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ON TWO COUNTER-EXAMPLES OF NON-COOPERATIVE GAMES WITHOUT NASH EQUILIBRIUM

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SUMMARY. This paper deals with two counter examples which answer some interesting questions related to non-zero sum games in connection with an extension of a theorem of Sion (1958) and a problem posed by Nowak (1988).

1. Introduction

In this paper, we give two examples of two-person non-zero sum games which answer some interesting questions related to game thory. The first example is connected with a very natural question related to an extension of Sion's (1958) theorem which says about the existence of optimal strategies in a two-person zero-sum game under certain assumptions That is, whether every n-person non-zero sum game with compact sets of pure strategies for the players and with quasi-concave and upper semi-continuous pay-off functions has a Nash equalibrium in pure strategies This is answered affirmatively in an otherwise excellent monograph of Vorobev (1984). However, because of a slight overlooking in the proof, the problem was remaining open. Ginchev (1986) tried to answer it negatively, but his counter-example is false (he did not notice that (0, 0) is a Nash equilibrium in his counter example). Example 1 in our paper definitely answers this question negatively.

The second example is related to an open problem posed by Nowak (1988). He proved that every n-person game with compact sets of pure strategies and with upper semi-continuous pay-off functions always has a correlated weak ϵ-equilibrium in mixed strategies (for definition and interesting properties and examples, see (Aumann, 1974, 1987, Moulin and Vial, 1978). He asks the following question : whether such a game has a Nash equilibrium (ϵ-Nash equilibrium) in mixed strategies. Example 2 of this paper answers this question only partially. Namely, it proves that Nash equilibrium may

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not exist, but it is still an open problem if e-Nash equilibrium in mixed strategies must exist in all such games.

Now we are ready to present our counter-examples :

Example 1 : The two-person non-zero sum game $< I, \{X_i\}_{i \in I}, \{F_i\}_{i \in I} >$, where $I = \{1, 2\}$ is the set of players, with $X_1 = X_2 = [0, 1]$, as the set of strategies, and $F_i : X_1 \times X_2 \to x$, $i = 1, 2$. as the pay off functions for each player determined by

$$F_1(x, y) = \begin{cases} 1-y+x & \text{if } x \leqslant y \leqslant 1 \\ 1-x+y & \text{if } 0 \leqslant y < x \end{cases}$$

and

$$F_2(x, y) = \begin{cases} \dfrac{y}{1-2x} & \text{if } 0 \leqslant x \leqslant \dfrac{1}{2}, 0 \leqslant y < -2x+1 \\ \dfrac{2x+3y-1}{2y} & \text{if } 0 \leqslant x < \dfrac{1}{2}, -2x+1 \leqslant y \leqslant 1 \\ \dfrac{3}{2} & \text{if } \dfrac{1}{2} \leqslant x \leqslant 1, \ y > 0 \\ 2 & \text{if } \dfrac{1}{2} \leqslant x \leqslant 1, \ y = 0. \end{cases}$$

is an example which does not have any Nash equilibrium in pure strategies, with compact sets of pure strategies and quasi-concave and upper semicontinuous pay-off functions.

Proof : It is easy to prove that the functions F_1 and F_2 defined above are upper semi-continuous and quasi-concave, by showing that for all real c, the sets $\{(x, y) : x, y \in [0, 1], F_i(x, y) \geqslant c\}$, $i = 1, 2$, are closed and convex. Let us define

$$\phi_1 = \{(x, y) : y \in [0, 1] \text{ and } \max_{0 \leqslant \tilde{x} \leqslant 1} F_1(\tilde{x}, y) \leqslant F_1(x, y)\}$$

$$\phi_2 = \{(x, y) : x \in [0, 1] \text{ and } \max_{0 \leqslant \tilde{y} \leqslant 1} F_2(x, \tilde{y}) \leqslant F_2(x, y)\}.$$

It is clear that

$$\phi_1 = \{(x\ y) : x, y \in [0, \ 1] \text{ and } x = y\}$$

$$\phi_2 = \{(x, y) : x, y \in [0, 1] \text{ and } y = 1 \text{ when } 0 \leqslant x < \frac{1}{2} \text{ and }$$

$$y = 0 \text{ when } \frac{1}{2} \leqslant x \leqslant 1\}.$$

Hence $\Phi_1 \cap \Phi_2 = \phi$, Therefore, there is no Nash equalibrium in pure strategies, ending the proof.

