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ON METIVIER-PELLAUMAIL INEQUALITY, EMERY TOPOLOGY AND PATHWISE FORMULAE IN STOCHASTIC CALCULUS

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SUMMARY. Using Métivier-Pellaumail inequality, a notion of dominating process of a semimartingale is introduced and its usefulness in study of the Emery topology and Pathwise formulae is exhibited. Pathwise formulas for solution to stochastic differential equations and for multiplicative integral are obtained.

1. Introduction

Métivier-Pellaumail (1979) proved an inequality for square integrable martingales, which they called a stopped-Doob's inequality. In the same article they established its usefulness for study of stochastic differential equations (SDE). This inequality makes all semimartingales amenable to L^2-theory. In Karandikar (1981a), it was shown that using the technique of *random time change*, all continuous semimartingales can be treated via the L^2-theory and various pathwise formulae were obtained.

In this article, we introduce a notion of a dominating process of a semimartingale, which is a modification of the notion of a control process introduced by Métivier (1982). The rest of the paper tries to establish that the dominating process is a very useful tool in the study of stochastic integrals and SDE's. The claim is that *a dominating process does for a semimartingale what the total variation process does for a process with bounded variation paths.*

After establishing elementary properties, we give a new metric for the Emery topology (see Emery, 1979a) and then introduce the notion of *fast convergence* in the Emery topology, which we denote by $\xrightarrow{*}$.

We then prove stability of solution to SDE's in the sense of $\xrightarrow{*}$ which implies stability results in Emery topology. This notion of $\xrightarrow{*}$ is particularly useful for pathwise formulae for stochastic integrals, solutions to SDE's, multiplicative integrals etc.

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Among the results proved are : that the pathwise formula in Karandikar (1981a) for solution to SDE is valid for r.c.l.l. semimartingales, that the Euler-Peano scheme given in Bichteler (1981) yields a pathwise formula for larger class of coefficients, which may depend on the whole history and analogues of results in Karandikar on multiplicative integrals for r.c.l.l. semimartingals. In all these results, the pathwise formulae are in terms of paths of X^n's, which approximate the driving semimartingale X in the $\xrightarrow{*}$ sense.

We would like to remark that proofs involving dominating process and $\xrightarrow{*}$ are very natural and simple because they do not involve decomposing the semimartingale into a martingale and a process with bounded variation paths nor does it involve separating large jumps of the semimartingale and treating it separately.

All processes we consider are defined on a fixed complete probability space $(\Omega, \mathcal{F}, P)$ and are adapted to a filtration $(\mathcal{F}_t)$ assumed to satisfy usual hypothesis.

$\mathcal{M}^2_{loc}$ will denote as usual, the class of all locally square integrable martingales and for $M \in \mathcal{M}^2_{loc}$, $[M, M]$ will denote the quadratic variation process of M and $< M, M >$ the dual predictable projection of M. $\mathcal{V}^+$ will denote the class of all increasing processes V with $V_0 \geqslant 0$ and $\mathcal{V}$ will denote processes B which can be written as $B = A_1 - A_2$, $A_1, A_2 \in \mathcal{V}^+$. For $B \in \mathcal{V}$, $|B| \in \mathcal{V}^+$ will denote the total variation process of B. For $U, V \in \mathcal{V}^+$, we say $U \ll V$ if $V - U \in \mathcal{V}^+$. $\mathcal{SM}$ will denote the class of all semimartingales. Recall that if $X \in \mathcal{SM}$, it can be decomposed as $X = M + A$, $M \in \mathcal{M}^2_{loc}$, $A \in \mathcal{V}$. The class of all r.c.l.l. adapted processes will be denoted by $\mathcal{Z}$ and $\mathcal{J}$ will stand for class of predictable processes f such that $|f|^* \in \mathcal{V}^+$. Here, (and in the sequel).

$$|f|^*_t = \sup_{0 \leqslant s \leqslant t} |f(s)|.$$

All statements regarding processes are to be interpreted modulo P-null sets. For $f \in \mathcal{J}$, $X \in \mathcal{SM}$, $\int f dX$ will denote the stochastic integral. For all unexplained terminology, definitions etc, see Dellacherie-Meyer (1980) and Jacod (1979).

2. The dominating process of a semimartingale

The following is a consequence of the Métivier-Pellaumail inequality, which suffices for our purpose.

Theorem 2.1 : *Let $M \in \mathcal{M}^2_{loc}$ and σ be a stop time. Then*

$$E|M|^{*2}_{\sigma-} \leqslant 4E\{< M, M >_{\sigma-} + [M, M]_{\sigma-}\}. \qquad \ldots (2.1)$$

From (2.1) it follows that for $f \in \mathcal{J}$, $N \in \mathcal{M}^2_{loc}$ and a stop time σ,

$$E|\int f dN|^{*2}_{\sigma-} \leqslant E\{(\int f^2 dB)_{\sigma-}\} \qquad \dots \quad (2.2)$$

where $B = 4 < N, N > +4[N, N]$. On the other hand, if $A \in \mathcal{V}$ and $f \in \mathcal{J}$, then for all σ,

$$E|\int f dA|^{*2}_{\sigma-} \leqslant E\{(\int |f| d|A|)^2_{\sigma-}\}. \qquad \dots \quad (2.3)$$

Combining (2.2) and (2.3), Métivier-Pellaumail introduced the notion of a control process of a semimartingale which is defined as follows : $U \in \mathcal{V}^+$ is said to be a control process for a semimartingale X if for all $f \in \mathcal{J}$,

$$E|\int f dX|^{*2}_{\sigma-} \leqslant E\{U_{\sigma-}(\int f^2 dU)_{\sigma-}\}. \qquad \dots \quad (2.4)$$

Such a process U always exists, one choice being

$$U = 8(1+ < M, M > +[M, M]+|A|),$$

where $X = M+A$ is a decomposition of X with $M \in \mathcal{M}^2_{loc}$ and $A \in \mathcal{V}$. Using this notion of control process, the theory of stochastic integral can be developed and existence, uniqueness of solution to SDE's in the Lipschitz case can be proved in a routine way via successive approximation and a Gronwall type lemma.

We will show that a more efficient combination of (2.2), (2.3) yields a lot more. Let $\theta : \mathcal{J} \times \mathcal{V}^+ \to \mathcal{V}^+$ be defined by

$$\theta_t(f, V) = \sqrt{2}\Big\{\Big(\int_0^t |f|^2 dV^2\Big)^{\frac{1}{2}} + \int_0^t |f| dV\Big\} \qquad \dots \quad (2.5)$$

The following properties of θ can be verified easily. Here $f, g \in \mathcal{J}$, U, $V \in \mathcal{V}^+$ and α, β are real numbers

$$\theta_t(f, V) \leqslant 3|f|^*_t V_t \qquad \dots \quad (2.6)$$

$$|f| \leqslant |g| \Rightarrow \theta_t(f, V) \leqslant \theta_t(g, V) \qquad \dots \quad (2.7)$$

$$U \leqslant V \Rightarrow \theta_t(f, U) \leqslant \theta_t(f, V). \qquad \dots \quad (2.8)$$

Now a combination of (2.2), (2.3) implies that the process

$$V = 2(< M, M > +[M, M])^{\frac{1}{2}} + |A|$$

dominates the semimartingale $X = M+A(M \in \mathcal{M}^2_{loc}, A \in \mathcal{V})$ in the following sense :

$$E|\int f dX|^{*2}_{\sigma-} \leqslant E\{\theta^2_{\sigma-}(f, V)\}. \qquad \dots \quad (2.9)$$

This inequality leads us to the following definition.

Definition : A process $V \in \mathcal{V}^+$ is said to be a dominating process for a semimartingale, X, written as $V \succ\!\!\succ X$ or $X \prec\!\!\prec V$, if there exists a decomposition $X = M+A$ with $M \in \mathcal{M}^2_{loc}$, $A \in \mathcal{V}$ such that

$$\{2(<M, M> +[M, M])^{\frac{1}{2}}+|A|\} \ll V. \qquad \dots \quad (2.10)$$

The inequality (2.9) can be rewritten as follows : if $X \prec\!\!\prec V$ then for all $f \in \mathcal{J}$, for all stop times σ,

$$\|\,|\int f dX|^*_{\sigma-}\|_2 \leqslant \|\theta_{\sigma-}(f, V)\|_2 \qquad \dots \quad (2.11)$$

and in particular

$$\|\,|X|^*_{\sigma-}\|_2 \leqslant 3\|V_{\sigma-}\|_2 \qquad \dots \quad (2.12)$$

where $\| \ \|_2$ denotes the L^2-norm on $(\Omega, \mathcal{F}, P)$.

