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CONSTRUCTION OF SOME SERIES OF ORTHOGONAL ARRAYS

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SUMMARY. A few general methods are developed in the present paper for constructing orthogonal arrays of various strengths, particularly of strength two and three. Some new and interesting series of orthogonal arrays constructed in the paper as illustrations are as follows :

(*i*) $OA((s-1)^m s^{n+m}, (s^{n-1}+s^{n-2} \ldots +s+1)\ (s-1)^{2m}+(s-1)^{2(m-1)}+\ldots+1,\ s,\ 2)$ for all $n \geqslant 2$ and $m \geqslant 1$, when s and $s-1$ are both primes or prime powers.

(*ii*) $OA(2(s-1)^m s^{n+m}, \{2(s^{n-1}+s^{n-2} \ldots +s)+1\}(s-1)^{2m}+(s-1)^{2(m-1)} \ldots +1, s, 2)$, when s and $s-1$ are both primes or prime powers, $n \geqslant 2$ and $m \geqslant 1$.

(*iii*) $OA(s^{p+3}, r, s, 3)$, $p \geqslant 0$ always implies the existence of $OA(\lambda^3 s^{p+3}, sr, \lambda s, 3)$, when both λ and s are powers of same prime.

(*iv*) $OA(\lambda^{t-1} s^t, \lambda s, s, t)$, $t \geqslant 3$, when both λ and s are powers of the same prime.

1. Introduction

Orthogonal arrays (in short, OA's) were first introduced by Rao (1946). Since then, efforts have been made by several authors to construct OA's of various strengths and indices, and to provide suitable upper and/or lower bounds to the number of constraints for an array with a given index, level and strength. The use of orthogonal arrays as fractional factorial experiments is too well known to be repeated. Bose (1960) first pointed out the interrelationship between orthogonal arrays and error correcting codes. Importance of the study of properties and of the construction of OA's has been greatly enhanced, because of their application in error correcting and error detecting codes.

In the present paper, a few methods are developed for the construction of orthogonal arrays, particularly those of strength two and three and some new series are constructed by those methods, as illustrations.

2. Definitions and notation

The definitions of an orthogonal array (OA) and balanced array (BA) are to be found in Rao (1973) and we follow the symbols therein. A rectangular array (N, r, s) is an $r \times N$ matrix with elements from Σ of $s \geqslant 2$ elements. OA (N, r, s, t) represents a (N, r, s) array which is orthogonal of strength t. We shall use the shorter notation OA (r, t) to denote an OA (N, r, s, t) where the other parameters of the array are obvious from the context.

Extending the definition of a completely resolvable orthogonal array of strength two in Bose and Bush (1952), we define an OA $(\mu\lambda s^t, r, s, t)$ to be μ-resolvable, if it is the juxtaposition of μs different arrays OA $(\lambda s^{t-1}, r, s, t-1)$, $t \geqslant 2$.

We shall call an OA $(\lambda s^t, r, s, t)$, $t \geqslant 2$ completely decomposable, if it is the justaposition of λs^{t-1} different arrays OA $(s, r, s, 1)$.

The rows of an array (N, r, s) are also called its constraints and N, the number of columns of the array, its size. The maximum number of constraints, r of an OA of size N, levels s and strength t is denoted by $f(N, s, t)$.

Let Σ be a finite module of s elements, viz., the null element e_0 and other elements $e_1, e_2, \ldots, e_{s-1}$. For $t \geqslant 2$, let us consider the s distinct t-tuples formed by the elements of Σ. They can be divided into s^{t-1} sets, $M_1, M_2, \ldots, M_{s^{t-1}}$, each consisting of s distinct t-tuples such that given any t-tuple in a set, say M_i, all the s t-tuples in the set can be obtained by adding successively the elements $e_0, e_1, \ldots, e_{s-1}$ of Σ to each element of the given t-tuple. A set of t-tuples, M_i satisfying this property will be called a closed set in future discussions. Suppose that it is possible to find an array B of r rows and $n = \mu s^{t-1}$ columns with elements belonging to Σ such that in every t-rowed submatrix of B, the number of t-tuples belonging to each M_i is the same and equals μ. Such an array B will be denoted by $S_t(\mu s^{t-1}, r, s)$, $t \geqslant 2$. Moreover, if the array B is also orthogonal of strength $t-1$, we will denote it as $S_t{}^*(\mu s^{t-1}, r, s)$. The shorter notation $S_t(r)$ and $S_t{}^*(r)$ will be used where there is no scope for confusion. The arrays of the type $S_t(r)$ and $S_t{}^*(r)$ were considered by Seiden (1954) and Seiden and Zemach (1966).

