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NOTES

TWO REMARKS ON VAGUE CONVERGENCE

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SUMMARY. The purpose of this note is to place on record two items of information concerning vague convergence of probability measures (p.m.'s) on the real line, which may have some novelty, curiosity and classroom value.

(1) *The Stirling approximation to the Gamma function as a vague convergence result* : As is well-known, this approximation is given by

$$\Gamma(\alpha) \sim \sqrt{2\pi}\, e^{-\alpha}\alpha^{\alpha-1/2} \text{ as } \alpha \to \infty \qquad \text{.. (1)}$$

where $\sim$ indicates that the ratio RHS/LHS $\to 1$ as $\alpha \to \infty$. Titchmarsh (1939, p. 58), for instance shows how this can be derived with additional argument from the Stirling approximation to $n! = \Gamma(n+1)$. Khan (1974) and Wong (1977) used vague convergence arguments to derive the approximation for $n!$. The traditional arguments (for $n!$) establish (1), first without the constant $\sqrt{2\pi}$ being specified (Feller, 1968, p. 52 for instance) and then using Wallis' formula for π to identify it (for a variant, see Feller, 1968, p. 180). The following effortless derivation of (1) using vague convergence arguments streamlines the Khan-Wong arguments for $n!$ to the more general (1). (This approximation is needed in particular to show that the probability density function (p.d.f.) of the Student t-distribution with n degrees of freedom converges to the p.d.f. of the standard normal distribution as $n \to \infty$, whence we can conclude using Scheffe's 'useful convergence theorem' that the said distribution converges vaguely to the standard normal uniformly over the whole real line.) As pointed out by the referee, the litertature on Stirling's formula is extensive : see, for instance, also Cramer (1946, Section 12.5), Knopp (1928, p. 543), Namias (1986), and Diaconis and Freedman (1986). In particular, it would be desirable to obtain vague convergence arguments to prove the asymptotic formula for $\log\Gamma(\alpha)$ involving the Bernoulli numbers, usually derived by applying the Euler-MacLaurin summation formula ; vide Cramer (1946) or Knopp (1928).

Let X_α be a r.v. with p.d.f. given by $e^{-x}x^{\alpha-1}/\Gamma(\alpha)$ for $x > 0$. A standard argument using characteristic functions readily shows that (the standardized form of X_α with its sign changed :) $Y_\alpha = (\alpha - X_\alpha)/\sqrt{\alpha} \xrightarrow{d} Z_0$, where Z_0 is a 'dummy' random variable with the standard normal distribution : $Z_0 \sim \Phi$. Since $x \to x^+$ is a continuous map from R^1 into R^1, it follows that $Y_\alpha^+ \xrightarrow{d} Z_0^+$. Noting that $E(Y_\alpha^+)^2 \leqslant EY_\alpha^2 \underset{\alpha}{=} 1$,

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a standard vague convergence argument (see for instance Chung, 1974, Th. 4.5.2, p. 95) shows that $EY_\alpha^+ \to EZ_0^+$. The d.f. of Z_0^+ being given by : 0 for $x < 0$, $\Phi(x)$ for $x \geqslant 0$, we have $EZ_0^+ = 1/\sqrt{2\pi}$ while

$$EY_\alpha^+ = \left(\int_0^\alpha (\alpha - x)e^{-x}x^{\alpha-1}dx\right)/\{\sqrt{\alpha}\Gamma(\alpha)\} = [e^{-x}x^\alpha]_0^\alpha/\{\sqrt{\alpha}\Gamma(\alpha)\}$$

$$= e^{-\alpha}\alpha^{\alpha-1/2}/\Gamma(\alpha)$$

and we have (1). The constant $\sqrt{2\pi}$ associated with the normal law as well as with the Stirling approximation thus acquires transparent significance in this derivation.