Before proceeding let us note that if X is an r.c.l.l. process and $V \in \mathcal{V}^+$ is such that (2.11) holds for all *bounded, simple predictable* processes f; for all stop times σ then the Dellacherie-Meyer-Mokobodzki theorem (see Métivier-Pellaumail (1980) and also Bichteler (1981)) implies that X is a semimartingale.

The following result gives two important properties of $\prec\!\!\prec$.

Theorem 2.2 : *Let $X, Y \in \mathcal{SM}$ and $f \in \mathcal{J}$. Suppose $X \prec\!\!\prec U$, $Y \prec\!\!\prec V$.*

Then $$\exists\, W \succ\!\!\succ (X+Y) \text{ such that } W \leqslant U+V \qquad \dots \quad (2.13)$$

and $$\exists\, D \succ\!\!\succ \int f dX \text{ such that } D \leqslant \theta(f, U). \qquad \dots \quad (2.14)$$

Proof : Let $X = M+A$, $Y = N+B$ be decompositions of X, Y with $M, N \in \mathcal{M}^2_{loc}$; $A, B \in \mathcal{V}$ with

$$2(<M, M> +[M, M])^{\frac{1}{2}}+|A| \ll U \qquad \dots \quad (2.15)$$

$$2(<N, N> +[N, N])^{\frac{1}{2}}+|B| \ll V. \qquad \dots \quad (2.16)$$

Take $M' = M+N$, $A' = A+B$, $W = 2(<M', M'> +[M', M'])^{\frac{1}{2}}+|A'|$. Then $X+Y = M'+A'$ and thus $(X+Y) \prec\!\!\prec W$. Now $W \leqslant (U+V)$ follows from $(<M', M'> +[M', M'])^{\frac{1}{2}} \leqslant (<M, M> +[M, M])^{\frac{1}{2}}+(<N, N> +[N, N])^{\frac{1}{2}}$.

This proves (2.13). For (2.14), take $N' = \int f dM$, $B' = \int f dA$

and $$D = 2(<N', N'> +[N', N'])^{\frac{1}{2}}+|B'|.$$

Then $\int f dX \prec\!\!\prec D$. Further

$$|B'| \leqslant \int |f|\, d|A| \leqslant \int |f|\, dU$$

and $$4(<N', N'> +[N', N']) = 4\int |f|^2\, d(<M, M> +[M, M])$$
$$\leqslant \int |f|^2\, dU^2.$$

These inequalities together give $D \leqslant \theta(f, U)$.

A metric for the Emery topology on $\mathcal{SM}$. The Emery topology (see Emery, 1979a) on the space of semimartingales is given by the metric

$$d_{sm}(X, Y) = \sup\{r_{cp}(\int f d(Y-X)) : f \in \mathcal{J}, |f|^* \leqslant 1\}$$

where
$$r_{cp}(Z) = \sum_{n=1}^{\infty} 2^{-n} E\{1 \wedge |Z|_n^*\}.$$

Recall that $r_{cp}(Z^k) \to 0$ is equivalent to saying that $|Z^k|_t^* \to 0$ in probability for every t.

For semimartingales X, Y, let

$$\rho(X, Y) = \inf\{r_{cp}(V) : V \succ\!\!\succ (Y-X)\}. \qquad \ldots \quad (2.17)$$

The observation (2.12) implies that $X = Y$ if $\rho(X, Y) = 0$ and (2.13) along with the observation that $W \leqslant U+V$ implies $r_{cp}(W) \leqslant r_{cp}(U)+r_{cp}(V)$ gives the triangle inequality for ρ. Thus ρ is a metric on $\mathcal{SM}$. It follows from (2.13) that under ρ, $\mathcal{SM}$ is a linear topological space. We will prove that ρ and d_{sm} are equivalent metrics. A first step is :

Lemma 2.3 : *$\rho(X^n, X) \to 0$ implies $d_{sm}(X^n, X) \to 0$.*

Proof : Since $\rho(Z, Y) = \rho(Z-Y, 0)$ and $d_{sm}(Z, Y) = d_{sm}(Z-Y, 0)$, we can assume that $X = 0$. Using usual subsequence arguments, suffices to prove that

$$\sum_{k=1}^{\infty} \rho(Y^k, 0) < \infty \Longrightarrow d_{sm}(Y^k, 0) \to 0.$$

Let $V^k \succ\!\!\succ Y^k$, $r_{cp}(V^k) \leqslant \rho(Y^k, 0)+2^{-k}$ be such that $\sum_{k=1}^{\infty} r_{cp}(V^k) < \infty$. It can be seen that this gives

$$V_t = \sum_{k=1}^{\infty} V_t^k < \infty.$$

Get $\sigma_m \uparrow \infty$ such that $V_{\sigma_m -} \leqslant m$. Then for any $f \in \mathcal{J}$ with $|f|^* \leqslant 1$, we have

$$\begin{aligned} r_{cp}(\int f dY^k) &= \sum_{n=1}^{\infty} 2^{-n} E\{1 \wedge |\int f dY^k|_n^*\} \\ &\leqslant \sum_{n=1}^{\infty} 2^{-n}\{P(\sigma_m \leqslant n) + E|\int f dY^k|_{\sigma_m -}^*\} \\ &\leqslant \sum_{n=1}^{\infty} 2^{-n} P(\sigma_m \leqslant n) + \|\theta_{\sigma_m -}(f, V^k)\|_2 \\ &\leqslant \sum_{n=1}^{\infty} 2^{-n} P(\sigma_m \leqslant n) + 3\|V^k_{\sigma_m -}\|_2. \end{aligned}$$

Using $\lim_k \|V^k_{\sigma_m-}\|_2 = 0$ (as $V^k_{\sigma_m-} \to 0$ and is bounded by m) for each m, we get

$$\limsup_{k\to\infty} \Big[\sup_{|f|^* \leqslant 1} r_{\sigma_0}(\int f dY^k) \Big] \leqslant \sum_{n=1}^{\infty} 2^{-n} P(\sigma_m \leqslant n). \qquad (2.18)$$

Now taking limit as $m \to \infty$ and using $\sigma_m \uparrow \infty$ we get that the left hand side of (2.18) is zero completing the proof.

Lemma 2.4 : *$(\mathcal{SM}, \rho)$ is a complete metric space.*

Proof : Suffices to show that if X^n is a sequence such that $\rho(X^{n+1}, X^n) \leqslant 2^{-n}$, then X^n converges to some X in $\mathcal{SM}$. Get $V^n \succ\!\!\succ (X^{n+1}-X^n)$ such that $V_t = \sum_{n=1}^{\infty} V^n_t < \infty$ as in the previous lemma. Let $Y^n = X^{n+1}-X^n$ and let $Y^n = N^n+B^n$ be a decomposition of Y^n with $N^n \in \mathcal{M}^2_{loc}$, $B^n \in \mathcal{V}$ with

$$[2(<N^n, N^n>+[N^n, N^n])^{\frac{1}{2}}+|B^n|] \leqslant V^n.$$

Let $M^n = \sum_{k=1}^{n} N^k$, $A^n = \sum_{k=1}^{n} B^k$ then $X^{n+1}-X^1 = M^n+A^n$. Now

$$\sum_{n=1}^{\infty} |B^n|_t < \infty$$

for all t implies that $\lim_n A^n_t = A_t$ exists, $A \in \mathcal{V}$ and $|A^n - A|_t \to 0$. Let

$$U_t = \sum_{n=1}^{\infty} (<N^n, N^n>)^{\frac{1}{2}}.$$

Then $U \in V^+$, is predictable and $U_0 = 0$. Hence we can get stoptimes σ_m such that $U_{\sigma_m} \leqslant m$ and $\sigma_m \uparrow \infty$. Then it follows that

$$(<M^j-M^i, M^j-M^i>_{\sigma_m})^{\frac{1}{2}} \leqslant \sum_{k=i+1}^{j} (<N^k, N^k>_{\sigma_m})^{\frac{1}{2}}$$

and hence goes to zero as $i, j \to \infty$, and is bounded by m. The dominated convergence theorem now gives

$$E<M^j-M^i, M^j-M^i>_{\sigma_m} \to 0$$

as $i, j \to \infty$ for every m. This gives existence of $M \in \mathcal{M}^2_{loc}$ such that

$$E<M^j-M, M^j-M>_{\sigma_m} \to 0. \qquad \ldots \quad (2.19)$$

The relation (2.19) also gives $E[M^j-M, M^j-M]_{\sigma_m} \to 0$ and hence

$$<M^j-M, M^j-M>_{\sigma_m}+[M^j-M, M^j-M]_{\sigma_m} \to 0$$

in probability. This implies M^n+A^n converges to $X = M+A$ in the ρ metric. Thus X^{n+1} converges to $X+X^1$. This completes the proof. □