Standard notation is used for partially balanced incomplete block designs and particularly, group divisible (GD) designs. The definition of a resolvable GD design is to be found in Raghavarao (1971).

The matrix operations which will be frequently used in the construction of OA's in the subsequent sections are described below :

Let $A_{r_1 \times n_1} = (a_{ij})$ and $B_{r_2 \times n_2} = (b_{ij})$ be two matrices written with elements from the finite module Σ.

(i) Let $r_1 = r_2 = r$. Then, we define $A \oplus B$ as an $r \times n_1 n_2$ matrix where for any column of A, say α_i and any column of B, say β_j, we define a column $\alpha_i + \beta_j$ of $A \oplus B$, $i = 1, 2, \ldots, n_1$; $j = 1, 2, \ldots, n_2$. Here the symbol $+$ stands for the usual vector addition.

(ii) The Kronecker sum, $A \bigcup B$ denotes an $r_1 r_2 \times n_1 n_2$ matrix of the form $(A(ij))$, where $A(ij) = A + b_{ij}.J_{r_1 \times n_1}$ and $J_{r_1 \times n_1}$ stands for a $r_1 \times n_1$ matrix of all 1's, $i = 1, 2, \ldots, r_2$, $j = 1, 2, \ldots, n_2$.

3. Construction of S_t's and $S_{t^\bullet}$'s

In the present section, construction of some series of S_t's and $S_{t^\bullet}$'s is taken up. They will be utilised in the two subsequent sections in constructing orthogonal arrays of various strengths and that is the main aim of the paper.

3.1 *S_2's and $S_{2^\bullet}$'s.* Let s be a prime or a prime power $\geqslant 2$ and let $o = e_0, e_1, \ldots, e_{s-1}$ be the s elements of $GF(s)$. In this case the multiplication table B of the elements is obviously an $S_2(s, s, s)$ and deleting the row E_{1s} from B, when $s > 2$ we obtain an $S_{2^\bullet}(s, s-1, s)$. E_{mn} denotes an $m \times n$ matrix with all elements e_0.

The existence of a resolvable GD $(s, s, s, 0, 1)$ is always ensured when s is a prime or a prime power (Raghavarao, 1971). Let $N_s^{(2)}$ be the incidence matrix of the resolvable GD design. We can write $N_s^{(2)}$ as $N_s^{(2)} = [N_1' . N_2' \ldots N_s']'$, where $N_i = [N_{i1}\, N_{i2} \ldots N_{is}]$, every N_{ij} is an $s \times s$ matrix with a single unity in each row and column, and the remaining elements o, $i, j = 1, 2, \ldots, s$. One such construction was given by Federer, Joiner and Raghavarao (1974), where their M is our $N_s^{(2)}$. Moreover, $N_i . N_i' = sI_s$, $i = 1, 2, \ldots, s$ and $N_i . N_{i'}' = J_{ss}$ $i \neq i'$, $i, i' = 1, 2, \ldots, s$.

It is observed that the matrix B which constitutes an $S_2(s, s, s)$ is of the form $(E_{s1} : S)$. Deleting this first column from B, we are left with S, an $s \times s-1$ matrix.

Let S_{ij} be obtained from N_{ij}, $i, j = 1, 2, \ldots, s$ of $N_s^{(2)}$ in the following manner :

If the (f, g)-th cell of N_{ij} contains unity, f-th row of N_{ij} is to be replaced by g-th row of S. Let us replace each N_{ij} by the corresponding S_{ij} in $N_s^{(2)}$, $i, j = 1, 2, \ldots, s$ and denote the resulting matrix by $S_s^{(2)}$.

