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ON FAMILIES OF DISTRIBUTIONS CLOSED UNDER EXTREMA

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SUMMARY. Order statistics from nonidentically but independently distributed random variables are not easy to deal with. But, when these belong to families of random variables closed under maximum or minimum elegant simplifications are possible. We consider such families and derive formulas for expectations of functions of single order statistics and deduce some recurrence relations.

1. Introduction

If A is a $n \times n$ matrix, then the permanent of A, denoted by per A, is defined as

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where S_n is the set of permutations of $1, 2, \ldots, n$. Thus the definition of the permanent is similar to that of the determinant except that all terms in the expansion get a positive sign. The book "Permanents" by Minc (1978) and the survey papers by Minc (1983, 1987) provide an excellent source of reference on permanents.

If $a_1, a_2, \ldots$ are column vectors, then

$$[\underset{i_1}{a_1} \quad \underset{i_2}{a_2} \ldots]$$

will denote the matrix obtained by taking i_1 copies of a_1, i_2 copies of a_2 and so on.

Let $X_1, X_2, \ldots, X_n$ be independent random variables with distribution functions $F_1, F_2, \ldots, F_n$ and densities $f_1, f_2, \ldots, f_n$ respectively and let $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n:n}$ denote the corresponding order statistics. Vaughan and Venables (1972) have shown that the density of any order statistic or the

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joint density of several order statistics is conveniently expressed in terms of a permanent. For example, the density of $X_{r:n}$ $(1 \leqslant r \leqslant n)$ is given by

$$h_{r:n}(x) = \frac{1}{(r-1)!\,(n-r)!} \operatorname{per} \begin{bmatrix} F_1(x) & 1-F_1(x) & f_1(x) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ F_n(x) & 1-F_n(x) & f_n(x) \end{bmatrix}, \quad -\infty < x < \infty$$
$$\qquad\qquad\qquad r-1 \qquad\quad n-r \qquad\quad 1$$

Similarly the distribution function of $X_{r:n}$ $(1 \leqslant r \leqslant n)$ or that of a subset of $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ may be expressed in terms of permanents. For example, the distribution function of $X_{r:n}$ $(1 \leqslant r \leqslant n)$ is given by (Bapat and Beg, 1989).

$$H_{r:n}(x) = \sum_{i=r}^{n} \frac{1}{i!\,(n-i)!} \operatorname{per} \begin{bmatrix} F_1(x) & 1-F_1(x) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ F_n(x) & 1-F_n(x) \end{bmatrix}, \quad -\infty < x < \infty$$
$$\qquad\qquad\qquad i \qquad\quad n-i$$

The following notation will be used throughout this paper. If $S \subseteq N = \{1, 2, \ldots, n\}$ then S' will denote the complement of S in N and $|S|$ will denote the cardinality of S. Let $X_{r:S}$ denote the r-th order statistic for $\{X_i \mid i \in S\}$ and $H_{r:S}(x)$, the distribution of $X_{r:S}$. When there is no confusion we will replace S by its cardinality. For convenience, for fixed x, $\boldsymbol{F}$ will denote the column vector $(F_1(x), F_2(x), \ldots, F_n(x))'$ and $\mathbf{1}$ the column vector of all ones. We will denote by $A[S|.)$ the matrix obtained from A by taking all the rows whose indices are in S.

Explicit expressions for moments of order statistics for a number of distributions, when all X_i's are independent and identically distributed (i.i.d), are available in the literature. A good number of these have been documented as exercises in David (1981). Balakrishnan *et al.* (1988) have reviewed several recurrence relations and identities available for the single and product moments of order statistics from some specific continuous distributions. All of these are for the case of i.i.d. random variables.

If it is desired to incorporate one or more outliers in $X_1, X_2, \ldots, X_n$ then it naturally leads to the situation where $X_1, X_2, \ldots, X_n$ are nonidentically distributed. It is a common practice to restrict the analysis to the case of one outlier since, for more outliers, the treatment becomes complicated. The permanent representation plays an important role in dealing with such situations. In some instances $F_1, F_2, \ldots, F_n$ may be believed to be of the same functional form but with different values of the parameters involved.

