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NONUNIFORM RATES OF CONVERGENCE TO THE POISSON DISTRIBUTION

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SUMMARY. Convergence rates of binomial and negative binomial distribution to the Poisson distribution are studied extending the earlier results. A zone is computed where the ratio of the probabilities approaches to one.

1. Introduction and the results

A number of distribution have limiting law as Poisson distribution, e.g., the binomial and the negative binomial distribution under appropriate assumption converge to the Poisson distribution. Rates of such convergence are provided in Kerstan (1964), Vervaat (1969) among others. Simons and Johnson (1971) obtained another result for binomial to Poisson convergence and showed $\lim\limits_{n} \sum\limits_{r=0}^{\infty} e^{tr} | b_r(n, \lambda/n) - p_r(\lambda) | = 0, \forall\ t$ where symbols have their usual meaning. This cannot be readily extended to the negative binomial distribution since the ratio of probabilities as considered therein turns out to be decreasing in the opposite direction in such a case. In this note we prove a similar result for the negative binomial distribution as well. Also we compute a zone of the integer r for which the ratio of the probabilities b_r/p_r approaches one. This is similar to the excessive deviation results in the Central Limit Theorem. It turns out that the zone remains the same for binomial and negative binomial distribution and is of the order $\log n/\log \log n$.

First consider the negative binomial distribution

$$B_n(x, p) = {}^{x+n-1}C_{n-1}\, p^n (1-p)^x = \binom{x+n-1}{n-1} P^x (1+P)^{-(x+n)} \qquad \ldots \quad (1)$$

where $P = \dfrac{1}{p} - 1$.

If $P \to 0,\ n \to \infty,\ Pn = \lambda$, then

$$B_n(x, p) \to P(x, \lambda) = e^{-\lambda}\, \lambda^x / x\,!$$

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Let $a_x = P(x, \lambda)/B_n(x, p)$ then $a_x/a_{x-1} = (n+\lambda)/(n+x-1) \downarrow$ as $x \uparrow$. Consider

$$\sum_x h(x) |B_n(x) - P(x)|$$

$$= \sum_x h(x) B_n(x) |P(x)/B_n(x) - 1| < (S+1) \sum_x h(x) B_n(x) \quad \ldots \quad (2)$$

where $S = \sup_{n,x} a_x < \infty$.

Also,

$$B_n(x) < \left(1+\frac{\lambda}{n}\right)^{-n} (n+x)^x \left(\frac{\lambda}{n}\right)^x / x! \ll x^{-\frac{1}{2}} (\lambda e)^x \left(\frac{n+x}{nx}\right)^x,$$

$x = 1, 2, \ldots$ and $B_n(0) < 1$.

Therefore with an application of Dominated Convergence Theorem to (2) with an appropriate choice of $h(x)$, one gets

$$\sum_x (1+x)^{-\delta} \exp\left[x \log \{nx/(\lambda e(n+x))\}\right] |B_n(x) - P(x)|$$

$$\ll \sum_x (1+x)^{-\delta-1/2} < \infty \text{ if } \delta > 1/2. \quad \ldots \quad (3)$$

We adopt the convention $0 \log 0 = 0$. If $nP_n = \lambda + o(1)$ then $a_x/a_{x-1} = (n+\lambda)(1+o(1))/(n+x-1)$ and repeating the above steps we obtain :

Theorem 1. *For the netgative binomial distribution* (1) *with* $nP_n = \lambda + o(1)$

$$\sum_x (1+x)^{-\delta} \exp\left[x \log \{nx/((1+o(1)) \lambda e(n+x))\}\right] |B_n(x) - P(x, \lambda)| < \infty \quad \ldots \quad (4)$$

In particular this implies $\sum_x e^{tx} |B_n(x) - P(x)| < \infty$ for every fixed t if $n \geq n_0(t)$.

Next for $nP_n = \lambda + C_n$, $C_n \to 0$ we get from Vervaat (1969)

$$\sum_x |B_n(x, P_n) - P(x, nP_n)| = O(P_n) = O\left(\frac{1}{n}\right). \quad \ldots \quad (5)$$

Also,

$$\sum_x |P(x, nP_n) - P(x, \lambda)| \leq (nP_n - \lambda) \sum_x \left(\frac{x-\lambda^*}{\lambda^*}\right) P(x, \lambda^*),\ \lambda^* \in (\lambda, \lambda + C_n),$$

$$\ll C_n \sum_x \left|\frac{x-\lambda}{\lambda}\right| P(x, \lambda)$$

$$= O(|C_n|). \quad \ldots \quad (6)$$

Combining (5) and (6)

$$\sum_x |B_n(x, P_n) - P(x, \lambda)| = O\left(|C_n| + \frac{1}{n}\right) \quad \ldots \quad (7)$$

with an application of Hölders inequality one gets from (4) and (7)

$$\sum_x (1+x)^{-\delta a} \exp\{ax \log(nx/(1+o(1))\lambda e(n+x))\} \; |B_n(x) - P(x, \lambda)|$$

$$= O\left(|C_n| + \frac{1}{n}\right)^{1-a}, \quad \ldots \quad (8)$$

where $0 < a < 1$ and $\delta > \frac{1}{2}$. This provides a nonuniform bound depending on both n and x.