It is easy to see that the matrix $C = [E_{s^2 1} : S_s^{(2)} : E_{s1} \bigcup S]$ is an $S_2(s^2, s^2, s)$ i.e. an S_t matrix with $\mu = s$ and $t = 2$. Then, we have the following result :

Theorem 3.1.1 : *If s is a prime or a prime power, there always exists an $S_2(s^p, s^p, s)$, $p = 1, 2$. If, in addition $s \geqslant 3$, there exists an $S_{2^\bullet}(s, s-1, s)$.*

Let $s-1$ too be a prime or a prime power in addition to s. We can start with a resolvable GD$(s-1, s-1, s-1, 0, 1)$, whose incidence matrix $N^{(2)}_{s-1}$ is partitioned in the same manner as $N^{(2)}_s$. Then, deleting any single column from S, we get an $(s-1)\times(s-1)$ matrix, say, denoted by $S^{(1)}$. Working with $S^{(1)}$ and the partitioned incidence matrix $N^{(2)}_{s-1}$ and following exactly the same procedure as in the construction of $S^{(2)}_s$, we get an $(s-1)^2\times(s-1)^2$ matrix $S^{(2)}_{s-1}$. Now, it is easy to see that

$$D = [S^{(2)}_{s-1} : E_{s-1,1} \bigcup S^{(1)}] \text{ is an } S_2((s-1)s, (s-1)^2, s).$$

Thus, we have the following theorem.

Theorem 3.1.2 : *If s and $s-1$ are both primes or prime powers, there exists an $S_2((s-1)s, (s-1)^2, s)$.*

3.2 *General S_t's and S_{t^*}'s, $t \geqslant 2$.* In general when s is a prime or a prime power, by modifying slightly the method of construction of OA$(s^t, s+1, s, t)$, $s > t$ given in Bush (1952a), we can obtain an $S_{t^*}(s^{t-1}, s-1, s)$. The modification to be effected is as follows :

Let us consider s^{t-1} polynomials $y_j(x) = a_{t-1}x^{t-1}+a_{t-2}x^{t-2}+\ldots+a_1x$ in some order, where the coefficients range over the field GF(s). Let us construst an array with column subscript j ranging from 0 to $(s^{t-1}-1)$ and the row subscript range from 1 to $s-1$. An $s-1$ by s^{t-1} is thus formed by writing in the i-th row and j-th column the element e_u, where $y_j(e_i) = e_u$, where $o = e_0, e_1, e_2, \ldots, e_{s-1}$ are the elements of GF(s). It is easy to see that the array so formed is an $S_{t^*}(s-1)$.

Now for any s, given an $S_{t^*}(\mu s^{t-1}, r, s)$, $t \geqslant 2$, an $S_t(\mu s^{t-1}, r+1, s)$ can always be obtained by adding one more row, viz., $E_{1,\mu s^{t-1}}$ to the former. Hence,

Theorem 3.2.1 : *If s is a prime or a prime power $\geqslant t$, there always exists an $S_t(s^{t-1}, s, s)$. There also exists an $S_{t^*}(s^{t-1}, s-1, s)$, when $s \geqslant t \geqslant 2$ and $s > 2$.*

The following theorem will be useful in obtaining S_t's and S_{t^*}'s, when s is not a prime or a prime power.

Theorem 3.2.2 : *The existence of $S_t(\mu_i s_i^{t-1}, r_i, s_i)$, $i = 1, 2, \ldots, m$ implies the existence of $S_t(\mu s^{t-1}, r, s)$, where $\mu = \mu_1, \mu_2, \ldots, \mu_m$, $s = s_1 s_2, \ldots, s_m$, $r = min(r_1, r_2, \ldots, r_m)$.*

Proof: The result is analogous to the product of OA's as discussed in Bush (1952b) and is not difficult to arrive at by following similar arguments. Hence, the proof is omitted. The result of the theorem 3.2.2 remains valid, if the S_t's are replaced by $S_{t}\bullet$'s.

Another theorem which will be very useful in constructing orthogonal arrays in the subsequent sections is stated and proved in the following lines.