In this paper we consider the case where $F_1, F_2, \ldots, F_n$ are not necessarily identical. In Section 2, we express the distribution function of $X_{r:n}$ $(1 \leq r \leq n)$ in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, \ldots, X_n\}$ where X_i's are arbitrary but independent random variates. In Section 3, we obtain exact and explicit expressions for expectation of functions of single order statistics, using the identities of Section 2. Finally, in Section 4 some applications to specific discrete and continuous distributions are given. Some known recurrence relations, when X_i's are i.i.d. are also deduced.

2. Identities

In this section we prove the following identities.

Theorem 2.1. *For arbitrary distributions* $F_1, F_2, \ldots, F_n$ *and* $n \geq 2$,

$$\text{(a)} \quad H_{r:n}(x) = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} H_{|S|:S}(x)$$

$$\text{(b)} \quad H_{r:n}(x) = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} H_{1:S}(x)$$

Proof. (a) The distribution function of $X_{r:n}$ $(1 \leq r \leq n)$ is given by (Bapat and Beg, 1989).

$$H_{r:n}(x) = \sum_{i=r}^{n} \frac{1}{i!\,(n-i)!} \operatorname{per} \, [\underset{i}{F} \;\; \underset{n-i}{1-F}]$$

$$= \sum_{i=r}^{n} \frac{1}{i!\,(n-i)!} \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n-i}{j} \operatorname{per} \, [\underset{n-j}{F} \;\; \underset{j}{1}]$$

$$= \sum_{i=r}^{n} \frac{1}{i!\,(n-i)!} \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n-i}{j} \sum_{|S|=n-j} j! \operatorname{per} \, [\underset{n-j}{F}]\,[S|.)$$

$$= \sum_{i=r}^{n} \frac{1}{i!\,(n-i)!} \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n-i}{j} \sum_{|S|=n-j} j!(n-j)!\, H_{|S|:S}(x)$$

$$= \sum_{j=0}^{n-r} \sum_{i=r}^{n-j} (-1)^{n-i-j} \binom{n-j}{i} \sum_{|S|=n-j} H_{|S|:S}(x).$$

Since,

$$\sum_{i=r}^{n-j} (-1)^{n-j-i} \binom{n-j}{i} = (-1)^{n-j-r} \binom{n-j-1}{n-j-r},$$

we get

$$H_{r:n}(x) = \sum_{j=0}^{n-r} (-1)^{n-j-r} \binom{n-j-1}{n-j-r} \sum_{|S|=n-j} H_{|S|:S}(x)$$

and the proof is complete.

It is easy to see that (b) follows from (a) by considering $-X_1$, $-X_2, \ldots, -X_n$ instead of $X_1, X_2, \ldots, X_n$.

Corollary 2.1. *Allowing $x \to \infty$, Theorem 2.1 gives*

$$\sum_{j=0}^{n-r} (-1)^{n-j-r} \binom{n-j-1}{n-j-r} \binom{n}{j} = 1$$

and

$$\sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \binom{n}{j} = 1$$

Corollary 2.2. *For the p-outlier model, that is, $F_1 = F_2 \ldots = F_{n-p} = F$ and $F_{n-p+1} = \ldots = F_n = G$ (outlier distribution), Theorem 2.1 yields*

$$H_{r:n}(x) = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{m=0}^{p} \binom{p}{m} \binom{n-p}{n-j-m} H_{n-j:n-j,m}(x)$$

and

$$H_{r:n}(x) = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{m=0}^{p} \binom{p}{m} \binom{n-p}{n-j-m} H_{1:n-j,m}(x)$$

where $X_{r:n,a}$ denotes the r-th order statistic from a sample of size n of which 'a' are outliers.

