), Jacod (1979)) the local time at a time t is given by

$$\tfrac{1}{2}\varphi(t, a) = (X_t - a)^+ - (X_0 - a)^+ - \int_0^t I_{(a,\infty)}(X_s)\, dX_s. \qquad \dots \quad (8)$$

Similarly we have

$$\frac{1}{2}\,\varphi(t, a) = (X_t - a)^- - (X_0 - a)^- + \int_0^t I_{(-\infty, a)}(X_s)\, dX_s. \qquad \dots \quad (9)$$

With $X_0 = y < c$, (8) implies

$$\frac{1}{2}\,\varphi(t \wedge \tau_c, a) = (X_{t\wedge\tau_c} - a)^+ - (y - a)^+ - \int_0^{t\wedge\tau_c} I_{(a,\infty)}(X_s) dX_s \qquad \dots \quad (10)$$

We note that for $a \geqslant c$, the RHS in (10) is zero almost surely. For $a < c$, Lemma 2 allows us to take the limit in (10) as $t \to \infty$ and we get

$$\frac{1}{2}\,\varphi(\tau_c, a) = (c - a)^+ - (y - a)^+ - \int_0^{\tau_c} I_{(a,\infty)}(X_s)\, dX_s.$$

Taking expectations, the theorem is proved in the case $y < c$. The case $y > c$ is proved similarly using (9).

As promised in the beginning of this section, we now give two applications (Corollaries 2.1 and 2.2) of our formula on crossings. With the same set up as in Theorem 2, let $\psi(x, y, c) = E\varphi(\tau_c, x)$.

Let
$$L_X = \left\{ f : E \int_0^{\tau_c} f^2(X_s) dA_s < \infty \right\}$$

$$L'_X = \left\{ f : \int_{\mathcal{R}} f^2(x)\psi(x, y, c) dx < \infty \right\}.$$

Corollary 2.1: $L_X = L'_X$.

Proof: The proof is immediate from the fact (see Meyer (1976), Jacod (1979)) that $\forall\, t$,

$$\int_0^t f(X_s) dA_s = \int_{-\infty}^{\infty} f(x)\varphi(t, x) dx \text{ almost surely.}$$

Corollary 2.2 : *Let (X_t) be a continuous square integrable martingale as in Theorem 2. Let f be a C^2-function. Then the following are equivalent :*

(a) *$f(X_t)$ is a square integrable martingale.*

(b) *$f(x) = ax+b$ for some a and b.*

Proof : (b) $\Rightarrow$ (a) is trivial. By Ito's formula (a) $\Rightarrow$

$$\int_0^{\tau_c} f''(X_s)\, dA_s = 0 \;\forall\; c \text{ almost surely.}$$

Taking expectations and using Theorem 2,

$$\int_{-\infty}^{\infty} f''(x)\psi(x, y, c)dx = 0 \;\forall\; c > y$$

i.e.
$$\int_{-\infty}^{c} f''(x)\,(c-x) \wedge (c-y)\, dx = 0 \;\forall\; c > y.$$

For $c_1 > c_2 > y$ this gives us,

$$\int_{-\infty}^{c_2} f''(x)dx = \frac{1}{c_1-c_2} \int_{c_2}^{c_1} f''(x)\,(c_1-x)dx$$

Letting $c_1 \to c_2$, we get

$$\int_{-\infty}^{c_2} f''(x)dx = 0 \;\forall\; c_2 > y.$$

i.e.
$$f''(c) = 0 \;\forall\; c > y.$$

The case $c < y$ is similar. This proves (a) $\Rightarrow$ (b).

Remark 1 : The above corollary can also be obtained using time change arguments.

Remark 2 : By using the usual stop time arguments Corollary. 2.2 can be strengthened as follows : If (X_t) is a continuous local martingale with $< X >_t \uparrow \infty$, and f is a C^2-function for which $f(X_t)$ is also a local martingale then f must necessarily be linear. For a related result see McGill, Rajeev and Rao (1988).

Remark 3 : Theorem 2 deals with martingales $Y_t = X_{t\wedge\tau_c}$ which has the property that Y_t is eventually c. In such a situation we were able to let $t \to \infty$ in Tanaka's formula and take expectations in the limit. This raises the question of whether the same is possible for martingales with a more general limit *ex.* X_∞ is only bounded. In other words, we have to show that stochastic integral appearing in Tanaka's formula viz. $\int_0^\infty I_{(a,\infty)}(X_s)dX_s$

is well defined, which is the same as asking that $E\mu_A\{s : X_s \in (a, \infty)\} < \infty$. In our case we showed this by using the equivalence between $E\mu_A\{s : X_s \in (a, \infty)\}$ and $E\, C^{X,\infty}_{[a,b]}$ (Theorem 1). Here the fact that $X_\infty =$ const. simplified matters a great deal. But we believe that an extension to more general X_∞'s should be possible with a little more effort.

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