Theorem 3.2.3: *The existence of an $S_t(\lambda s^{t-1}, r, s)$ implies the existence of an $S_t(\mu s_2^{t-1}, r, s_2)$, where $s = s_1 s_2$ and $\mu = \lambda s_1^{t-1}$.*

Proof: Let $e_0^{(i)}, e_1^{(i)}, \ldots, e_{s_i-1}^{(i)}$ denote the s_i elements of the module Σ_i, $i = 1, 2$. The module Σ of $s_1 s_2 = s$ elements can be represented by the $s_1 s_2$ ordered pairs $(e_i^{(1)}, e_j^{(2)})$, $i = 0, 1, \ldots, s_1-1$, $j = 0, 1, \ldots, s_2-1$. The addition of two elements $(e_{i_1}^{(1)}, e_{i_2}^{(2)})$ and $(e_{j_1}^{(1)}, e_{j_2}^{(2)})$ in Σ is defined as usual by $(e_{i_1}^{(1)}+e_{j_1}^{(1)}, e_{i_2}^{(2)}+e_{j_2}^{(2)})$.

Now, let $M_1^{(i)}, M_2^{(i)}, \ldots, M^{(i)}{}_{s_i^{t-1}}$ represent the closed sets of t tuples formed by the elements of Σ_i, $i = 1, 2$. Similarly, let M_{ij}'s, $i = 1, 2, \ldots, s_1^{t-1}$ and $j = 1, 2, \ldots, s_2^{t-1}$ represent the closed sets of t-tuples formed by the elements of Σ, where if α is any t-tuple belonging to M_{ij} for some i and j, the t-tuple formed by only the first coordinates of the elements of α belongs to $M_i^{(1)}$ and the t-tuple formed by only the second coordinates of the elements of α belongs to $M_j^{(2)}$.

Now, let the $S_t(\lambda s^{t-1}, r, s)$ be written with the elements of Σ. Then, let us introduce the mapping

$$f: \Sigma \to \Sigma_2$$

$$f(e_i^{(1)}, e_j^{(2)}) = e_j^{(2)}, \text{ for all } i, j,$$

and thus rewrite the array $S_t(r)$ in terms of the elements of Σ_2. Then, for any t-tuple in $S_t(\lambda s^{t-1}, r, s)$ belonging to the set M_{ij}, for some i and j, the corresponding t-tuple in the new array will belong to $M_j^{(2)}$. So, in any t-rowed submatrix of the new array there are $\mu = \lambda s_1^{t-1}$, t-tuples belonging to any $M_j^{(2)}$, $j = 1, 2, \ldots, s_2^{t-1}$

Hence, the theorem is proved.

Making use of Theorems 3.1.1, 3.1.2 and 3.2.1 along with the Theorem 3.2.3, we obtain the following corollaries.

Corollary 3.2.1 : *There always exists an* S_2 $(s_1^p s_2^p, s_1^p s_2^p, s_2)$. $p = 1, 2,$ *when* s_1 *and* s_2 *are powers of the same prime.*

Corollary 3.2.2 : *There exists an* S_2 $((s_1s_2-1)\, s_1s_2, (s_1s_2-1)^2, s_2)$, *when* s_1 *and* s_2 *are powers of the same prime and* $s-1 = s_1s_2-1$ *is a prime or a prime power.*

Corollary 3.2.3 : *There exists an* $S_t(s_1^{t-1}\, s_2^{t-1} \cdot s_1s_2, s_2)$, $t \geqslant 2$, *when* s_1 *and* s_2 *are powers of the same prime.*

4. Construction of orthogonal arrays

Let us state 3 lemmas which will be needed to prove the main results of this section.

Lemma 4.1 : *Given an array B with entries from a module* Σ *of s elements, which is* $S_t(\mu s^{t-1}, r, s)$ *and a vector* $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ *with* $m = qs$ *such that among the elements of* α *each element of* Σ *occurs q times,*

$$A = \alpha' \bigcup B \text{ gives an } OA(q\mu s^t, r, s, t).$$

Lemma 4.2 : *If A is an* $OA(\lambda s^t, t, s, t)$ *and* $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ *is any t-tuple with all elements* $\epsilon\Sigma$, $\alpha \oplus A$ *is also an* $OA(\lambda s^t, t, s, t)$.

