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ON ASYMPTOTIC REPRESENTATION AND APPROXIMATION TO NORMALITY OF L-STATISTICS-I

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SUMMARY. Asymptotic representation has been obtained for suitable linear functions of order statistics of i.i.d. observations which strengthen the previously known results. The representation has been made use of to obtain some non-uniform rates of convergence to normality of the statistics.

1. Introduction

Let $\{X, X_1, X_2 \ldots\}$ be a sequence of i.i.d. random variables with $E|X| < \infty$ and X_i having continuous distribution function F. We define the e.d.f. (empirical distribution function $F_n(\cdot)$ at the n-th stage as

$$F_n(x) = n^{-1} \sum_{i=1}^{n} I_{(X_i \leq x)}, \quad -\infty < x < \infty.$$

For some bounded function w on [0, 1], consider the following linear combination of order statistics

$$L_n = \int_{-\infty} xw(F_n(x))dF_n(x)$$

and the corresponding parametric value

$$L = \int_{-\infty}^{\infty} xw(F(x))dF(x).$$

Let $V_n(u)$ denote the e.d.f. of $U_i = F(X_i)$, which are of course distributed uniformly on [0, 1], and G any inverse of F. Then, we can also write

$$L_n = \int_0^1 G(u)w(V_n(u))dV_n(u) \quad \text{a.s.}$$

$$L = \int_0^1 G(u)w(u)du.$$

Further, let us define

$$Z_i = \int_0^1 (u - I_{(U_i \leq u)} w(u) dG(u)).$$

Note that the Z_i's and L are well defined in view of the condition $E|X| < \infty$. By Z we shall denote a r.v. having the same distribution as Z_i's.

Moore (1968) provides an elegant proof of asymptotic normality of $n^{\frac{1}{2}}(L_n - L)$ by showing that if w is sufficiently smooth, then

$$R_n = \left| (L_n - L) - n^{-1} \sum_{i=1}^{n} Z_i \right| = o_p(n^{-\frac{1}{2}}).$$

Later Ghosh (1972) proves that if w has bounded second derivative (this condition is slightly stronger than those in Moore, 1968) and

$$\int_0^1 [u(1-u)]^{1/(2+\delta)} dG(u) < \infty$$

for some $\delta > 0$ (this is equivalent to assuming that $E|X|^{2+\delta} < \infty$ for some $\delta > 0$, see Lemma 2.2 of this paper), then

$$R_n \ll n^{-1}(\log n)^2 \quad \text{a.s.}$$

(Vingrodov Symbol $\ll$ is used for 0 whenever it is convenient.) In particular this later result trivially yields the law of the iterated logarithm for L_n.

Section 2 of this paper contains some results on representation of L-statistics which strengthen the result of Ghosh (1972), and provide answer to a question raised there. In Theorem 2 of this section, Moore's technique has been combined with quantile representation which leads to the representation of L-statistics for a much wider class than the one described above.

Turning to rates of convergence to normality we would like to mention the papers of Rosenkrantz and Reilly (1972), Bjerve (1977) and Holmers (1977) which obtain Berry-Esseen type bounds for L-statistics. Rosenkrantz and Reilly (1972) used Skorohod's representation to show that the rate of convergence for trimmed type L-statistics in $n^{-1/4}$. Bjerve (1977) uses Fourier-transform method and shows that the rate is actually $n^{-\frac{1}{2}}$. However, Bjerve's method heavily depends upon the i.i.d. structure and the statistics considered are again of the trimmed type.

Helmers (1977) shows that the rate of convergence is $n^{-\frac{1}{2}}$ for L-statistics with certain smooth weight functions which are not necessarily of the trimmed type. In Section 3 of this paper we establish some non-uniform rates of convergence to normality and few interesting corollaries of it under the above set up. The set up is similar to that of Helmers (1977) but the results are not directly comparable.

In a subsequent paper, Singh (1979) we extend the results of this paper to general mixing random variables.

Throughout, l_n^a and ll_n^a denote $(\log n)^a$ and $(\log\log n)^a$ respectively and b_i's are absolute constants.

2. Representation of L-statistics

Let us say that a point $x \in (0, 1)$ is a jump point of the function w if x is a discontinuity point of w but w is either left continuous or right continuous at a.