Remark 1 : Using arguments given above it can be checked that

$$\rho(X^n, X) \to 0$$

iff there exists a decomposition $X^n - X = M^n + A^n$ with $|A^n|_t^* \to 0$ in probability and $< M^n, M^n >_t \to 0$ in probability for every t. We now have

Theorem 2.5 : *ρ, d_{sm} are equivalent metrics on $\mathcal{SM}$.*

Proof : Follows from the facts that ρ, d_{sm} are complete metrics on $\mathcal{SM}$, that under each of them, $\mathcal{SM}$ is a linear topological space, and Lemma 2.1, which implies that the identity mapping from $(\mathcal{SM}, \rho)$ into $(\mathcal{SM}, d_{sm})$ is continuous. The rest is open mapping theorem.

The characterisation of convergence in the Emery topology as $X^n \to X$ iff $\exists A^n \succ\!\!\succ (X^n - X)$ with $r_{cp}(A^n) \to 0$ yields simple proofs of various results on the topology given in Emery (1979b). A similar characterisation of convergence in Emery topology via control processes is given in Emery (1980). However, it does not yield a metric for the topology.

With the aim of obtaining pathwise formulae, we make the following definitions.

Definition : For processes g^n, g, say that $g^n \xrightarrow{\circ} g$ if

$$\sum_{n=1}^{\infty} |g^n - g|_t^{*2} < \infty \quad \forall t.$$

Clearly, $g^n \xrightarrow{\circ} g$ implies that $|g^n - g|_t^* \to 0$ as for all t, i.e. g^n converges to g as in the ucc topology (uniform convergence on compacts). The analogous notion for semimartingales is :

Definition : For semimartingales X^n, X say that $X^n \xrightarrow{*} X$ if $\exists V^n \succ\!\!\succ (X^n - X)$ such that $V^n \xrightarrow{\circ} 0$.

Remark 2 : It is easy to see that if $r_{cp}(V^n) \to 0$ then there exists a subsequence $\{V^{n_k}\}$ such that $V^{n_k} \xrightarrow{\circ} 0$. From this it follows that if $\rho(X^n, X) \to 0$, then there exists a subsequence $\{X^{n_k}\}$ such that $X^{n_k} \xrightarrow{*} X$. Since $X^n \xrightarrow{*} X$ clearly implies $\rho(X^n, X) \to 0$, we have the following. For $\rho(X^n, X)$ to converge to zero, it is necessary and sufficient that every subsequence of X^n contains a further subsequence converging to X in the sense of $\xrightarrow{*}$. It is clear that the convergence $\xrightarrow{*}$ does not correspond to convergence in any topology.

However, it may be noted that for a subset $\mathcal{A}$ of $\mathcal{SM}$, its closure consists of all $X \in \mathcal{SM}$ for which there exists a sequence X_n in $\mathcal{A}$ with $X_n \xrightarrow{*} X$.

The following result gives the importance of $\xrightarrow{*}$ for pathwise formulae.

Theorem 2.6 : *Suppose $X^n \xrightarrow{*} X$. Then*

(*i*) $X^n \xrightarrow{\circ} X$

(*ii*) *X^n converges a.s. to X in the ucc topology.*

Proof : Let $V^n \succ\!\!\succ (X^n - X)$ be such that

$$U_t = \sum_{n=1}^{\infty} (V_t^n)^2 < \infty$$

and get stop times $\sigma_i \uparrow \infty$ such that $U_{\sigma_i -} \leqslant i$. Then (2.12) implies

$$\| |X^n - X|^*_{\sigma_i -} \|_2^2 \leqslant 9 \| V^n_{\sigma_i -} \|_2^2$$

and hence

$$E \sum_{n=1}^{\infty} |X^n - X|^{*2}_{\sigma_i -} \leqslant 9 \sum_{n=1}^{\infty} E(V^n_{\sigma_i -})^2$$

$$\leqslant 9i$$

$$< \infty.$$

This proves

$$\sum_{n=1}^{\infty} |X^n - X|^{*2}_{\sigma_i -} < \infty \text{ a.s.}$$

which along with $\sigma_i \uparrow \infty$ yields $X^n \xrightarrow{\circ} X$. The second part follows from this.

We now state some useful results whose proofs are elementary and hence are omitted. The first one relates the notion of $\xrightarrow{\circ}$, $\xrightarrow{\circ}$ with stochastic integration.

Theorem 2.7 : *Let X^n, Y^n, X, $Y \in \mathcal{SM}$, $f^n, f \in \mathcal{J}$. Then we have*

$$X^n \xrightarrow{*} X,\ \ Y^n \xrightarrow{*} Y \implies X^n + Y^n \xrightarrow{*} X + Y \qquad \ldots \quad (2.20)$$

$$X^n \xrightarrow{*} X \implies \int f dX^n \xrightarrow{*} \int f dX \qquad \ldots \quad (2.21)$$

$$f^n \xrightarrow{\circ} f \implies \int f^n\, dX \xrightarrow{*} \int f\, dX \qquad \ldots \quad (2.22)$$

$$f^n \xrightarrow{\circ} f,\ X^n \xrightarrow{*} X \implies \int f^n\, dX^n \xrightarrow{*} \int f\, dX. \qquad \ldots \quad (2.23)$$

Remark 3 : Using the usual subsequential arguments and Remark 2, it can be shown that (2.23) implies

$$r_{cp}(f^n - f) \to 0,\ \rho(X^n, X) \to 0 \implies \rho(\int f^n dX^n, \int f dX) \to 0. \qquad \ldots \quad (2.24)$$

The next result connects the quadratic variation process with $\xrightarrow{*}$.

Theorem 2.8 : *Let X^n, X, Y be semimartingales. Then*

$$V \succ\!\!\succ X \Longrightarrow [X, X] \leqslant V^2 \qquad \dots \quad (2.25)$$

$$X^n \xrightarrow{*} 0 \quad \Longrightarrow [X^n, X^n]^{\frac{1}{2}} \xrightarrow{*} 0 \qquad \dots \quad (2.26)$$

$$X^n \xrightarrow{*} X \quad \Longrightarrow [X^n, Y] \xrightarrow{*} [X, Y] \qquad \dots \quad (2.27)$$

$$X^n \xrightarrow{*} X \quad \Longrightarrow [X^n, X^n] \xrightarrow{*} [X, X]. \qquad \dots \quad (2.28)$$

Using the change of variable formula in the form given in Lemma 8 in Emery (1979a) and the observation that if h is a locally Lipschitz function, then $Z^n \xrightarrow{o} Z$ implies $h(Z^n) \xrightarrow{o} h(Z)$, we can prove Theorem 2.9.

Theorem 2.9 : *Let h be twice continuously differentiable function and suppose that its second derivative h'' is locally Lipschitz. Then $X^n \xrightarrow{*} X$ implies $h(X^n) \xrightarrow{*} h(X)$.*

With a little work, a vector version of this result can also be proved.

3. Stability of solution to SDE

Let d be a fixed integer and $L(d)$ be the class of all $d \times d$ matrices. An $L(d)$ valued process $X = (X^{ij})$ is said to be a semimartingale if $X^{ij} \in \mathcal{SM}$ for all i, j. This is written as $X \in \mathcal{SM}(L(d))$ or simply $X \in \mathcal{SM}$. Similarly, for an $L(d)$ valued process $f = (f^{ij})$, $f \in \mathcal{B}$(respectively $\mathcal{J}$, $\mathcal{V}$) if $f^{ij} \in \mathcal{B}$(respectively $\mathcal{J}$, $\mathcal{V}$).