Lemma 4.3 : *Let A be a* $t \times n$ *array* $(n = \lambda_1 s^{t-1})$, *where first* $(t-1)$*-rows constitute an* $OA(t-1, t-1)$ *and the t-th row is identical with the* $(t-1)$*-th row. Let B be a* $t \times m$ *array* $(m = \lambda_2 s)$ *the last two rows of which constitute an* $S_2(2)$. *All the elements of the arrays* $\epsilon\, \Sigma$. *Then,*

$$B \oplus A \text{ is } OA(t, t).$$

Proof : The first two lemmas are obvious and we consider only the proof of the third lemma. Let us consider any column β of B and let its last two elements be b_{t-1} and b_t. In $\beta \oplus A$, the first $t-1$ rows constitute an OA$(t-1, t-1)$ by Lemma 4.2 and in all the t-tuples of $\beta \oplus A$ the t-th element differs from $(t-1)$-th element by $b_t - b_{t-1}$. As the last two rows of B constitute an $S_2(2)$, in the difference series of the last two rows of B (t-th row minus $(t-1)$-th row), each element of Σ occurs λ_2 times. Hence, $B \oplus A$ is an OA(t, t) in which each t-tuple occurs $\lambda_1\lambda_2$ times.

4.1 *Orthogonal arrays of strength two* : A general result in the construction of orthogonal arrays of strength two is contained in the following theorem.

Theorem 4.1.1 : *The existence of an* $OA(\lambda s^2, r_1, s, 2)$ *and* $S_2(\mu s, r_2, s)$ *implies the existence of a* μ*-resolvable* $OA(\lambda\mu s^3, r_1r_2, s, 2)$. *Moreover, if the*

$OA(\lambda s^2, r_1, s, 2)$ is completely decomposable, the resulting OA is also completely decomposable.

Proof: Let the arrays be written with elements from the module Σ of s elements. Let the array A with r_1 rows and $m = \lambda s^2$ columns represent the $OA(\lambda s^2, r_1, s, 2)$ and let the array B with r_2 rows and $n = \mu s$ columns represent the $S_2(\mu s, r_2, s)$.

Let $C = A \bigcup B$. Then, C can be easily proved to be an $OA(r_1 r_2, 2)$. with the help of the Lemmas 4.2 and 4.3.

But C is the juxtaposition of μs arrays each of which is an $OA(r_1 r_2, 1)$. So, the array is μ-resolvable. Moreover, if the $OA(\lambda s^2, r_1, s, 2)$ is completely decomposable, the resulting array given by C is also obviously completely decomposable.

It is to be noted that if any $OA(\lambda \mu s^t, r, s, t)$, $t \geqslant 2$ is μ-resolvable, we can write the corresponding array, say, C decomposed into μs blocks as

$$C = [C(1) : C(2) : \dots : C(\mu s)],$$

where each of $C(i)$'s is an $OA(r, t-1)$, $i = 1, 2, \dots, \mu s$. We can add one more row to C by writing e_0 for all the columns of the first blocks $C(1) \dots C(\mu)$, e_1 for all the columns of the next blocks $C(\mu+1), \dots, C(2\mu)$ and so on e_{s-1} for all the columns of the last blocks, $C((s-1)\mu+1), \dots C(s\mu)$. Thus, we obtain an $OA(\lambda \mu s^t, r+1, s, t)$.

So, in the Theorem 4.1.1, because the resulting $OA(\lambda \mu s^3, r_1 r_2, s, 2)$ is μ-resolvable, we can always add one more row to it to obtain an $OA(\lambda \mu s^3, r_1 r_2+1, s, 2)$.