Theorem 1 : *Let us assume that w has bounded second derivative throughout on $(0, 1)$ except possibly at finitely many points $a_1, a_2, \ldots, a_k$ which are all jump points. Further, assume that, in a neighborhood of each of the points $G(a_1), G(a_2), \ldots, G(a_k)$, F admits a density which is bounded away from zero. Then*

$$R_n \ll n^{-1} ll_n \quad a.s.$$

if $E|X|^{1+\delta} < \infty$ for some $\delta > 0$ and

$$R_n \ll n^{-1}(l_n)^{1+\gamma} \quad a.s.$$

for all $\gamma > 0$ if $E|X| < \infty$.

(We do not require any density condition on F if there is no jump point). We begin with a few lemmas.

Lemma 2.1 : *If*

$$E_r = \sup_{t \in (0,1)} (t(1-t))^{-r} | V_n(t) - t$$

for $0 \leqslant r \leqslant \frac{1}{2}$, then

$$E_{1/2} \ll n^{-\frac{1}{2}} l_n^{1/2+\gamma} a.s. \qquad \ldots \quad (2.1)$$

for all $\gamma > 0$ and for any $\frac{1}{2} \geqslant \beta > 0$

$$E_{\frac{1}{2}-\beta} \ll n^{-\frac{1}{2}} ll_n^{1/2} a.s. \qquad \ldots \quad (2.2)$$

(2.1) follows from Theorem 3.1(11) of Csáki (1975) and (2.2) from the theorem in James (1975). Alternative proofs can be found in Singh (1978) which are quite flexible for weak dependence structures. Csörgö and Revész (1975) and Csáki (1977) also contain stability results for weighted empirical processes based on i.i.d. r.v.'s

Lemma 2.2 : (i) $E|X| < \infty \Longleftrightarrow \int_0^1 u(1-u)dG(u) < \infty.$

(ii) $\int_0^1 (u(1-u))^{r-\delta}dG(u) < \infty$, *for some* $r > 0$ *and* $0 < \delta < r$, *implies* $E|X|^{1/r} < \infty$.

(iii) $E|X|^{r+\delta} < \infty$, *for some* r *and* δ *positive, implies*

$$\int_0^1 (u(1-u))^{1/r}dG(u) < \infty.$$

The assertions seem to be well known. A proof for (ii) and (iii) can be found in Singh (1978) (see Lemma 4.4.1).

Lemma 2.3 : *Let* $F_n^{-1}(\cdot)$ *denote the right continuous inverse of* F_n. *If* $E_{**} = \sup_{0 \leqslant t \leqslant 1} |F_nF_n^{-1}(t)-t|$, *then* $E_{**} \leqslant 1/n$ *a.s.*

Proof: Since $0 \leqslant F_n(F_n^{-1}(s))-s \leqslant \sup_{t \in [0,1]} |F_n(t)-F_n(t-)|$, it follows in view of the independence that

$$|F_nF_n^{-1}(s)-s| \leqslant 1/n \quad \text{a.s.}$$

for all $s \in [0, 1]$.

Proof of Theorem 1 : For the sake of simplicity in writing, we assume that w satisfies the smoothness condition everywhere except at a jump point a, $0 < a < 1$, where it is right continuous. Under these conditions it follows that w and w' have both left and right limits at a. It is plain that the proof works for all weight functions with finitely many jump points.

Let us define $w^*(u) = w'(u)$ for $u \neq a$ and $w^*(a) = 0$. Similarly $w^{**}(u) = w''(u)$ for $u \neq a$ and $w^{**}(a) = 0$. Following Moore (1968) and Ghosh (1972), we write

$$L_n - L = I_{n1} + I_{n2} + I_{n3}$$

where

$$I_{n1} = \int_0^1 G(u)w^*(u)(V_n(u)-u)du + \int_0^1 G(u)w(u)d(V_n(u)-u)$$

$$I_{n2} = \int_0^1 G(u)[w(V_n(u))-w(u)-(V_n(u)-u)w^*(u)]dV_n(u)$$

$$I_{n3} = \int_0^1 G(u)w^*(u)(V_n(u)-u)d(V_n(u)-u).$$

Now, we argue in two steps as follows.