A process $V \in \mathcal{V}^+$ (real valued) is said to be a *dominating process* for $X = (X^{ij}) \in \mathcal{SM}(L(d))$ if $\exists$ $V^{ij} \in \mathcal{V}^+$, $V^{ij} \succ\!\!\succ X^{ij}$ and $V = \sum_{ij} V^{ij}$, and is written as $X \prec\!\!\prec V$ or $V \succ\!\!\succ X$.

For $f \in \mathcal{J}$ $(L(d))$ and $X \in \mathcal{SM}(L(d))$, $Y = \int f dX$ is defined by

$$Y^{ik} = \sum_j \int f^{ij} \, dX^{jk}$$

and it can be checked that if $X \prec\!\!\prec V$, then

$$E \| \int f dX \|^{*2}_{\sigma-} \leqslant d^4 \theta^2_{\sigma-}(\|f\|, V) \qquad \dots \quad (3.1)$$

where $\| \ \|$ is the Hilbert-Schmidt norm on $L(d)$. The constant d^4 in (3.1) can perhaps be improved.

For $L(d)$ valued processes X^n, X, we will say $X^n \xrightarrow{*} X$ (or $X^n \xrightarrow{o} X$) if $X^{n,ij} \xrightarrow{*} X^{ij}$ (or $X^{n,ij} \xrightarrow{o} X^{ij}$) for all i, j, where $X^n = (X^{n,ij})$ and $X = (X^{ij})$.

We will consider stochastic differential equations (SDE) of the form

$$Z = Y + \int G(Z)dX \qquad \ldots \quad (3.2)$$

where $Y \in \mathcal{D}$, $X \in \mathcal{SM}$, both $L(d)$ valued and

$$G : \mathcal{D}(L(d)) \to \mathcal{A}(L(d))$$

satisfies, for some $A \in \mathcal{V}^+$

$$\|G(Z_1) - G(Z_2)\|_t^* \leqslant A_{t-}\|Z_1 - Z_2\|_{t-}^* \; ; Z_1, Z_2 \in \mathcal{D}. \qquad \ldots \quad (3.3)$$

We will denote the class of G's satisfying (3.3) by $\mathcal{L}(A)$.

By taking rows or columns of zero's, we can convert equations of the type (3.2) with Y, Z $a \times b$ matrix valued, G mapping $a \times b$ matrix valued processes to $a \times c$ matrix valued process and X-an $c \times b$ matrix valued semimartingale to the $L(d)$ valued case.

The existence and uniqueness of solutions to (3.2) is well known. In Métivier (1982), these equations are treated via a Grownwall type Lemma which we state next. The first part is lemma 29.1 in the reference cited above and the second part of the lemma follows from the first part and the observation that

$$\theta_t^2(f, V) \leqslant 4(1 + V_t) \int_0^t |f|^2 \, d(V_t^2 + V_t).$$

Lemma 3.1 : *Let A, B, $V \in \mathcal{V}^+$ and τ be a stop time such that*

$$EA_{\tau-} < \infty, \; EB_{\tau-} < \infty, \; V_{\tau-} \leqslant a.$$

(*i*) *Suppose that for all stop times $\sigma \leqslant \tau$,*

$$EA_{\sigma-} \leqslant b + E\{(\int A_- dV)_{\sigma-}\}.$$

Then $EA_{\tau-} \leqslant bC_a$, where C_a is a constant which depends only on a.

(*ii*) *Suppose that for all stop times $\sigma \leqslant \tau$,*

$$EB_{\sigma-}^2 \leqslant b + E\{\theta_{\sigma-}^2(B_-, V)\},$$

then $EB_{\tau-}^2 \leqslant bC'_a$, where the constant C'_a depends only on a.

We now give an outline of proof of existence and uniqueness of solutions to (3.2). Apart from being simple, and on the lines of proof's in the case of equations driven by Brownian motion, it also shows that the method of successive approximation converges in the sense $\xrightarrow{2}$.

Theorem 3.2 : *Let $X \in \mathcal{SM}$, $Y \in \mathcal{D}$, both $L(d)$ valued and $G \in \mathcal{L}(A)$. Define Z^n by*

$$Z^0 = Y$$
$$Z^{n+1} = Y + \int G(Z^n)dX.$$

Then, $Z^n - Z \xrightarrow{*} 0$, where Z is a solution to (3.2). *Further, if Z' is any solution to* (3.2), *then*

$$P(Z_t = Z'_t \text{ for all } t) = 1.$$

Proof: Let $V \succ\!\!\succ X$. Let $B^n_t = \|Z^{n+1} - Z^n\|_t^{*2}$, $n \geqslant 0$. Let τ_i be stop times increasing to ∞ such that $V_{\tau_i-} \leqslant i$, $A_{\tau_i-} \leqslant i$, $B^0_{\tau_i-} \leqslant i$. Using (3.3) for G, we get for $\sigma \leqslant \tau_i$, with $U = V^2 + V$,

$$\begin{aligned} EB^{n+1}_{\sigma-} &\leqslant d^{4,2}_i E\theta^2_{\sigma-}(\sqrt{B^n_-}, V) \\ &\leqslant d^4_i 4(1+i)E(\textstyle\int B^n_- dU)_{\sigma-} \end{aligned}$$

for $n \geqslant 0$. Using $B^0_{\tau-} \leqslant i$ and $U_{\tau_i-} \leqslant i^2 + i$ it follows that $b_n = EB^n_{\tau_i-} < \infty$ for all $n \geqslant 0$.

Let $H^m_t = \sum_{n=0}^{m} 4^n B^n_t$. Then $EH^m_{\tau_i-} < \infty$ and for stop times $\sigma \leqslant \tau_i$ we have for some constants K_{1i}, K_{2i} depending only on i,

$$\begin{aligned} EH^{m+1}_{\sigma-} &\leqslant i + K_{1i}E(\textstyle\int H^m dU)_{\sigma-} \\ &\leqslant i + K_{1i}E(\textstyle\int H^{m+1} dU)_{\sigma-} \end{aligned}$$

and thus by Lemma 3.1,

$$EH^{m+1}_{\tau-} \left(= \sum_{n=0}^{m+1} 4^n b^n \right) \leqslant K_{2i}.$$

Since this holds for all n, we get

$$\sum_{n=0}^{\infty} 4^n b^n \leqslant K_{2i} < \infty.$$

In particular, $b^n \leqslant 4^{-n}$ for large n and as a consequence $\|(Z^{n+1} - Z^n)_{\sigma_i-}\|_2 \leqslant 2^{-n}$ for large n. This yields existence of $Z \in \mathfrak{D}$ such that $\|Z^n - Z\|^*_t \to 0$ in probability, and such that

$$\| \|Z^n - Z\|^*_{\sigma_i-} \|_2 \leqslant 2^{-(n-1)}$$

for large n. This yields $Z^n \xrightarrow{e} Z$. Lipschitz property for G implies $G(Z^n) \xrightarrow{e} G(Z)$ and hence by Theorem 2.7,

$$\int G(Z^n)dX \xrightarrow{*} \int G(Z)dX.$$

Thus, Z is a solution to (3.2) and $Z^n - Z \xrightarrow{*} 0$.

For uniqueness, let Z, Z' be solutions to (3.2). Get stop times $\sigma_i \uparrow \infty$, with $V_{\sigma_i-} \leqslant i$, $A_{\sigma_i-} \leqslant i$ and $\|Z-Z'\|_{\sigma_i-} \leqslant i$. Then for all stop times $\tau \leqslant \sigma_i$,

$$E\|Z-Z'\|^{*2}_{\tau-} \leqslant K_{3i}E\{(\int \|Z-Z'\|^{*2}_{-} dU)_{\sigma-}\}$$

for $U = V^2+V$ and a constant K_{3i}. Then Lemma 3.1 yields $E\|Z-Z'\|^{*2}_{\sigma_i-}=0$ for all i and hence

$$P(Z_t = Z'_t \text{ for all } t) = 1.$$

Remark 4 : Bichteler (1981) had shown that $Z^n \xrightarrow{*} Z$ and Emery (1979b) had shown that $Z^n \to Z$ in the ρ-topology.

The next result gives a refinement of a result of Emery vis-a-vis*-convergence.