Now, if the resulting array C in Theorem 4.1.1 is completely decomposable, we can write C decomposed into $\lambda \mu s^3$ blocks as

$$C = [C(1) : C(2) : \dots : C(\lambda \mu s^3)],$$

where each one of the blocks is an $OA(r_1 r_2, 1)$. If there exists an $OA(\lambda \mu s^3, r_3, s, 2)$ and representing the array corresponding to the $OA(\lambda \mu s^2, r_3, s, 2)$ by D, we can show the matrix

$$E = \begin{bmatrix} E_u \bigcup D \\ C \end{bmatrix} \text{ is an OA of strength 2.}$$

So, in this case we get an $OA(r_1 r_2 + r_3, 2)$. Thus, we have

Corollary 4.1.1 : *The existence of an* $OA(\lambda s^2, r_1, s, 2)$ *and an* $S_2(\mu s, r_2, s)$ *implies the existence of an* $OA(\lambda\mu s^2, r_1r_2+1, s, 2)$. *Moreover, if the original* $OA(\lambda s^2, r_1, s, 2)$ *is completely decomposable and there exists an* $OA(\lambda\mu s^2, r_3, s, 2)$, *the existence of an* $OA(\lambda\mu s^2, r_1r_2+r_3, s, 2)$ *is ensured.*

By Theorem 3.2.2 and Corollary 4.1.1, we have the following results:

Corollary 4.1.2 : *The existence of an* $OA(\lambda s^2, r, s, 2)$ *implies the existence of an* $OA(\lambda s^2, rs_0+1, s, 2)$ *and by repeated applications of the procedure, implies the existence of* $OA(\lambda s^{p+2}, rs_0^p+s_0^{p-1}+\ldots+s_0+1, s, 2)$ *for all* $p \geqslant 1$, *where* $s \geqslant 2$ *and* $s_0 = min(p_1^{n_1}, p_2^{n_2}, \ldots, p_k^{n_k})$ *and* $s = p_1^{n_1}p_2^{n_2}\ldots p_k^{n_k}$ *is the prime power decomposition of s.*

Clearly, $s_0 = s$ in the Corollary 4.1.2 if s is a prime or a prime power.

Obviously, an $OA(\lambda s^t, r, s, t)$ is also an $S_{t^*}(\lambda s^t, r, s)$. Suppose, there exist $OA(\lambda_i s_i^2, r_i, s, 2)$, $i = 1, 2$. Treating the first OA as OA and the second OA as $S_{2^*}(r_2)$, which implies an $S_2(r_2+1)$ and making use of Theorem 4.1.1, we have by writing A for the first array and B for the second array,

$$C = A \bigcup B \text{ is } OA(\lambda_1\lambda_2 s^4, r_1(r_2+1), s, s)$$

Some more rows can be added to C in the following manner. Let us write,

$$D = \begin{bmatrix} E_{1n} \bigcup B \\ C \end{bmatrix}, \text{ where } n = \lambda_1 s^2.$$

D, can be easily shown to be an $OA(\lambda_1\lambda_2 s^4, (r_1+1)(r_2+1)-1, s, 2)$. Hence, we have,

Corollary 4.1.3 : *The existence of* $OA(\lambda_i s^2, r_i, s, 2)$, $i = 1, 2$, *implies the existence of* $OA(\lambda_1\lambda_2 s^4, r, s, 2)$, *where* $r = (r_1+1)(r_2+1)-1$.

It may be noted that Corollary 4.1.3 is an improvement upon Theorem 4 of Shrikhande (1964). The general results proved so far are utilised in constructing some series of orthogonal arrays of strength 2.

(i) Addelman and Kempthorne (1961) have given a method of construction of $OA(2s^n, 2(s^{n-1}+s^{n-2}\ldots+s)+1, s, 2)$, $n \geqslant 2$ and s, a prime or a prime power. Let us borrow their method of construction for the smallest n i.e., $n = 2$ and that is enough. Then, the general result will follow easily from Theorem 3.1.1 and Corollary 4.1.2. In Addelman and Kempthorne's (1961) method, there exists an $OA(2s^2, 2s, s, 2)$ which is completely decomposable.

So, by a repeated application of Corollary 4.1.2 in conjunction with Theorem 3.1.1, we have a completely decomposable OA$(2s^n, 2s^{n-1}, s, 2)$, $n > 2$. Assuming an OA$(2s^2, 2s+1, s, 2)$ exists, we have by induction the existence of OA$(2s^n, 2(s^{n-1}+s^{n-2}+\ldots+s)+1, s, 2)$ for all $n > 2$, by a repeated application of Corollary 4.1.2.