Step I : We first prove the theorem assuming that w is continuous at a but not necessarily differentiable. In this case, since

$$\int_0^1 G(u)w^*(u)(V_n(u)-u)du = \int_0^1 G(u)(V_n(u)-u)dw(u),$$

it follows, by integration by parts, that

$$I_{n1} = -\int_0^1 w(u)(V_n(u)-u)dG(u) = n^{-1}\sum_{i=1}^{n} Z_i.$$

To analyze I_{n2}, let us fix a $\beta > 0$ such that

$$\limsup_{n\to\infty} n^{\frac{1}{2}}ll_n^{-1/2}E_0 < \beta/2 \quad \text{a.s.}$$

and define $\beta_n = \beta n^{-\frac{1}{2}}ll_n^{1/2}$. We can express I_{n2} as

$$I_{n2} = \left(\int_{(0,\, a-\beta_n]} + \int_{(a-\beta_n,\, a+\beta_n]} + \int_{(a+\beta_n,\, 1)}\right) G(u)$$

$$[w(V_n(u))-w(u)-(V_n(u)-u)w^*(u)]dV_n(u)$$

$$= I+II+III \quad \text{(say)}.$$

By the choice of β_n it follows that, for all n sufficiently large,

$$I+III \ll E_0^2 \int G(u)dV_n(u) \ll n^{-1}ll_n \quad \text{a.s.}$$

Further, since w satisfies the Lipschitz condition of order 1 on (0, 1), we have that

$$II \ll E_0 \int_{(a-\beta_n,\, a+\beta_n]} G(u)dV_n(u)$$

$$\ll E_0[V_n(a+\beta_n)-V_n(a-\beta_n)]$$

($G(u)$ is bounded on the interval for all sufficiently large n)

$$= E_0[(V_n(a+\beta_n)-a-\beta_n)-(V_n(a-\beta_n)-a+\beta_n)+2\beta_n]$$
$$\ll n^{-1}ll_n \quad \text{a.s.}$$

Thus if $E|X| < \infty$, one has $I_{n2} \ll n^{-1}ll_n$. Coming to I_{n3}, let us note that, a.s.,

$$I_{n3} = \frac{1}{2}\int_0^1 G(u)w^*(u)d(V_n(u)-u)^2 + \frac{1}{2n}\int_0^1 G(u)w^*(u)dV_n(u).$$

Clearly, if $E|X| < \infty$,

$$\frac{1}{n}\int_0^1 G(u)w\ (u)dV_n(u) \ll n^{-1}, \quad \text{a.s.},$$

and it follows by integration by parts that

$$\int_0^1 G(u)w^*(u)d(V_n(u)-\ u)^2$$
$$= -\int G(u)(V_n(u)-u)^2dw^*(u) - \int w^*(u)(V_n(u)-u)^2dG(u)$$
$$= -\int G(u)(V_n(u)-u)^2w^{**}(u)du - G(a)(V_n(a)-a)^2(w'(a+)-w'(a-))$$
$$-\int w^*(u)(V_n(u)-u)^2dG(u)$$
$$\ll E_0^2+E_r^2\int_0^1 |w^*(u)|(u(1-u))^{2r}dG(u)$$

for any $0 \leqslant r \leqslant \frac{1}{2}$. The theorem now follows in this case in view of Lemmas 2.1 and 2.2.

Step 2 : We finally relax the continuity of w at a. Let us suppose $w(a)-w(a-) = b \neq 0$. Define a new continuous weight function w_1 as follows :

$$w_1(u) = w(u) \qquad \text{for } w \in (0, a)$$
$$= w(u)-b \qquad \text{for } w \in [a, 1).$$

Using the result of step 1, if

$$L_n' = \int G(u)w_1(V_n(u))dV_n(u) \text{ and } L' = \int G(u)w_1(u)du,$$

we have the a.s. representation,

$$L_n'-L'+\int(V_n(u)-u)w_1(u)dG(u)$$
$$\ll n^{-1}l_n^{1+\gamma} \text{ for all } \gamma > 0, \text{ if } E|X| < \infty,$$
$$\ll n^{-1}ll_n \text{ if } E|X|^{1+\delta} < \infty \text{ for some } \delta > 0.$$

Now, the claim is established by showing that if $E|X| < \infty$,

$$(L_n - L) - (L_n' - L') + \int (\dot{V}_n(u) - u)(w(u) - w_1(u))dG(u) \ll n^{-1}ll_n \quad \text{a.s.}$$