Theorem 3.3 : *Let X^n, $X \in \mathcal{S}\mathcal{M}$, Y^n, $Y \in \mathcal{D}$ and G^n, $G \in \mathcal{L}(A)$ for some $A \in \mathcal{V}^+$. Let Z^n, Z be solutions to*

$$Z^n = Y^n + \int G^n(Z^n)dX^n$$
$$Z = Y + \int G(Z)dX.$$

Suppose further that $X^n \xrightarrow{*} X$, $Y^n \xrightarrow{o} Y$ and

$$G^n(Z) \xrightarrow{o} G(Z). \qquad \ldots \quad (3.4)$$

Then $Z^n \xrightarrow{o} Z$. If further, $Y^n \xrightarrow{*} Y$, then $Z^n \xrightarrow{*} Z$.

Proof : Let $V^n \succ\!\!-\!\!\succ (X^n-X)$ be such that $B^2_t = \sum_n (V^n_t)^2 < \infty$ for all t and let $V^0 \succ\!\!-\!\!\succ X$. Using Theorem 2.2, get $U^n \succ\!\!-\!\!\succ X^n$ such that $U^n_t \leqslant V^n_t + V^0_t$. Let

$$D_t = A^2_t + (V^0_t)^2 + B^2_t + \|F(Z)\|^{*2}_t + \sum_{n=1}^{\infty} \|G^n(Z)-G(Z)\|^{*2}_t + \sum_{n=1}^{\infty} \|Y^n-Y\|^{*2}_t.$$

Let $\tau_i = \inf\{t : D_t \geqslant i\}$. We then have $D_{\tau_i-} \leqslant i$ and as a consequence, $(U^n_{\tau_i-})^2 \leqslant 2i$, $A^2_{\tau_i-} \leqslant i$, $(V^0_{\tau_i-})^2 \leqslant i$, $|F(Z)|^{*2}_{\tau_i-} \leqslant i$. Using these inequalities and writing

$$Z^n - Z = Y^n - Y + \int(G^n(Z^n)-G^n(Z))dX^n + \int(G^n(Z)-G(Z))dX^n$$
$$+\int G(Z)d(X^n-X)$$

we get, for any stoptime $\sigma \leqslant \tau_i$

$$(4d^4)^{-1}E\|Z^n-Z\|^{*2}_{\sigma-} \leqslant E\|Y^n-Y\|^{*2}_{\sigma-} + iE\theta^2_{\sigma-}(\|Z^n-Z\|^*_{-}, U^n)$$
$$+2iE\|G^n(Z)-G(Z)\|^{*2}_{\sigma-} + iE(V^n_{\sigma-})^2$$

Hence, for constants $K_t < \infty$, δ_{nt} such that $\Sigma_n \delta_{nt} < \infty$, we have

$$E\|Z^n - Z\|^{*2}_{\sigma-} \leqslant \delta_{nt} + K_t E\theta^2_{\sigma-}(\|Z^n - Z\|_{-}, U^n)$$

for all stop times $\sigma \leqslant \tau_t$. Using $(U^n_{\tau_t-})^2 \leqslant 2t$ we conclude from Lemma 3.1 that for a suitable constant K_{4t}

$$E\|Z^n - Z\|^{*2}_{\tau_t-} \leqslant \delta_{nt} K_{4t}$$

and hence
$$E \sum_{n=1}^{\infty} \|Z^n - Z\|^{*2}_{\tau_t-} \leqslant K_{4t} \sum_{n=1}^{\infty} \delta_{nt} < \infty.$$

This proves $Z^n \xrightarrow{o} Z$. For the second part, first note that

$$G^n(Z^n) - G(Z) \xrightarrow{o} 0 \qquad \ldots \quad (3.5)$$

which follows from the fact that G^n, $G \in \mathscr{L}^0(A)$ and that

$$G^n(Z^n) - G(Z) = G^n(Z^n) - G^n(Z) + G^n(Z) - G(Z).$$

Thus,
$$\int (G^n(Z^n) - G(Z))dX \xrightarrow{*} 0. \qquad \ldots \quad (3.6)$$

Now $G(Z) \in \mathscr{I}$ and (3.5) imply

$$\sup_n \|G^n(Z^n)\|^{*2}_t < \infty. \qquad \ldots \quad (3.7)$$

Applying Theorem 2.2 to each component, get

$$W^n \succ\!\!\succ \int G^n(Z^n) d(X^n - X)$$

such that
$$W^n_t \leqslant d^4\theta_t(\|G^n(Z^n)\|, V^n)$$
$$\leqslant 3d^4 \|G^n(Z^n)\|^*_t V^n_t.$$

Now (3.7) and $\sum_n (V^n_t)^2 < \infty$ yields $\sum_n (W^n_t)^2 < \infty$. Thus

$$\int G^n(Z^n) d(X^n - X) \xrightarrow{*} 0$$

which along with (3.6) gives $\int G^n(Z^n) dX^n \xrightarrow{*} \int G(Z) dX$. Now $Y^n \xrightarrow{*} Y$ implies $Z^n \xrightarrow{*} Z$.

Remark 5 : In Emery (1979b), it is proved that in the set up of Theorem 3.3, if $\rho(X^n, X) \to 0$ and $r_{cp}(G^n(Z) - G(Z)) \to 0$ (instead of assumptions involving $\xrightarrow{*}, \xrightarrow{o}$) then $r_{cp}(Z^n - Z) \to 0$ and if $\rho(Y^n, Y) \to 0$ then $\rho(Z^n, Z) \to 0$. This can be deduced from the theorem given above via usual subsequential arguments and Remark 2.

Remark 6 : In Theorem 3.3 if one assumes

$$G^n(Z^n) - G(Z^n) \xrightarrow{o} 0 \qquad \ldots \quad (3.8)$$

instead of (3.3), then the conclusions are still valid. This can be proved by writing

$$Z^n - Z = Y^n - Y + \int G(Z)d(X^n - X) + \int (G(Z^n) - G(Z))dX^n$$
$$+ \int (G^n(Z^n) - G(Z^n))dX^n$$

and using arguments analogous to the ones in the proof of Theorem 3.3. In fact we do not need to assume $G^n \in \mathcal{L}(A)$.

4. Pathwise formulae

Bichteler (1981) proved that for $X \in \mathcal{SM}$, $Y \in \mathcal{D}$, the integral $\int Y_- dX$ can be evaluated pathwise, i.e. $(\int_0^T Y_- dX)(\omega)$ can be written as an explicit function of the ω-paths $\{Y(s, \omega), X(s, \omega), 0 \leqslant s \leqslant T\}$. Such pathwise formulae are important for statistical applications.

Bichteler also gave a pathwise formula for solution to an SDE. A different formula was obtained by Karandikar (1981a) for SDE's driven by continuous semimartingales using elementary methods.

In this section, we will show that the notion of dominating process introduced earlier yields these results and moreover, gives elegant conditions on the coefficients when the Euler-Peano scheme or the modified successive approximation scheme yield a pathwise solution to the SDE in question.

A sequence $\{\tau_i\}$ of stop times will be said to be a partition (of $[0, \infty)$) if $\tau_0 = 0$, $\tau_i \leqslant \tau_{i+1}$ for all i and $\tau_i \to \infty$.

For a partition $\{\tau_i\}$, we define $J : \mathcal{D} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{J}$ by

$$JX = \sum_{i=0}^{\infty} X_{\tau_i} 1_{[\tau_i, \tau_{i+1})}$$

$$HX = \sum_{i=0}^{\infty} X_{\tau_i} 1_{(\tau_i, \tau_{i+1}]}.$$

Note that given $\varepsilon > 0$ and $X^1, \ldots, X^k \in \mathcal{D}(\mathrm{L}(d))$; $\{\tau_i\}$ defined by

$$\tau_0 = 0$$

$$\tau_{i+1} = \inf\{t \geqslant \tau_i : \|X^j_t - X^j_{\tau_i}\| \geqslant \varepsilon \text{ for some } j, 1 \leqslant j \leqslant k\}$$

is a partition and $\|JX^j - X^j\| \leqslant \varepsilon$. The sequence $\{\tau_i\}$ defined above will be called that the canonical ε-partition for $X^1, X^2, \ldots, X^k$. Note that the canonical ε-partition for $X^1, \ldots, X^k$ is defined pathwise i.e. $\tau_i(\omega)$ is an explicit function of the paths $\{X^j(t, \omega) : 1 \leqslant j \leqslant k ; t \geqslant 0\}$.