(ii) By Theorem 3.1.2 and Lemma 4.1, we have the existence of OA$((s-1)s^2, (s-1)^2+1, s, 2)$, when both s and $s-1$ are primes or prime powers. By using Corollary 4.1.2 along with the Theorem 3.1.2, we have the existence of OA$((s-1)s^3, \{(s-1)^2+1\}s+1, s, 2)$, OA$((s-1)s^n, \{s^{n-3}+s^{n-4}+\ldots+s+1\}(s-1)^2+1, s, 2)$ for all $n \geqslant 4$ and more generally, OA$((s-1)^m s^{n+m}, \{s^{n-1}+s^{n-2}+\ldots+s+1\}.(s-1)^{2m}+(s-1)^{2(m-1)}\ldots+(s-1)^2+1, s, 2)$ for all $n \geqslant 2$ and $m \geqslant 1$, when s and $s-1$ are both primes or prime powers.

(iii) Using Addelman and Kempthorne's (1961) OA series along with Theorem 3.1.2 yields the existence of the series of OA$(2(s-1)^m s^{n+m}, \{2(s^{n-1}+s^{n-2}+\ldots+s)+1\}.(s-1)^{2m}+(s-1)^{2(m-1)}\ldots+(s-1)^2+1, s, 2)$, when both s and $s-1$ are primes or prime powers.

(iv) Proceeding in a similar manner and making use of the theorems of this section, it is observed that if s and s^2-1 are both primes or prime powers, we can construct an OA$((s^2-1)s^3, (s^2-1)^2+s+1, s, 2)$. By making use of $S_2(s, s, s)$ and/or $S_2((s-1)s, (s-1), s)$ repeatedly (the latter exist when $s-1$ too is a prime or a prime power), via Corollary 4.1.1, we can construct many more series of orthgonal arrays of strength two from it.

(v) From Corollary 3.2.3 we know the existence of an $S_2(\lambda s, \lambda s, s)$, when s and λ are powers of the same prime and hence via Lemma 4.1, in this case we have a resolvable OA$(\lambda s^2, \lambda s, s, 2)$. This result was proved by Bose and Bush (1952). But, our result is more general. By Corollary 4.1.2, we have the existence of OA$(\lambda s^n, \lambda s^{n-1}+s^{n-2}+\ldots+s+1, s, 2)$ for all $n \geqslant 2$, when λ and s are both powers of the same prime. By a repeated use of $S_2(\lambda s, \lambda s, s)$ along with OA$(\lambda s^2, \lambda s, s, 2)$ we have the existence of OA$(\lambda^{n-1}s^n, (\lambda s)^{n-1}+(\lambda s)^{n-2}+\ldots+(\lambda s)+1, s, 2)$, for all $n \geqslant 2$, when both s and λ are powers of the same prime. Again by applying $S_2(\lambda s, \lambda s, s)$ to the series of OA$(2s^n, 2(s^{n-1}+s^{n-2}+\ldots+s)+1, s, 2)$, $n \geqslant 2$. via Corollary 4.1.2, we have the existence of OA$(2\lambda s^{n+1}, \lambda s\{2(s^{n-1}+s^{n-2}+\ldots+s)+1\}+1, s, 2)$, when both λ and s are powers of the same prime.

One more point that needs to be mentioned at this stage is that the Theorem 3.2.2 in the previous section is more general than the comparable result on orthogonal arrays proved by Bush (1952b).

By Bush's (1952b) result it is known that $f(2\cdot 10^3, 10, 2) \geqslant 15$. But Theorem 3.2.2 used along with Theorem 3.1.1 and Corollary 3.2.3 gives $f(2\cdot 10^3, 10, 2) \geqslant 18$.

It is to be noted that in all the new series of OA's of strength two considered in the present paper the number of constraints are considerably large, even if they don't attain the known upper bounds.

4.2 *Orthogonal arrays of strength three*: The method of construction employed in Section 4.1 in the construction of orthogonal arrays of strength two can be extended with a little modification to the construction of orthogonal arrays of strength three. A general result proved in that direction is contained in the following theorem.