Remark 1 : Notice that the game in example 1 satisfies stronger assumptions than the analogous Sion's minimax theorem ; namely, the pay-off functions are upper semi-continuous and quasi-concave on $X_1 \times X_2$ not only in each variable. However, this game possesses an ϵ-Nash equilibrium in pure strategies. One can verify that the family $\left\{\frac{1}{2}-\frac{1}{n}, \frac{1}{2}\right\}$ under $n \to \infty$, of pairs of players' pure strategies constitutes some ϵ-Nash equilibrium with pay-off value (1, 3/2). But it is an open problem, whether this is true in general under the assumptions mentioned above.

Remark 2 : Notice that in example 1, a Nash equilibrium exists in mixed strategies. One can check that the pair $\mu_1^* = \frac{1}{3}\delta_{\{0\}} + \frac{2}{3}\delta_{\{1\}}, \mu_2^* = \frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}}$, (here, $\delta_{\{x\}}$ denotes the probability measure concentrated at the point x) is a Nash equilibrium with pay off value $\left(\frac{1}{2}, \frac{4}{3}\right)$. However, we do not know whether there always exists a Nash equilibrium in mixed strategies, under the assumption of upper semi-continuity and quasi-concavity with compact sets of pure strategies. At the end, notice that the pay off value $\left(\frac{1}{2}, \frac{4}{3}\right)$ in mixed strategies is less useful for both players than payoff value $\left(1, \frac{3}{2}\right)$ in pure strategies (Remark 1).

Example 2 : The two-person non-zero sum game $< I, \{F_i\}, \{F_i\}_{i \in I} >$, $I = \{1, 2\}$, with $X_1 = X_2 = [0, 1]$ and with F_i defined on $X_1 \times X_2$ by

$$F_1(x, y) = \begin{cases} 0 & \text{if } 0 < y < 1 \\ x & \text{if } y = 1 \\ 1-x^2 & \text{if } y = 0, \end{cases}$$

$$F_2(x, y) = \begin{cases} 0 & \text{if } 0 < y < 1 \text{ or } (y = 0, x < 2/3) \\ 2 & \text{if } y = 0, x \leqslant 2/3 \\ 1 & \text{if } y = 1, \end{cases}$$

is an example of a game with compact sets of players' pure strategies and with upper semi-continuous pay-off functions on $X_1 \times X_2$ which possesses ϵ-Nash equilibrium but there is no Nash equilibrium.

Proof: Clearly F_1 and F_2 are upper semi-continuous. Let $\mathscr{M}$ be the set of probability measures on [0, 1]. Suppose on the contrary, that there is an Nash equilibrium in mixed strategies. This means that there exists $(\mu_1^*, \mu_2^*) \in \mathscr{M} \times \mathscr{M}$ such that

$$F_1(\mu_2^*, \mu_2^*) \geqslant F_1(\mu_1, \mu_2^*) \text{ for all } \mu_1. \qquad \ldots \quad (1)$$

$$F_2(\mu_1^*, \mu_2^*) \geqslant F_2(\mu_2^*, \mu_2) \text{ for all } \mu_2. \qquad \ldots \quad (2)$$