Corollary 2.3. For the case of a sample of n independent and identically distributed random variables $X_1, X_2, \ldots, X_n$ having distribution function $F(x)$, Theorem 2.1 simply reduces to

$$F_{r:n}(x) = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \binom{n}{j} F_{n-j:n-j}(x)$$

and

$$F_{r:n}(x) = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \binom{n}{j} F_{1:n-j}(x)$$

where $F_{r:n}(x)$ denotes distribution function of $X_{r:n}(1 \leqslant r \leqslant n)$.

3. Main results

In this section we make use of the identities of Theorem 2.1 to obtain expressions for expectations of functions of order statistics.

Suppose the random variable X has an arbitrary distribution function $F(x)$. Define the following two families of distribution functions with a positive parameter λ.

Family I. $F^{\lambda}(x) = [F(x)]^{\lambda},\ \lambda > 0$

and

Family II. $F_{\lambda}(x) = 1-[1-F(x)]^{\lambda},\ \lambda > 0.$

Let $X^{(\lambda)}$ have distribution function $F^{\lambda}(x)$. Let $X_1, X_2, \ldots, X_n$ be independently distributed as $X^{(\lambda_1)}, X^{(\lambda_2)}, \ldots X^{(\lambda_n)}$ respectively. Then

$$H_{|S|:S}(x) = \prod_{i\epsilon S} F^{\lambda_i}(x) = \prod_{i\epsilon S} [F(x)]^{\lambda_i}$$

$$= [F(x)]^{\lambda_S} = F^{\lambda_S}(x),\ \lambda_S = \sum_{i\epsilon S} \lambda_i$$

and from (a) of Theorem 2.1, we have

$$H_{r:n}(x) = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} F^{\lambda_S}(x). \qquad \ldots \quad (1)$$

Let $X_{(\lambda)}$ have distribution function $F_{\lambda}(x)$. If $X_1, X_2, \ldots, X_n$ are distributed independently as $X_{(\lambda_1)}, X_{(\lambda_2)}, \ldots, X_{(\lambda_n)}$ then

$$H_{1:S}(x) = 1-\prod_{i\epsilon S} [1-F_{\lambda_i}(x)] = 1-\prod_{i\epsilon S} [1-F(x)]^{\lambda_i}$$

$$= 1-[1-F(x)]^{\lambda_S} = F_{\lambda_S}(x),\ \lambda_S = \sum \lambda_i$$

and from (b) of Theorem 2.1, we have

$$H_{r:n}(x) = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} F_{\lambda_S}(x) \qquad \ldots \quad (2)$$

Let $g(.)$ be a Borel measurable function from $\mathcal{R}$ to $\mathcal{R}$. Assume that $E\{g(.)\}$ exists. Then, from (1) and (2), we get

$$E\{g(X_{r:n})\} = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} g^*(\lambda_S) \qquad (3)$$

and

$$E\{g(X_{r:n})\} = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} g_*(\lambda_S) \quad \ldots \quad (4)$$

where

$$g^*(\lambda) = E\{g(X^{(\lambda)})\}$$

and

$$g_*(\lambda) = E\{g(X_{(\lambda)})\}.$$

From (3), for $n \geqslant 1$ and $r = n$,

$$E\{g(X_{n:n})\} = g^*(\lambda_N),\ \lambda_N = \sum_{i \in N} \lambda_i \quad \ldots \quad (5)$$

and for $r \in M = \{1, 2, \ldots, n-1\}$

$$E\{g(X_{r:n})\} = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \Big\{ \sum_{\substack{|S|=n-j \\ S \subset M}} g^*(\lambda_s) + \sum_{\substack{|S|=n-j-1 \\ S \subset M}} g^*(\lambda_S + \lambda_n) \Big\}$$

$$= \sum_{j=1}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{\substack{|S|=n-j \\ S \subset M}} g^*(\lambda_S)$$

$$+ \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{\substack{|S|=n-j-1 \\ S \subset M}} g^*(\lambda_S + \lambda_n)$$

$$= E\{g(X_{r:n-1})\} + \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{\substack{|S|=n-j-1 \\ S \subset M}} g^*(\lambda_S + \lambda_n) \quad \ldots \quad (6)$$