Theorem 4.1.2: *The existence of an* $OA(\lambda s^3, r_1, s, 3)$ *and an* $S_3(\mu s^2, r_2, s)$ *implies the existence of an* $OA(\lambda\mu s^5, r_1 r_2, s, 3)$.

Proof: All the arrays are written with elements from a module Σ of s elements. Let the array A with r_1 rows and $m = \lambda s^3$ columns represent the $OA(\lambda s^3, r_1, s, 3)$ and B with r_2 rows and $n = \mu s^2$ columns represent the $S_3(\mu s^2, r_2, s)$. Then, invoking Lemmas 4.2 and 4.3, it can be proved easily that $C = A \bigcup B$ is an $OA(r_1 r_2, 3)$.

Hence, the resulting array C is $OA(r_1 r_2, 3)$.

Theorem 4.2.2: *The existence of an* $OA(\lambda s^3, r, s, 3)$ *implies the existence of* $OA(\lambda s^4, 2r, s, 3)$.

Proof: Let Σ be the class of residues Mod(s). Let A be the array for $OA(\lambda s^3, r, s, 3)$ with elements from Σ.

Let

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & s-1 \end{bmatrix}.$$

Then $C = A \bigcup B$ is $OA(2r, 3)$.

Assuming s to be a prime or a prime power, Theorems 4.2.1 and 4.2.2 do not help us in improving the lower bounds on $f(s^p, s, 3)$, $p \geqslant 5$, that have been obtained by Mukhopadhyay (1978). But the theorems have other uses.

The existence of an $S_3(\lambda^2 s^2, \lambda s, s)$ is ensured by Corollary 3.2.3, when λ and s are both powers of the same prime. So, by making use of lower bounds given in Mukhopadhyay (1978) via Theorems 4.2.1 and 4.2.2, we obtain the following lower bounds when s and λ are both powers of the same prime, for all $p \geqslant 0$.

$$f(\lambda^2 s^5, s, 3) \geqslant \lambda s(s+1), s \text{ odd and } \geqslant \lambda s(s+2), s \text{ even}$$

$$f(\lambda^2 s^{6+3p}, s, 3) \geqslant \lambda s\{s^{2(p+1)}+s^{2p} \ldots +s^2+1\}$$

$$f(\lambda^2 s^{7+3p}, s, 3) \geqslant 2\lambda s\{s^{2(p+1)}+s^{2p} \ldots +s^2+1\}$$

$$f(\lambda^2 s^{8+3p}, s, 3) \geqslant \lambda s(s+1)\{s^{2(p+1)}+s^{2p} \ldots +s^2+1\}, \ s \text{ odd,}$$

and
$$\geqslant \lambda s(s+2)\{s^{2(p+1)}+s^{2p} \ldots +s^2+1\}, \ s \text{ even.}$$

Moreover, Theorems 4.2.1 and 4.2.2 can also be used to construct series of orthogonal arrays of strength three, when the number of levels, s is not a prime or a prime power.

4.3 *General orthogonal arrays of strength t (>3)*: By Corollary 3.2.3, we know that $S_t(\lambda^{t-1}s^{t-1}, \lambda s, s)$ exists for all $t \geqslant 2$, when both λ and s are powers of the same prime. Hence, via Lemma 4.1, there always exists an $\text{OA}(\lambda^{t-1}s^t, \lambda s, s, t)$, $t \geqslant 2$, when both λ and s are powers of the same prime. In particular, writing $\lambda = s^p$, $p \geqslant 1$, when s is a prime or a prime power, we have the existence of $\text{OA}(s^{t+p(t-1)}, s^{p+1}, s, t)$, $t \geqslant 2$, or writing $\lambda^{t-1} = s^p$ i.e., $\lambda = s^{p/t-1}$, assuming $p/t-1$ is an integer, we have the existence of $\text{OA}(s^{t+p}, s^{p/(t-1)+1}, s, t)$, $t \geqslant 2$, when s is a prime or a prime power. For instance, we have $f(8^5, 8, 4) \geqslant 16$, $f(s^{11}, s^3, 3) \geqslant s^4$, $f(s^{15}, s^3, 4) \geqslant s^4$, $f(s^{24}, s^4, 5) \geqslant s^5$, when s is a prime or a prime power.

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