Since all the pure strategies $o < y < 1$ for player 2 are dominated by pure strategy $y = 1$, μ_2^* is totally concentrated at two points $y = 0$ and $y = 1$. Hence $\max\limits_{\mu_2 \in \mathscr{M}} F_2(\mu_1^*, \mu_2) = \max\limits_{0 \leqslant \beta \leqslant 1} F_2(\mu_1^*, \beta\delta_{\{0\}} + (1-\beta)\delta_{\{1\}})$. But the right hand side of the last equation is equal to

$$\max_{0 \leqslant \beta \leqslant 1} \beta\{2\mu_1^*[2/3, 1]-1\}+1 = 2\beta\mu_1^*[2/3, 1]+(1-\beta)\,\mu_1^*[0, 1]$$

Therefore $\max\limits_{\mu_2 \in \mathscr{M}} F_2\,((\mu_1^*, \mu_2)$ is achieved exactly at $\mu_2 = \mu_2^*$ where

$$\mu_2^* = \begin{cases} \delta_{\{0\}} & \text{if } \mu_1^*[2/3, 1] > \frac{1}{2} \\ \delta_{\{1\}} & \text{if } \mu_1^*[2/3, 1] < \frac{1}{2} \\ \underbrace{\gamma\delta_{\{0\}} + (1-\gamma)\delta_{\{1\}}}, & 0 \leqslant \gamma \leqslant 1 \text{ if } \mu_1^*\,[2/3, 1] = \frac{1}{2}.\ \gamma\delta(0)+(1-\gamma)_{\{1\}}. \end{cases} \qquad \ldots \quad (3)$$

Further, we easily get

$$F_1(\mu_1, \beta\delta_{\{0\}}+(1-\beta)\delta_{\{1\}}) = \int_0^1 [(1-\beta)x+\beta(1+x^2)]d\mu_1$$

Besides for every β, the function $(1-\beta)x+\beta(1-x^2)$ has a unique maximum at the point x_{max} where

$$x_{max} = \begin{cases} \dfrac{1-\beta}{2\beta} & \text{if } \beta > 1/3 \\ 1 & \text{if } \beta \leqslant 1/3. \end{cases}$$

Hence, $\max\limits_{\mu_1 \in \mathscr{M}} F_1(\mu_1, \beta\delta_{\{0\}}+1-\beta\delta_{\{0\}})$ is achieved exactly at $\mu_1 = \mu_1^*$ where

$$\mu_1^* = \begin{cases} \delta\left\{\dfrac{1-\beta}{2\beta}\right\} & \text{if } \beta \geqslant 1/3 \\ \delta_{\{1\}} & \text{if } 0 \leqslant \beta < 1/3. \end{cases} \qquad \ldots \quad (4)$$

Now, a very simple analysis leads to the conclusion that there exists no pair $(\mu_1, \mu_2) \in \mathscr{M} \times \mathscr{M}$ satisfying (3) and (4), and, thereby (1) and (2).

Therefore, there does not exist a Nash equilibrium in mixed strategies. To prove that there is ϵ-Nash equilibrim, it suffices to consider, e.g. the family $\{\mu_1^{(n)}, \mu_2^{(n)}\}$ as $n \to \infty$, where, for all n,

$$\mu_1^{(n)} = \frac{1}{2}\,\delta_{\left\{\frac{2}{8}-\frac{1}{n}\right\}} + \frac{1}{2}\,\delta_{\left\{\frac{1}{3}+\frac{1}{n}\right\}}$$

$$\mu_2^{(n)} = \frac{3}{7}\,\delta_{\{0\}}, + \frac{4}{7}\delta_{\{1\}},$$

and the pay-off value is equal to $\left(\frac{13}{21}, 1\right)$. So, the proof has been completed.

Remark 3 : Unfortunately, we do not know any other example of the game under the assumptions of example 2 on the strategies of the two players and on the pay-off functions, which does not possess an ϵ-Nash equilibrium. This problem is intriguing in view of the result of Nowak (1988), (as also being an extension of Glicksberg's therem (1952), since, we do not know as yet any game having correlated weak ϵ-equilibrium but not ϵ-Nash equilibrium.

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