Next note that from (8), $B_n(x) \sim P(x, \lambda)$, $x \to \infty$ if

$$(1+x)^{\delta a} \exp\{-ax \log(nx/(1+o(1))\lambda e(n+x))\} \left(|C_n| + \frac{1}{n}\right)^{1-a}$$

$$= o(P(x, \lambda)) = o(x^{-x-1/2} (\lambda e)^x). \quad \ldots \quad (9)$$

For $C_n = O(1/n)$ this gives after some straightforward computation, $x \log x \leqslant \log n - \frac{\log x}{2(1-a)} + x \log(\lambda e) - M_n$, $M_n \to \infty$.

A conservative region of x is : $x \leqslant \log n/\log \log n + M_n^*$ where

$$M_n^* = O\left(\frac{\log n \log_3 n}{\log_2 n(\log_2 n - \log_3^n)}\right) \to \infty, \; \log_2 n = \log \log n \,; \log_3 n = \log \log \log n.$$

In general the solution turns out to be

$$x \leqslant -\log(|C_n| + 1/n)/\log(-\log(|C_n| + 1/n)) + M_n^{**}, \quad \ldots \quad (10)$$

for some $M_n^{**} \to \infty$.

Hence the following theorem.

Theorem 2. *For the negative binomial distribution with* $nP_n = \lambda + C_n$, $C_n \to 0$ *one has* $B_n(x) \sim P(x, \lambda)$ *in the region defined in* (10).

Next consider the binomial distribution

$$b_r(n, p_n) = {}^nC_r p_n^r (1-p_n)^{n-r} \to \frac{e^{-\lambda} \lambda^r}{r\,!} = p_r(\lambda)$$

where $np_n = \lambda + o(1) = \lambda + C_n$.

A straightforward modification of Simons and Johnson (1970) states

$$\lim_n \Sigma h(r) |b_r(n, p_n) - p_r(\lambda)| < \infty \Leftrightarrow \Sigma h(r) p_r(\lambda) < \infty \quad \ldots \quad (11)$$

Note that r.h.s. of (11) is true if one takes

$$h(r) = (1+r)^{-\delta} p_r^{-1}(\lambda) \quad \ldots \quad (12)$$

where $\delta > 1$.

Using the result from Vervaat (1969) and proceeding as in (5) and (6) one gets

$$\sum_r |b_r(n, p_n) - p_r(\lambda)| = O\left(|C_n| + \frac{1}{n}\right). \quad \ldots \quad (13)$$

This and the l.h.s. of (11) with the choice of h as in (12), with an application of Hölders inequality gives

$$\Sigma\, [(1+r)^{\delta} p_r(\lambda)]^{-a} |b_r(n, p_n) - p_r(\lambda)|$$

$$= O\left(|C_n| + \frac{1}{n}\right)^{1-a} \quad \ldots \quad (14)$$

where $0 < a < 1$.

The above provides a nonuniform bound in the binomial case depending on both n and r similar to (8).

Proceeding as in the case of negative binomial distribution one obtains the following :

Theorem 3. *For the binomial distribution with*

$$np_n = \lambda + C_n,\ C_n \to 0$$

$$\sum_r [(1+r)^{\delta} p_r(\lambda)]^{-a} |b_r(n, p_n) - p_r(\lambda)| = O\left(|C_n| + \frac{1}{n}\right)^{1-a} \quad 0 < a < 1, \delta > 1$$

and $b_r(n, p_n) \sim p_r(\lambda)$ *for* r *in the zone*

$$r \leqslant -\log\left(|C_n| + \frac{1}{n}\right) \Big/ \log\left(-\log\left(|C_n| + \frac{1}{n}\right)\right) + M,\ M > 0.$$

REFERENCES

SIMONS, G. and JOHNSON, N. L. (1971). On the convergence of binomial to Poisson distribution. *Ann. Math. Stat.*, **42**, 1735-1736.

KERSTAN, J. (1964). Verallgemeinerung eines Satzes von Prochrow und Lecam. *Z Warh. Verw. Geb*, **2**, 173-179.

VERVAAT, W. (1969). Upper bounds for the distance in total variation between the binomial or negative binomial and the Poisson distribution. *Statistica Neerlandica*, **23**, 79-86.

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