In what follows, we will be considering a sequence $\{\tau_i^n : i \geq 0\}$, $n \geq 1$ of partitions. We will denote the corresponding operators J, H by J^n, H^n respectively. More specifically, we will be dealing with $\{\tau_i^n\}$ such that

$$\|J^n X^n - X^n\| \xrightarrow{o} 0 \qquad \ldots \quad (4.1)$$

where $X^n \in \mathfrak{D}$ is a given sequence. Note that if $\{\tau_i^n\}$ is the canonical ε_n-partition for X^n and $\Sigma \varepsilon_n^2 < \infty$, then it follows that (4.1) holds.

Let $Y \in \mathfrak{D}$ and $\{\tau_i^n\}$ be such that

$$\|J^n Y - Y\| \xrightarrow{o} 0$$

and suppose $X^n \xrightarrow{*} X$. Then $H^n Y \xrightarrow{*} Y_-$ and thus $\int (H^n Y) dX^n \xrightarrow{*} \int Y_- dX$. Taking $\{\tau_i^n\}$ to be the ε_n-canonical partition of Y, we get the following :

Theorem 4.1 : *Let $Y \in \mathfrak{D}$ and $X^n \rightarrow X$. Let $\{\tau_i^n\}$ be the canonical ε_n-partition for Y with $\Sigma\, \varepsilon_n^2 < \infty$. Let*

$$Z^n(t, \omega) = \sum_{i=0}^{\infty} Y(\tau_i^n(\omega), \omega)\, (X^n(\tau_{i+1}^n(\omega) \wedge t, \omega) - X^n(\tau_i^n(\omega) \wedge t, \omega)).$$

Let $\Omega_0 = \{\omega : Z^n(., \omega)$ converges in u.c.c. topology$\}$ and

$$Z(t, \omega) = \begin{cases} \lim\limits_n Z^n(t, \omega), \text{ if } \omega \in \Omega_0 ; \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Then $P(\Omega_0) = 1$, $Z = \int Y \xrightarrow{*} dX$ and $Z^n \xrightarrow{*} Z$.

This gives a pathwise formula for $\int Y_- dX$ when X is only approximately known. If we take $X^n = X$, we get Bitchteler's formula.

We now turn to Pathwise formulae for the solution to SDE of the type considered in the last section. We assume that the functional $G : \mathfrak{D} \rightarrow \mathcal{J}$ is of the form

$$G(Z)\,(t, \omega) = F(Z)\,(t-, \omega) \qquad \ldots \quad (4.2)$$

where $F : \mathfrak{D} \rightarrow \mathfrak{D}$ is in turn a pathwise mapping

$$F(Z)\,(t, \omega) = b(t, \omega, Z(\omega)) \qquad \ldots \quad (4.3)$$

and $b : [0, \infty) \times \Omega \times D([0, \infty), L(d))$ is a mapping, such that $F(Z)$ defined by (4.3) belongs to $\mathfrak{D}$ for all $Z \in \mathfrak{D}$. Here $D([0, \infty), L(d))$ is the space of all r.c.l.l. functions from $[0, \infty)$ into $L(d)$. We further assume that

$$\|b(t, \omega, \phi_1) - b(t, \omega, \phi_2)\| \leqslant A(t, \omega) \|\phi_1 - \phi_2\|_t^* \qquad \ldots \quad (4.4)$$

for some $A \in \mathcal{V}^+$. Then G defined via (4.2)–(4.3) belongs to $\mathcal{L}(A)$.

Our object is, given $Y^n \xrightarrow{*} Y$, $X^n \xrightarrow{*} X$, to obtain a formula for the solution Z to the SDE (3.2) where for $\omega \in \Omega$ fixed, $Z(t, \omega)$ is given explicitly in terms of the paths $\{Y^n(., \omega)\,;\, X^n(., \omega)\,;\, b(., \omega,.),\ n \geqslant 1\}$. Note that since we know that the successive approximation method (in case $X^n \equiv X$, $Y^n \equiv Y$) converges uniformly a.s. and that Z^{n+1} can be computed pathwise from Z^n using Theorem 4.1, we already have a pathwise formula, However, this involves repeated limits and is thus unsuitable for computational purposes. The importance of obtaining a formula involving a single limit is self evident from the point of view of applications.

The first formula we give was obtained by Karandikar (1981a) when X is a continuous semimartingale and $X^n \equiv X$, $Y^n \equiv Y$.

Theorem 4.2 : *Let b satisfy* (4.4). *Let $Y^n \xrightarrow{*} Y$ and $X^n \xrightarrow{*} X$. Fix $\varepsilon_n > 0$ such that $\sum_n \varepsilon_n^2 < \infty$. Put $Z^0 = Y^0$ and define $\{Z^n, \{\tau_i^n : i \geqslant 1\}\}$ inductively by*

$$\tau_0^n = 0,\ Z_0^n = Y_0^n$$

$$\tau_{i+1}^n(\omega) = \inf\{t \geqslant \tau_i^n(\omega) : \|b(t, \omega, Z^{n-1}(\omega)) - b(\tau_i^n(\omega), \omega, Z^{n-1}(\omega))\| \geqslant \varepsilon_n\}$$

and for $\tau_i^n < t \leqslant \tau_{i+1}^n$,

$$Z_t^n = Z_{\tau_i^n}^n + Y_t^n - Y_{\tau_i^n}^n + b(\tau_i^n,., Z^{n-1})(X_t^n - X_{\tau_i^n}^n).$$

Then $Z^n \xrightarrow{*} Z$ where Z is solution to (3.2), *where G, b are related via* (4.2), (4.3). *As a consequence, Z is limit of Z^n in the u.c.c. topology, a.s.*

Proof : Let J^n, H^n correspond to $\{\tau_i^n\}$. It is easy to verify that

$$Z^n = Y^n + \int H^n(F(Z^{n-1}))dX^n.$$

Then

$$\begin{aligned} Z^n - Z = Y^n - Y &+ \int (H^n(F(Z^{n-1})) - G(Z^{n-1}))dX^n \\ &+ \int (G(Z^{n-1}) - G(Z))dX^n \\ &+ \int G(Z)d(X^n - X). \end{aligned}$$

Using $\|H^n(F(Z^{n-1})) - G(Z^{n-1})\| \leqslant \varepsilon_n$ and $X^n \xrightarrow{*} X$, it follows from Theorem 2.7 that

$$\int \{H^n(F(Z^{n-1})) - G(Z)\}dX^n \xrightarrow{*} 0.$$

Thus we have

$$Z^n - Z = W^n + \int (G(Z^{n-1}) - G(Z))dX^n,$$

where $W^n \xrightarrow{*} 0$. Now proceeding as in the proof of Theorem 3.3, it can be shown that $Z^n \xrightarrow{*} Z$. One needs to use that $\exists D \in \mathcal{V}^+$ such that $\forall_n\ \exists U^n \succ\!\!\succ X^n$ with $U_t^n \leqslant D_t$. This fact was noted in the proof of Theorem 3.3.

Under additional conditions on b, one could get other approximation schemes. We give a condition under which Euler-Peano scheme gives a pathwise formula.

Theorem 4.3 : *Suppose b satisfies* (4.2) *and further that for all s and for all* $\phi \in D([0, \infty), L(d))$

$$\phi(t) = \phi(s \wedge t) \ \forall t \Longrightarrow b(t, \omega, \phi) = b(s \wedge t, \omega, \phi). \quad \ldots \quad (4.5)$$

Let ε_n be such that $\sum_n \varepsilon_n^2 < \infty$. *Let* $\{\tau_i^n, Z_t^n\}$ *be defined by*

$$\tau_0^n = 0, \ Z_0^n = Y_0^n$$

$$\tau_{i+1}^n = \inf\{t \geqslant \tau_i^n : \|Y_t^n - Y_{\tau_i^n}^n + b(\tau_i^n,., Z^n)(X_t^n - X_{\tau_i^n}^n)\| \geqslant \varepsilon_n\} \quad \ldots \quad (4.6)$$

and for $\tau_i^n < t \leqslant \tau_{i+1}^n$

$$Z_t^n = Z_{\tau_i^n}^n + Y_t^n - Y_{\tau_i^n}^n + b(\tau_i^n,., Z^n)(X_t^n - X_{\tau_i^n}^n). \quad \ldots \quad (4.7)$$

(*Note that* $b(\tau_i^n, \omega, Z^n)$ *can be evaluated once* $Z_t^n : t \leqslant \tau_i^n$ *is determined*).