(2) *A criterion for the vague convergence of p.m.'s on R^1 to a limit p.m.* : Chung (1974, p. 94), mentions a criterion for such convergence, namely, that for every $g \in C_B(R^1)$, the set of all bounded continuous maps from R^1 to R^1, the sequence $\{\int g d\mu_n\}$ should converge, $\{\mu_n\}$ being the given sequence of p.m.'s. The source is not specified, and in a personal communication it was merely stated that this result belongs to the 'folklore'. K. R. Parthasarathy has pointed out that one can be more precise : the problem was considered by Aleksandrov (1940, 41, 43) and in Varadarajan (1961), the idea of proof being that Λ below is a non-negative (hence in particular continuous) linear functional on $C_B(R^1)$ with $\Lambda(I) = 1$, and then the Riesz representation theorem can be invoked to make the desired conclusion. The following proof uses a summability lemma due to Steinhaus, quoted below, which also leads to an elegant proof of the Vitali-Hahn-Saks (VHS) theorem based on it : for proofs (of the lemma and of the VHS theorem), we refer the reader to Ash (1972, 42-43).

Lemma (Steinhaus) : ***Let $\{a_{ij}\}$ be a double sequence of real numbers such that***

(*i*) ***for every j, $a_{ij} \to 0$ as $i \to \infty$,***

(*ii*) ***$\Sigma_j |a_{ij}| \leqslant M$ for every i, and***

(*iii*) ***$\Sigma_j a_{ij} = 1$ for every i.***

Then there exists a sequence $\{x_j\}$ comprising 0's ***and*** 1's ***alone such that, if $y_i = \Sigma_j a_{ij}x_j$, then $\{y_i\}$ does not converge.***

Let it be given that $\{\mu_n\}$ is a sequence of p.m.'s on the real line such that $\{\int g d\mu_n\}$ converges for every $g \in C_B(R^1)$; denote the limit by $\Lambda(g)$. Let $\{\nu_n\}$ denote a vaguely convergent subsequence of $\{\mu_n\}$ and ν_0 the vague limit thereof. We shall show that ν_0 is necessarily a p.m. It will then follow that ν_0 is the limit p.m. for *every* vaguely convergent subsequence of $\{\mu_n\}$ and so $\mu_n \xrightarrow{v} \nu_0$. We shall use the fact that for $g \in C_0(R^1)$, i.e., for $g \in C_B(R^1)$ with $g(x) \to 0$ as $|x| \to \infty$, we have $\int g d\nu_n \to \int g d\nu_0$, as is well-known, or, $\Lambda(g) = \int g d\nu_0$ for $g \in C_0(R^1)$.

Let g_1 denote the usual "triangular function with $[-1, 1]$ as base", i.e., $g_1(x) = 1 - |x|$ for $|x| \leqslant 1$ and zero otherwise. For $n \geqslant 2$, let g_n denote the continuous function on R^1 made up of the two triangular functions with bases $[-n, -n+2]$ and $[n-2, n]$. Then, $h_n = g_1 + \dots + g_n$ is the trapezoidal function $:= 1$ on

$[-n+1, n-1]$, linear on $[-n, -n+1]$ and on $[n-1, n]$ and zero outside $[-n, n]$. We then have $\Sigma_j g_j \equiv 1$. Since the g_j are non-negative continuous functions and $\Sigma_j g_j$ is continuous, the well-known theorem of Dini's enables us to conclude that the convergence of this series is uniform on every compact interval; in turn, this implies that the sum-function of any sub-series of this series is continuous (and $\leqslant 1$). Thus, if $\{x_j\}$ is any sequence of 0's and 1's, then $g = \Sigma_j x_j g_j \in C_B(R^1)$.

Now suppose if possible that $\alpha = \nu_0(R^1) < 1$, and define: $a_{ij} = (\int g_j d\nu_i - \int g_j d\nu_0)/(1-\alpha)$. Then, since $\Sigma\, g_j \equiv 1$, conditions (i)—(iii) of Steinhaus' lemma are statisfied (for $M = (1+\alpha)/(1-\alpha)$). Hence there exists a sequence $\{x_j\}$ of 0's and 1's such that if $y_i = \Sigma_j a_{ij} x_j$ then $\{y_i\}$ is non-convergent. If, for this choice of $\{x_j\}$, we take $g = \Sigma_j x_j g_j$, we see that $y_i = (\int g d\nu_i - \int g d\nu_0)/(1-\alpha)$ converges to $(\Lambda(g) - \int g d\nu_0)/(1-\alpha)$. This contradiction shows that $\alpha < 1$ is impossible, so that ν_0 is a p.m., and, as argued earlier, $\nu_n \xrightarrow{v} \nu_0$ so that $(\int g d\mu_n \to \int g d\nu_0)$ or) $\Lambda(g) = \int g d\nu_0$ for every $g \in C_B(R^1)$.

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