To this end, let us note that

$$(L_n - L_n') - (L - L') = b \int_{[V_n^{-1}(a), 1)} G(u)dV_n(u) - b \int_{[a, 1)} G(u)du$$

$$= b \int_{[V_n^{-1}(a), 1)} G(u)d(V_n(u) - u) - b \left(\int_{[a, 1)} - \int_{[V_n^{-1}(a), 1)} \right) G(u)du.$$

Since G satisfies the Lipschitz condition of order 1 in a neighborhood of a, the second term in the above expression can be written as,

$$-bG(a)(V_n^{-1}(a) - a) + O(n^{-1}ll_n) \quad \text{a.s.}$$

Further, by integration by parts, we have

$$b \int_{[V_n^{-1}(a), 1)} G(u)d(V_n(u) - u) = -bG(V_n^{-1}(a))[V_n(V_n^{-1}(a)) - V_n^{-1}(a)]$$

$$-b \int_{[V_n^{-1}(a), 1)} (V_n(u) - u)dG(u).$$

Thus, up to $n^{-1}ll_n$ a.s.,

$$(L_n - L) - (L_n' - L') = -b \int_{[V_n^{-1}(a), 1)} (V_n(u) - u)dG(u)$$

$$+ b(G(V_n^{-1}(a)) - G(a))(V_n^{-1}(a) - a)$$

$$-bG(V_n^{-1}(a))(V_n(V_n^{-1}(a)) - a)$$

$$= b \int_{[a, 1]} (V_n(u) - u)dG(u)$$

using Lemma 2.3. The proof is complete.

In particular this result implies that trimmed mean can be linearlized up to a remainder of the order $n^{-1}ll_n$ which is impossible if one tries to do so using asymptotic representation of quantile processes. Also, one requires existence of density everywhere for the representation of quantile processes.

In the next theorem, we combine the above represetnation technique with Bahadur-Kiefer representation of quantile process to establish the representation for L-statistics with a much wider class of weight functions; however the remainder is less sharp.

Theorem 2 : *Assume that, for some real numbers* $0 < \alpha < \beta < 1$, *F satisfies the conditions* :

F'' exists on the interval $(G(\alpha)-\epsilon, G(\beta)+\epsilon)$, for some ϵ, where F' is bounded away from zero and F'' is bounded.

Let W be a function of bounded variation on [0, 1] such that $W' = w$ exists on $(0, \alpha+\gamma)\bigcup(\beta-\gamma, 1)$ for $\gamma > 0$ and has bounded second derivative on this set. Define

$$L_n^* = \int_0^1 F_n^{-1}(t)dW(t), \quad L^* = \int_0^1 G(t)dW(t)$$

and

$$Z_i^* = \int_\alpha^\beta [(u-I_{(U_i < u)})/F'(G(t))]dW(t)$$

$$+\left(\int_0^\alpha+\int_\beta^1\right)(u-I_{(U_i < u)})w(u)dG(u).$$

Then, if $E|X| < \infty$,

$$L_n^*-L^* = n^{-1}\sum_1^n Z_i^*+O(n^{-3/4}l_n^{\frac{1}{2}}ll_n^{1/4} \text{ a.s.}).$$

Proof : It follows from the representation of quantile processes (see Remark 4.2 of Babu and Singh 1978) that

$$\int_\alpha^\beta (F_n^{-1}(t)-G(t))dW(t) = \int_\alpha^\beta \frac{t-F_n(G(t))}{F'(G(t))} dW(t)+O(n^{-3/4}l_n^{1/2}ll_n^{1/4}) \text{ a.s.} \quad \ldots \quad (2.3)$$

Considering the integral $\int_0^\alpha (F_n^{-1}(t)-G(t))w(t)dt$, we note that, since w satisfies the Lipschitz condition of order 1 in $(0, \alpha+\gamma)$ and $E|X| < \infty$,

$$\int_0^\alpha F_n^{-1}(t)w(t)dt = \int_0^{G(t)} xw(F_n(x))dF_n(x)+\int_{G(t)}^{F_n^{-1}(t)} xw(F_n(x))dF_n(x)+O(n^{-1}) \text{ a.s.}$$

Now, following the proof of the previous theorem, we see that

$$\int_0^{F_n^{-1}(t)} xw(F_n(x))dF_n(x)-\int_0^{G(t)} xw(F(x))dx$$

$$=\int_0^\alpha G(u)w'(u)(V_n(u)-u)du+\int_0^\alpha G(u)w(u)d(V_n(u)-u)+O(n^{-1}l_n^2) \quad \text{a.s.}$$

$$=\int_0^\alpha w(u)(u-V_n(u))dG(u)+G(\alpha)w(\alpha)(V_n(\alpha)-\alpha)+O(n^{-1}l_n^2) \quad \text{a.s.}$$