Then Z^n converges in u.c.c topology (a.s) to Z-the solution to (3.2). *In fact,* $Z^n \xrightarrow{e} Z$.

Proof : As in Bichteler (1981) it can be argued that $\tau_i^n \to \infty$ (a.s) as $i \to \infty$ and hence that τ_i^n is a partition.

Let J^n, H^n correspond to $\{\tau_i^n\}$.

Then $$Z^n = Y^n + \int H^n F(Z^n) dX^n.$$

Thus, to prove that $Z^n \xrightarrow{e} Z$ in view of Theorem 3.2 suffices to prove that

$$H^n F(Z^n) - G(Z^n) \to 0$$

or equivalently, $$J^n F(Z^n) - F(Z^n) \xrightarrow{e} 0. \quad \ldots \quad (4.8)$$

The assumption (4.5) on b implies that $J^n F(J^n Z^n) = F(J^n Z^n)$ and hence

$$\begin{aligned} \|J^n F(Z^n) - F(Z^n)\|_t^* &\leqslant \|J^n F(Z^n) - J^n F(J^n Z^n)\|_t^* \\ &\quad + \|F(J^n Z^n) - F(Z^n)\|_t^* \\ &\leqslant 2A\|Z^n - J^n Z^n\|_t^* \\ &\leqslant 2A\varepsilon_n \end{aligned}$$

by choice of the partition $\{\tau_i^n\}$. Since $\sum_n \varepsilon_n^2 < \infty$, this proves (4.8).

Remark 7 : When $b(t, \omega, \phi) = g(\phi_t)$ for some Lipschitz function g, then, (4.5) holds. In this case the above result is proved in Bichteler (1981).

Suppose b does not satisfy (4.5). What does the Euler-Peano Scheme yield ? Here, in the definition of $\{\tau_i^n\}$, Z_t^n, one replaces $b(\tau_i^n,., Z^n)(X_t^n - X_{\tau_i^n}^n)$ by

$$\int 1_{(\tau_i^n, t]}\, b(s-,\ \omega, (Z^n)^{H^n(s)})dX_s^n \qquad \ldots \quad (4.9)$$

where $W_t^\sigma = W_{t\wedge\sigma}$ and $H^n(s) = \tau_i^n$ for $s \in (\tau_i^n, \tau_{i+1}^n]$.

Now, $$Z^n = Y^n + \int b(s-, ., (Z^n)^{H^n(s)})dX_s^n$$

and since $$\|b(s-, ., (Z^n)^{H^n(s)}) - b(s-, ., Z^n)\| \leqslant A(s, .)\|Z_u^n - (Z^n)^{H^n(u)}\|_s^* \leqslant A(s, .)\varepsilon_n$$

one can show that $Z^n \xrightarrow{*} Z$ in this case as well.

However, in this case, the integral in (4.9) involves a limiting operation and thus the resulting formula for Z involves iterated limits.

Thus, the Euler-Peano scheme converges in $\xrightarrow{*}$ sense if b satisfies the Lipschitz condition, and if further b satisfies (4.5), then it yields a pathwise formula involving a single limit.

Remark 8 : In Theorem 4.3 if $\{\tau_i^n\}$ is the canonical ε_n-partition for X^n instead of being defined by (4.6), but Z^n is defined via (4.7), even then it can be shown that $Z^n \xrightarrow{*} Z$.

The exponential equation. We will now look at the exponential equation and pathwise formulae for its solution. In this case, we can see the importance of obtaining a pathwise formula for the solution to (3.2) in terms of paths of X^n (such that $X^n \xrightarrow{*} X$).

For $X \in \mathcal{SM}(L(d))$, the solution Z to the SDE

$$Z = I + X_0 + \int Z_- dX \qquad \ldots \quad (4.10)$$

is called the exponential of X and we will denote it by $\varepsilon(X)$. It is also called the multiplicative integral $\Pi(I + dX)$ (see Emery, 1978 and Karandikar, 1981b).

The pathwise formulae for $\varepsilon(X)$ when X is continuous has been found to be useful in diverse contexts namely, Girsanov problem for a Lie group valued Brownian motion (Karandikar, 1982) and study of large deviations for products of random matrices (Watkins, 1987).

The next result is in preparation for pathwise formulae for $\varepsilon(X)$.

Theorem 4.4 : *Let $X^n \xrightarrow{*} X$ and $\varepsilon_n > 0$ be such that $\sum_n \varepsilon_n^2 < \infty$. Let $\{\tau_i^n\}$ be a partition such that the corresponding J^n satisfies for some $A \in \mathcal{V}^+$*

$$\|J^n X^n - X\| \leqslant \varepsilon_n A. \qquad \ldots \quad (4.11)$$

Define $\{Z_t^n\}$ *by* $Z_0^n = (I+X_0^n)$ *and*

$$Z_t^n = Z_{\tau_i^n}^n(I+X_t^n-X_{\tau_i^n}^n), \qquad \tau_i^n < t \leqslant \tau_{i+1}^n. \qquad \ldots \quad (4.12)$$

Then $Z^n \xrightarrow{*} Z = \mathcal{E}(X)$.

Proof: Let $G(W) = W_- + I1_{\{0\}}$ and $G^n(W) = H^nW + I1_{\{0\}}$. Then $Z = I + \int G(Z)dX$ and $Z^n = I + \int G^n(Z^n)\,dX^n$. Thus suffices to prove

$$\|G^n(Z^n) - G(Z^n)\| \xrightarrow{\circ} 0 \qquad \ldots \quad (4.13)$$

The conclusion $Z^n \xrightarrow{*} Z$ would then follow from Theorem 3.2 and Remark 6 following it. Note that

$$G^n(Z^n) - G(Z^n) = H^nZ^n - Z_-^n \qquad \ldots \quad (4.14)$$

so that (4.13) is same as

$$\|J^nZ^n - Z^n\| \xrightarrow{\circ} 0. \qquad \ldots \quad (4.15)$$

Now
$$\|Z^n - J^nZ^n\|_t^* = \|J^nZ^n(X^n - J^nX^n)\|_t^*$$
$$\leqslant \varepsilon_n A_t \|J^nZ^n\|_t^*. \qquad \ldots \quad (4.16)$$

To complete the proof, we will show that for suitable stop times $\sigma_i \uparrow \infty$,

$$\sup_n EA_{\sigma_i-}^2 \|J^nZ^n\|_{\sigma_i-}^{*2} < \infty,$$

which along with (4.16) gives (4.15).

Using $X^n \xrightarrow{*} X$, get $D \in \mathcal{V}^+$ and $U^n \succ\!\!-\!\!\succ X^n$ such that $U_t^n \leqslant D_t$, as in proof of Theorem 3.3. Let $\sigma_i \uparrow \infty$ be defined by

$$\sigma_i = \inf\{t \geqslant 0 : D_t \geqslant i \text{ or } A_t \geqslant i\}.$$

From
$$Z^n = I + \int G^n(Z^n)\,dX^n$$

we get, for any stop time $\tau \leqslant \sigma_i$,

$$E\|Z^n\|_{\tau-}^{*2} \leqslant K_{i,d} + i^2 E\theta_{\tau-}^2(\|Z^n\|_-^*, U^n)$$

which yields

$$E\|Z^n\|_{\sigma_i-}^{*2} \leqslant K_{i,d}'$$

where $K_{i,d}$, $K_{i,d}'$ are constants depending only on i, d. Now $A_{\sigma_i-} \leqslant i$ implies

$$\sup_n EA_{\sigma_i-}^2 \|Z^n\|_{\sigma_i-}^{*2} \leqslant i^2 K_{i,d}'.$$

This completes the proof.

Taking $X^n \equiv X$ and $\{\tau_i^n\}$ to be the canonical ε_n-partition for X, with $\sum_n \varepsilon_n^2 < \infty$, we get that the *random Riemann products*

$$Z_t^n = (I+X_0) \prod_{i=0}^{\infty} \left(I+X^n_{\tau^n_{i+1}\wedge t}-X^n_{\tau^n_i\wedge t}\right)$$

converges uniformly on compacts, a.s. to $Z = \varepsilon(X)$, indeed, $Z^n \xrightarrow{*} Z$.