From (4), for $n \geqslant 1$ and $r = 1$,

$$E\{g(X_{1:n})\} = g_*(\lambda_N),\ \lambda_N = \sum_{i \in N} \lambda_i \quad \ldots \quad (7)$$

and for $2 \leqslant r \leqslant n$,

$$E\{g(X_{r:n})\} = \sum_{j=1}^{n-r} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{\substack{|S|=n-j \\ S \subset M}} g_*(\lambda_S)$$

$$+ \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{\substack{|S|=n-j-1 \\ S \subset M}} g_*(\lambda_S + \lambda_n)$$

$$= E\{(X_{r-1:n-1})\} + \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{\substack{|S|=n-j-1 \\ S \subset M}} g_*(\lambda_S + \lambda_n). \quad \ldots \quad (8)$$

The relations (3), (4), (5), (6), (7) and (8) can be used to get recurrence relations involving single order statistics.

4. Applications

In this section we obtain exact and explicit expressions for expectations of functions of single order statistics for some specific distributions. Some known recurrence relations based on single order statistics, when all X_i's are i.i.d., are deduced.

Examples for Family I. (i) Consider

$$F(x) = q^{1-x},\ x = 0, 1\ ;\ 0 \leqslant q \leqslant 1$$

then $$F^{\lambda}(x) = q^{\lambda(1-x)},\ x = 0, 1\ ;\ 0 \leqslant q \leqslant 1,\ \lambda > 0$$

which is a Bernoulli distribution.

If $g(x) = e^{tx}$, then

$$g^{*}(\lambda) = \sum_{x=0}^{1} e^{tx} P(X^{\lambda} = x)$$

$$= q^{\lambda} + e^{t}(1-q^{\lambda})$$

$$= q^{\lambda} + (1-q^{\lambda}) \sum_{k=0}^{\infty} \frac{t^k}{k!}$$

Hence from (3), the mgf of $X_{r:n}$ is given by

$$\psi(t) = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} \left\{ q^{\lambda S} + (1-q^{\lambda S}) \sum_{k=0}^{\infty} \frac{t^k}{k!} \right\},\ \forall\ t \in \mathcal{R} \ldots \qquad \ldots (9)$$

From (9), for $k = 1, 2, \ldots,$

$$EX_{r:n}^{k} = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} (1-q^{\lambda S})$$

$$= 1 - \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} q^{\lambda S}, \qquad \ldots (10)$$

using Corollary 2.1.

Also, for $n \geqslant 1$ and $r = n$,

$$EX_{n:n}^{k} = 1 - q^{\lambda_N},\ \lambda_N = \sum_{i \in N} \lambda_i \qquad \ldots (11)$$

and for $1 \leqslant r \leqslant n-1$, (6) gives

$$EX^k_{r:n} = EX^k_{r:n-1} + \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{\substack{|S|=n-j-1 \\ S \subset M}} (1-q^{\lambda_S+\lambda_n})$$

$$= EX^k_{r:n-1} + \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \binom{n-1}{j}$$

$$- \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j-1} q^{\lambda_S+\lambda_n}$$

$$= EX^k_{r:n-1} - \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j-1} q^{\lambda_S+\lambda_n}. \qquad \ldots \quad (12)$$

(ii) Consider

$$F(x) = \frac{x}{\theta},\ 0 \leqslant x \leqslant \theta\ ;\ \theta > 0$$

then

$$F^\lambda(x) = \left(\frac{x}{\theta}\right)^\lambda,\ 0 \leqslant x \leqslant \theta\ ;\ \theta, \lambda > 0$$

which is a power function distribution.