Therefore, if we show that, a.s.,

$$\int_{G(\alpha)}^{F_n^{-1}(\alpha)} xw(F_n(x))dF_n(x)+w(\alpha)G(\alpha)(V_n(\alpha)-\alpha) \ll n^{-1}l_n^2 \quad \ldots \quad (2.4)$$

then it would follow that

$$\int_0^{\alpha} (F_n^{-1}(t)-G(t))w(t)dt = \int_0^{\alpha} (t-V_n(t))w(t)dG(t)+O(n^{-1}l_n^2) \quad \text{a.s.} \quad \ldots \quad (2.5)$$

To prove (2.4), we note that the following statements are true up to the order $n^{-1}l_n^2$ a.s.

$$\text{L.h.s. of } (2.4) = \int_{G(\alpha)}^{F_n^{-1}(\alpha)} [xw(F_n(x))-G(\alpha)w(\alpha)]dF_n(x)$$

$$\leqslant \int_{G(\alpha)}^{F_n^{-1}(\alpha)} [\,|x-G(\alpha)|\,|w(F_n(x))|+|G(\alpha)|\,|w(F_n(x))-w(\alpha)|\,]dF_n(x)$$

$$\ll |(F_n(G(\alpha))-\alpha)(F_n^{-1}(\alpha)-G(\alpha))|+(F_n(G(\alpha))-\alpha)^2+n^{-1}$$

$$\ll n^{-1}l_n^2 \text{ a.s.}$$

proving (2.4) and hence (2.5).

Similarly one shows that

$$\int_{\beta}^{1} [F_n^{-1}(t)-G(t)]w(t)dt = \int_{\beta}^{1} (t-V_n(t))w(t))dG(t)+O(n^{-1}l_n^2) \text{ a.s.} \quad \ldots \quad (2.6)$$

The statements (2.3), (2.5) and (2.6) yield the theorem.

3. Approximation to normality

Theorems 3 and 4 of this section, which study some rates of convergence to normality of L_n resemble Theorems 1 and 2 of Michel (1976) respectively which establish similar rates for sample mean of i.i.d. r.v.'s. The theorems are stated as follows.

Theorem 3 : *Let us assume that all the conditions of Theorem 1 about w and F hold. Further assume that $E|Z|^{2+c} < \infty$, $0 < \sigma^2 = V(Z)$ and $E|X|^{1+\frac{c}{2}+\delta} < \infty$ where c and δ are some positive numbers. Then for all x real such that $x^2 \leqslant (c+1)l_n$,*

$$|P(n^{\frac{1}{2}}(L_n-L) \leqslant \sigma x)-\Phi(x)|$$

$$\leqslant b_1 n^{-c^*} l_n \exp(-(1-\bar{c})x^2/2)+b_2 n^{-c/2}|x|^{-2(2+c)}+nP(|Z| > b_3 n^{\frac{1}{2}}|x|) \quad \ldots \quad (3.1)$$

where $c^* = min.(c, 1)/2$, $\bar{c} = c^*(c+1)^{-1}$ *and* b_1, b_2, b_3 *are prositive constants independent of* x *and* n.

Theorem 4 : *Let* w *and* F *be as in Theorem* 1. *If* $E|Z|^{2+c} < \infty$, $0 < \sigma^2 = V(Z)$ *and* $E|X|^{2+c+\epsilon_0} < \infty$ *for some* c *and* ϵ_0 *positive, then for all* x *such that* $x^2 \geqslant (c+1)l_n$,

$$|P(n^{\frac{1}{2}}(L_n - L) \leqslant \sigma x) - \Phi(x)| \leqslant b_4 n^{-c/2}|x|^{-2-c-\epsilon_0/2} + nP(|Z| > b_5 n^{-1/2}|x|)$$

where b_4 *and* b_5 *are positive constants independent of* x *and* n.

Proofs of the above theorems depend upon a few lemmas which we state and prove below.