It is useful to obtain analogous result with exp(s) replacing $(I+B)$. Such a result with Z^n converging in u.c.c. in probability was proved in Emery (1978). Almost sure convergence was proved in Karandikar (1981b) for continuous semimartingales.

We will be considering functions f from $L(d)$ into itself. A generic element of $L(d)$ will be denoted by $x = (x^{jk})$. We fix a function $f: L(d) \to L(d)$ such that $f(0) = 0$, the partial deviations

$$\frac{\partial}{\partial x^{jk}} f = f_{jk}, \; \frac{\partial}{\partial x^{lm}} f_{jk} = f_{jk,lm}$$

exist, are continuous and that $f_{jk,lm}$ are locally Lipschitz continuous for $1 \leqslant j, k, l, m \leqslant d$. Define $g : L(d) \times L(d) \to L(d)$ by

$$g(x, y) = f(x+y)-f(x)- \sum_{jk} f_{jk}(x)y^{jk}$$

$$-\frac{1}{2} \sum_{jk} \sum_{lm} f_{jk,lm}(x)y^{jk}y^{lm}, \qquad \ldots \quad (4.17)$$

We need the following estimates in the sequel :

Lemma 4.5 : *For $\alpha > 0$, there exists a constant C_α such that for $x, y, y_1, y_2 \in L(d)$ with $\|x\| \leqslant \alpha$, $\|y\| \leqslant \alpha$, $\|y_1\| \leqslant \alpha$, $\|y_2\| \leqslant \alpha$, we have*

$$\|g(x, y)-g(0, y)\| \leqslant C_\alpha \|x\| \, \|y\|^2, \qquad \ldots \quad (4.18)$$

$$\|g(x, y_1)-g(x, y_2)\| \leqslant C_\alpha \|y_1-y_2\| \, \|y_1\|^2. \qquad \ldots \quad (4.19)$$

Proof : Fix x, y, and for $0 \leqslant t \leqslant 1$, define

$$h(t) = f(x+ty)-f(x) - t \sum_{jk} f_{jk}(x)y^{jk}$$

$$-\frac{t^2}{2} \sum_{jk,lm} f_{jk,lm}(x)y^{jk}y^{lm}.$$

Then $h(0) = 0 = h'(0)$ and hence by Taylor's theorem

$$\begin{aligned} g(x, y) &= h(1) \\ &= \int_0^1 (1-t)h''(t)\, dt \\ &= \int_0^1 (1-t) \Big[\sum_{jk,lm} \{f_{jk,lm}(x+ty) - f_{jk,lm}(x)\} y^{jk} y^{lm} \Big]\, dt. \end{aligned}$$

The required properties follow from this and the assumptions on f.

Theorem 4.6 : *Let f be as above and let $e(A) = I + f(A)$. Let $Y^n \xrightarrow{e} Y$. Let $\{\tau_i^n\}$ be a partition such that the corresponding J^n satisfy, for some $A \in \mathcal{V}^+$,*

$$\|J^n Y^n - Y^n\| \leqslant \varepsilon_n A \qquad \ldots \quad (4.20)$$

with $\Sigma\, \varepsilon_n^2 < \infty$. Define X^n, Z^n by

$$X_t^n = \sum_{i=0}^{\infty} f\left(Y^n_{\tau^n_{i+1}\wedge t} - Y^n_{\tau^n_i \wedge t}\right)$$

and

$$Z_t^n = \prod_{i=0}^{\infty} e\left(Y^n_{\tau^n_{i+1}\wedge t} - Y^n_{\tau^n_i \wedge t}\right)$$

(These are really finite sum and product respectively. Also, we write $\prod_{i=0}^{\infty} B_i$ to mean $B_0 B_1 B_2 \ldots B_i B_{i+1} \ldots$)

Then $X^n \xrightarrow{e} X$ and $Z^n \xrightarrow{e} Z = e(X)$,

where

$$X_t = \sum_{jk} f_{jk}(0) Y_t^{jk} + \frac{1}{2} \sum_{jk,lm} f_{jk,lm}(0) Y_t^{jk} Y_t^{lm} + \sum_{s \leqslant t} g(0, \Delta Y_s). \qquad \ldots \quad (4.21)$$

Proof : It is easy to see that X^n, Z^n satisfy (4.12). Since $Y^n \xrightarrow{e} Y$, $\|Y^n\|_t^* \to \|Y\|_t^*$ a.s. and hence $\sup_n \|Y^n\|_t^* < \infty$ a.s. From this it follows that the partition $\{\tau_i^n\}$ satisfies (4.11). Thus if we show $X^n \xrightarrow{e} X$, then $Z^n \xrightarrow{e} Z$ would follow from Theorem 4.3.

Note that for any $S \in \mathcal{S}\mathcal{M}(L(d))$ and stop times $\sigma \leqslant \tau$, Ito's formula yields

$$\begin{aligned} f(S_\tau - S_\sigma) = &\sum_{jk} \int 1_{(\sigma,\tau]} f_{jk}(S_{u-} - S_\sigma)\, dS_u^{jk} \\ &+ \frac{1}{2} \sum_{jk,lm} \int 1_{(\sigma,\tau]} f_{jk,lm}(S_{u-} - S_\sigma)\, d[S^{jk}, S^{lm}]_u \\ &+ \sum_{\sigma < u \leqslant \tau} g(S_{u-} - S_\sigma, \Delta S_u). \end{aligned}$$

Using this for $S = Y^n$, $\tau = \tau_{i+1}^n \wedge t$, $\sigma = \tau_i^n \wedge t$ and summing over i, we get

$$Y^n = W^{1,n} + W^{2,n} + W^{3,n}$$

where

$$W^{1,n} = \sum_{jk} \int f_{jk}(Y^n - J^n Y^n)_- \, dY^{n,jk}$$

$$W^{2,n} = \sum_{jk,lm} \int f_{jk,lm}(Y^n - J^n Y^n)_- d[Y^{n,jk}, Y^{n,lm}]$$

$$W_t^{3,n} = \sum_{s \leq t} g((Y^n - J^n Y^n)_{s-}, \Delta Y_s^n).$$

Now, it follows from the local Lipschitz property of f_{jk} and $f_{jk,lm}$ that

$$f_{jk}(Y^n - J^n Y^n) \xrightarrow{o} f_{jk}(0)$$

$$f_{jk,lm}(Y^n - J^n Y^n) \xrightarrow{o} f_{jk,lm}(0)$$

and hence
$$W^{1,n} \xrightarrow{*} \sum_{jk} f_{jk}(0) Y^{jk} \quad \ldots \quad (4.22)$$

$$W^{2,n} \xrightarrow{*} \sum_{jk,lm} f_{jk,lm}(0)[Y^{jk}, Y^{lm}]. \quad \ldots \quad (4.23)$$

Let $U_t^n = W_t^{3,n} - \sum_{s \leq t} g(0, \Delta Y_s)$ and $V_t^{n,ij} = |U^{n,ij}|_t$. Then

$$\|V^n\|_t \leq \sum_{s \leq t} \|g((Y^n - J^n Y^n)_{s-}, \Delta Y_s^n) - g(0, \Delta Y_s)\|$$

$$\leq \sum_{s \leq t} \|g((Y^n - J^n Y^n)_{s-}, \Delta Y_s^n) - g((Y^n - J^n Y^n)_{s-}, \Delta Y_s)\|$$

$$+ \sum_{s \leq t} \|g((Y^n - J^n Y^n)_{s-}, \Delta Y_s) - g(0, \Delta Y_s)\|$$

$$\leq K \left\{ \sum_{s \leq t} \|\Delta Y_s^n - \Delta Y_s\| \, \|\Delta Y_s\|^2 + \|(Y^n - J^n Y^n)_{s-}\| \, \|\Delta Y_s\|^2 \right.$$

$$\leq 3\epsilon_n K A \sum_{jk} [Y^{jk}, Y^{jk}]_t,$$

where $K(t, \omega) = C_\alpha$, with $\alpha = \sup_n \|Y^n\|_t^*(\omega)$, C_α as in Lemma 4.5. This yields $V^n \xrightarrow{o} 0$ which along with (4.22), (4.23) implies $X^n \xrightarrow{*} X$. This completes the proof.

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