If $g(x) = e^{tx}$, then

$$g^*(\lambda) = \int_0^\theta e^{tx} d\left(\frac{x}{\theta}\right)^\lambda = \lambda \int_0^\theta \theta^{-\lambda} e^{tx} x^{\lambda-1} dx$$

Putting $y = \frac{x}{\theta}$, we get

$$g^*(\lambda) = \lambda \int_0^1 e^{\theta t y} y^{\lambda-1}\, dy = \lambda \sum_{k=0}^{\infty} \frac{\theta^k t^k}{(\lambda+k)k\,!}$$

Hence from (3), the mgf of $X_{r:n}$ is given by

$$\psi(t) = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{k=0}^{\infty} \frac{\theta^k t^k}{k\,!} \left\{ \sum_{|S|=n-j} \frac{\lambda_S \theta^k}{(\lambda_S+k)} \right\},\ \forall\, t \in \mathcal{R}$$

$$\ldots \quad (13)$$

From (13), for $k = 1, 2, \ldots,$

$$E\, X^k_{r:n} = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} \frac{\lambda_S \theta^k}{(\lambda_S+k)} \qquad \ldots \quad (14)$$

Also, for $n \geqslant 1$ and $r = n$,

$$E\,X^k_{n:n} = \frac{\lambda_N \theta^k}{(\lambda_N + k)}, \qquad \dots \quad (15)$$

$$E\,X^k_{n:n} = \theta\Big(1 - \frac{1}{\lambda_N + k}\Big) E\,X^{k-1}_{n:n},\ \lambda_N = \sum_{i \in N} \lambda_i \qquad \dots \quad (16)$$

and for $1 \leqslant r \leqslant n-1$, (6) gives

$$EX^k_{r:n} = EX^k_{r:n-1} + \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{\substack{|S| = n-j-1 \\ S \subset M}} \frac{(\lambda_S + \lambda_n)\theta^k}{(\lambda_S + \lambda_n + k)} \quad \dots \quad (17)$$

(iii) Consider

$$F(x) = \exp\{-e^{-(x-\xi)/\theta}\},\ -\infty < x < \infty,\ -\infty < \xi < \infty,\ \theta > 0$$

then

$$F^\lambda(x) = \exp\{-\lambda\, e^{-(x-\xi)/\theta}\},\ -\infty < x < \infty,\ -\infty < \xi < \infty\ ;\ \theta, \lambda > 0$$

which is an extreme value Type I distribution (See Johnson and Kotz, 1970, p. 272).

If $g(x) = e^{tx}$, then

$$g^*(\lambda) = \int_{-\infty}^{\infty} e^{tx}\, d[\exp\{-\lambda\, e^{-(x-\xi)/\theta}\}]$$

$$= \int_{-\infty}^{\infty} e^{tx}\, d[\exp\{-e^{-(x-\xi')/\theta}\}],\ \xi' = \xi + \theta \log \lambda$$

$$= e^{\xi' t}\, \Gamma(1-\theta t)$$

$$= \lambda^{\theta t} e^{t\xi}\, \Gamma(1-\theta t),\ \theta|t| < 1.$$

Hence from (3), the mgf of $X_{r:n}$ is given by

$$\psi(t) = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} \lambda_S^{t\theta} e^{t\xi}\, \Gamma(1-\theta t),\ \theta|t| < 1 \quad \dots \quad (18)$$

Examples for Family II. (i) Consider

$$F(x) = 1 - q^{x+1},\ x = 0, 1, 2, \dots\ ;\ 0 \leqslant q \leqslant 1,$$

then

$$F_\lambda(x) = 1 - q^{\lambda(x+1)},\ x = 0, 1, 2, \dots\ ;\ \lambda > 0$$

which is a geometric distribution.

If $g(x) = (1+t)^x$

$$= 1 + xt + \binom{x}{2} t^2 + \binom{x}{3} t^3 + \dots$$

$$= 1 + x^{(1)} t + x^{(2)} \frac{t^2}{2!} + x^{(3)} \frac{t^3}{3!} + \dots,\ t \in \mathcal{R}$$

then

$$g_*(\lambda) = \sum_{x=0}^{\infty} q^{\lambda x}(1-q^{\lambda})(1+t)^x$$

$$= (1-q^{\lambda})\sum_{x=0}^{\infty}(q^{\lambda}(1+t))^x = \frac{1-q^{\lambda}}{1-q^{\lambda}(1+t)}$$

$$= \frac{1-q^{\lambda}}{(1-q^{\lambda})-q^{\lambda}t} = \left(1-\frac{q^{\lambda}}{1-q^{\lambda}}\ t\right)^{-1}$$

$$= \sum_{k=0}^{\infty}\left(\frac{q^{\lambda}}{1-q^{\lambda}}\right)^k t^k.$$

Hence from (4), the factorial mgf of $X_{r:n}$ is given by

$$\sum_{j=0}^{r-1}(-1)^{r-j-1}\binom{n-j-1}{n-r}\sum_{|S|=n-j}\ \sum_{k=0}^{\infty}\left(\frac{q^{\lambda_S}}{1-q^{\lambda_S}}\right)^k t^k \quad \ldots \ (19)$$

From (19), for $k = 1, 2, \ldots,$ the factorial moment of $X_{r:n}$ is given by

$$EX_{r:n}^{(k)} = \sum_{j=0}^{r-1}(-1)^{r-j-1}\binom{n-j-1}{n-r}\sum_{|S|=n-j} k\,!\left(\frac{q^{\lambda_S}}{1-q^{\lambda_S}}\right)^k \quad \ldots \ (20)$$

Also for $n \geqslant 1$ and $r = 1$,

$$EX_{1:n}^{(k)} = k\,!\left(\frac{q^{\lambda_N}}{1-q^{\lambda_N}}\right)^k, \quad \ldots \ (21)$$

$$EX_{1:n}^{(k)} = k\left(\frac{q^{\lambda_N}}{1-q^{\lambda_N}}\right)EX_{1:n}^{(k-1)},\ \lambda_N = \sum_{i\in N}\lambda_i \quad \ldots \ (22)$$

and for $2 \leqslant r \leqslant n$, (8) gives

$$EX_{r:n}^{(k)} = EX_{r-1:n-1}^{(k)} + \sum_{j=0}^{r-1}(-1)^{r-j-1}\binom{n-j-1}{n-r}\sum_{\substack{|S|=n-j-1\\ S\subset M}} k\,!\left(\frac{q^{\lambda_S+\lambda_n}}{1-q^{\lambda_S+\lambda_n}}\right)^k \quad \ldots \ (23)$$

(ii) Consider

$$F(x) = 1-e^{-x},\ x \geqslant 0$$

then

$$F_{\lambda}(x) = 1\text{-}e^{-\lambda x},\ x \geqslant 0,\ \lambda > 0$$

which is an exponential distribution.

If $g(x) = e^{tx}$, then

$$g_*(\lambda) = \int_0^{\infty} e^{tx}d(1-e^{-\lambda x}) = \lambda\int_0^{\infty} e^{-(\lambda-t)x}\,dx$$

$$= \left(1-\frac{t}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty}\frac{k\,!\,t^k}{\lambda^k k\,!},\ \ |t| < \lambda.$$

Hence from (4), the mgf of $X_{r:n}$ is given by

$$\psi(t) = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} \left(1-\frac{t}{\lambda_S}\right)^{-1}, |t| < \lambda_S, \forall S \subset N \quad \ldots \quad (24)$$

Bapat and Beg (1989) obtained (24) by different method.

From (24), for $k = 1, 2, \ldots,$

$$EX_{r:n}^k = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} \frac{k!}{\lambda_S^k} \quad \ldots \quad (25)$$

Also, for $n \geqslant 1$ and $r = 1$,

$$EX_{1:n}^k = \frac{k!}{\lambda_N^k}, \quad \ldots \quad (26)$$

$$EX_{1:n}^k = \frac{k}{\lambda_N} EX_{1:n}^{k-1}, \; \lambda_N = \sum_{i \in N} \lambda_i \quad \ldots \quad (27)$$

and for $2 \leqslant r \leqslant n$, (8) gives

$$EX_{r:n}^k = EX_{r-1:n-1}^k + \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{\substack{|S|=n-j-1 \\ S \subset M}} \frac{k!}{(\lambda_S+\lambda_n)^k} \quad \ldots \quad (28)$$

(iii) Consider

$$F(x) = 1 - \left(\frac{\theta}{x}\right), \; x \geqslant \theta \,;\, \theta > 0$$

then

$$F_\lambda(x) = 1 - \left(\frac{\theta}{x}\right)^\lambda, \; x \geqslant \theta \,;\, \theta, \lambda > 0$$

which is a Pareto distribution.

If $g(x) = x^k$, $k = 1, 2, \ldots,$ then

$$g_*(\lambda) = \int_\theta^\infty x^k d\left[1-\left(\frac{\theta}{x}\right)^\lambda\right] = \lambda \int_\theta^\infty x^k \, \theta^\lambda \, x^{-(\lambda+1)} \, dx$$

$$= \frac{\lambda\theta^k}{(\lambda-k)} \int_\theta^\infty (\lambda-k) \, \theta^{\lambda-k} \, x^{-(\lambda-k+1)} \, dx$$

$$= \frac{\lambda\theta^k}{(\lambda-k)}, \; k < \lambda.$$

Hence from (4), we have

$$EX_{r:n}^k = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} \frac{\lambda_S \, \theta^k}{(\lambda_S-k)}, \; k < \lambda_S \quad \ldots \quad (29)$$

For $n \geqslant 1$ and $r = 1$,

$$EX_{1:n}^{k} = \frac{\theta^{k}\lambda_{N}}{(\lambda_{N}-k)}, \; \lambda_{N} = \sum_{i \in N} \lambda_{i}, \; k < \lambda_{N} \qquad \ldots \quad (30)$$

$$EX_{1:n}^{k} = \theta\left(1-\frac{1}{\lambda_{N}-k}\right) EX_{1:n}^{k-1} \qquad \ldots \quad (31)$$

and for $2 \leqslant r \leqslant n$, (8) gives

$$EX_{r:n}^{k} = EX_{r-1:n-1}^{k} + \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{\substack{|S|=n-j-1 \\ S \subset M}} \frac{\theta^{k}(\lambda_{S}+\lambda_{n})}{(\lambda_{S}+\lambda_{n}-k)} \ldots \quad (32)$$

(iv) Consider

$$F(x) = 1-e^{-x^{2}/2}, \; x > 0$$

then

$$F_{\lambda}(x) = 1-e^{-\lambda x^{2}/2}, \; x \geqslant 0, \; \lambda > 0$$

which is a Rayleigh distribution.

If $g(x) = e^{tx}$, then

$$g_{*}(\lambda) = \int_{0}^{\infty} e^{tx} d[1-e^{-\lambda x^{2}/2}]$$

$$= \lambda \int_{0}^{\infty} e^{-\lambda x^{2}/2} \sum_{k=0}^{\infty} \frac{t^{k}\, x^{k+1}}{k\,!} \, dx$$

$$= \lambda^{k} \sum_{k=0}^{\infty} \frac{t^{k}}{k\,!} \int_{0}^{\infty} e^{-\lambda x^{2}/2} x^{k+1} \, dx$$

Putting $y = \frac{x^{2}}{2}$, we get

$$g_{*}(\lambda) = \lambda \sum_{k=0}^{\infty} \frac{t^{k}}{k\,!} \int_{0}^{\infty} e^{-\lambda y} (2y)^{\frac{k+1}{2}} \frac{dy}{\sqrt{2y}}$$

$$= \lambda \sum_{k=0}^{\infty} \frac{t^{k}}{k\,!} \int_{0}^{\infty} e^{-\lambda y} (2y)^{k/2} \, dy$$

$$= \sum_{k=0}^{\infty} \frac{t^{k}}{k\,!} 2^{k/2} \frac{\Gamma\left(\frac{k}{2}+1\right)}{\lambda^{k/2}}, \; t \in \mathcal{R}$$

Hence from (4), the mgf of $X_{r:n}$ is given by

$$\psi(t) = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} \left\{ \sum_{k=0}^{\infty} \frac{t^{k} 2^{k/2}}{k\,!} \frac{\Gamma\left(\frac{k}{2}+1\right)}{\lambda_{S}^{k/2}} \right\}, \; t \in \mathcal{R} \qquad \ldots \quad (33)$$

From (33), for $k = 1, 2, \ldots$

$$EX^k_{r:n} = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} \frac{2^{k/2}\Gamma\left(\frac{k}{2}+1\right)}{\lambda_S^{k/2}} \quad \ldots \quad (34)$$

For $n \geqslant 1$ and $r = 1$,

$$EX^k_{1:n} = \frac{2^{k/2}\Gamma\left(\frac{k}{2}+1\right)}{\lambda_N^{k/2}} \quad \ldots \quad (35)$$

$$EX^{(k+2)}_{1:n} = \frac{(k+2)}{\lambda_N} EX^k_{1:n} \quad \ldots \quad (36)$$

and for $2 \leqslant r \leqslant n$, (8) gives

$$EX^k_{r:n} = EX^k_{r-1:n-1} + \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{\substack{|S|=n-j-1\\ S \subset M}} \frac{2^{k/2}\Gamma\left(\frac{k}{2}+1\right)}{(\lambda_S+\lambda_n)^{k/2}} \quad \ldots \quad (37)$$

(v) Consider

$$F(x) = 1-e^{-x^\xi}, \; x \geqslant 0, \xi > 0$$

then

$$F_\lambda(x) = 1-e^{-\lambda x^\xi}, \; x \geqslant 0 \; ; \xi > 0, \lambda > 0$$

which is a Weibull distribution. In particular, with $\xi = 2$, $\lambda = \lambda/2$ it reduces to the Rayleigh distribution and with $\xi = 1$ the exponential distribution.

If $g(x) = x^k$, $k = 1, 2, \ldots$, then

$$g_*(\lambda) = \int_0^\infty x^k d[1-e^{-\lambda x^\xi}] = \lambda \xi \int_0^\infty x^{k+\xi-1} e^{-\lambda x^\xi} dx$$

Putting $y = x^\xi$, we get

$$g_*(\lambda) = (\lambda) \int_0^\infty y^{k/\xi} e^{-\lambda y} dy = \frac{\Gamma\left(\frac{k}{\xi}+1\right)}{\lambda^{k/\xi}}$$

Hence from (4), we have

$$EX^k_{r:n} = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} \frac{\Gamma\left(\frac{k}{\xi}+1\right)}{\lambda_S^{k/\xi}} \quad \ldots \quad (38)$$

For $n \geqslant 1$ and $r = 1$

$$EX^k_{1:n} = \frac{\Gamma\left(\frac{k}{\xi}+1\right)}{\lambda_N^{k/\xi}}, \quad \ldots \quad (39)$$

$$EX^{k+\xi}_{1:n} = \left(\frac{\xi}{k}\right) \lambda_N EX^k_{1:n}, \; \xi = 1, 2, \ldots \quad \ldots \quad (40)$$

and for $2 \leqslant r \leqslant n$, (8) gives

$$EX^k_{r:n} = EX^k_{r-1:n-1} + \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{\substack{|S|=n-j-1 \\ S \subset M}} \frac{\Gamma\left(\frac{k}{\xi}+1\right)}{(\lambda_n+\lambda_S)^{k/\xi}} \quad \ldots \quad (41)$$

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