Lemma 3.1 : *For any given positive numbers* a, b *and* r *satisfying the relation*

$$0 \leqslant r < min((2a+2)^{-1}, b^{-1}), \quad \ldots \quad (3.2)$$

there exists a positive constant $k = k(a, b, r)$ *such that if* $y^2 \geqslant kl_n$

$$P(E_r \geqslant n^{-\frac{1}{2}}y) \leqslant n^{-a}y^{-b}.$$

Proof: Let us write $(0, 1) = \bigcup_{i=1}^{4} J_{in}$ where $J_{1n} = (0, n^{-1-a}y^{-b}]$, $J_{2n} = (n^{-1-a}y^{-b}, \frac{1}{2}]$, $J_{3n} = [\frac{1}{2}, 1-n^{-1-a}y^{-b})$ and $J_{4n} = [1-n^{-1-a}y^{-b}, 1)$. Now, $V_n(n^{-1-a}y^{-b}) = 0$ implies that for $y \geqslant 1$ and all n large enough

$$\sup_{t \in J_{1n}} (t(1-t))^{-r}|V_n(t) - t| \leqslant 2^r n^{-(1+a)(1-r)} y^{-b(1-r)} < n^{-\frac{1}{2}} \text{ (since } r < \tfrac{1}{2}).$$

Consequently

$$P\left(\sup_{t \in J_{1n}} (t(1-t))^{-r}|V_n(t) - t| \geqslant n^{-\frac{1}{2}}\right)$$

$$\leqslant P(V_n(n^{-1-a}y^{-b}) \geqslant n^{-1}) \leqslant n^{-a}y^{-b}$$

using Markov's inequality.

Dividing the interval J_{2n} into subintervals of length

$$\gamma_n = n^{-(5+3a)/2}\, y^{-3b/2}$$

and using some elementary approximations we arrive at

$$\sup_{t \in J_{2n}} (t(1-t))^{-r}|V_n(t) - t| \leqslant 2^r \max\{s^{-r}|V_n(s) - s| : s = n^{-1-a}y^{-b} + l\gamma_n,$$

$$l = 1, 2, \ldots, \gamma_n^{-1}/2\} + O(n^{-1})$$

which along with Bonferroni s inequality leads to the inequality

$$P\left(\sup_{t\in J_{2n}} (t(1-t))^{-r}|V_n(t)-t| \geq n^{-\frac{1}{2}}y\right)$$

$$\leq n^{(5+3a)/2}y^{3b/2} \sup_{t\in[n^{-1-a}y^{-b},\,3/4]} P(|V_n(t)-t| \geq \tfrac{1}{2}n^{-\frac{1}{2}}yt^r) \quad \dots \quad (3.3)$$

for all n large enough. For t in $[n^{-1-a}y^{-b}, 3/4]$, we conclude by using Markov's inequality that

$$P(n(V_n(t)-t) \geq \tfrac{1}{2}t^r n^{\frac{1}{2}}y) \leq \exp(-3(a+b+2)(l_n+l_y))[E\exp(z(V_1(t)-t))]^n \quad \dots \quad (3.4)$$

where $z = 6(a+b+2)t^{-r}n^{-\frac{1}{2}}y^{-1}(l_n+l_y)$, and l_y denotes $\log y$. Note that if $t\epsilon[n^{-1-a}y^{-b}, 3/4]$ and r satisfies (3.2), then z is bounded above for $n \geq 1$, $y \geq 1$; let us say it is bounded by k_1. Now using Taylor's expansion and the inequality $\log(1+\alpha) \leq \alpha$ for all $\alpha > -1$, we find that the above probability bound can not exceed

$$\exp[-3(a+b+2)(l_n+l_y)+18e^{k_1}(a+b+2)^2y^{-2}(l_n+l_y)^2].$$

If $y^2 \geq 18e^{k_1}(a+b+2)^2l_n$, then $18e^{k_1}(a+b+2)^2y^{-2}(l_n+l_y)^2 \leq 2(l_n+l_y)$ for all n large enough and hence l.h.s. of (3.4) $\ll n^{-3a-3}y^{-3b}$. Similar bound holds for $-n(V_n(t)-t)$ and hence for $n|V_n(t)-t|$ Thus, l.h.s. of (3.3) $\ll n^{-a}y^{-b}$ for an appropriate choice of k. We obtain similar estimates in the cases $t\epsilon J_{4n}$ and $t\epsilon J_{3n}$ imitating the above proofs to conclude the lemma.

Corollary 3.1 : *Taking $b = 2$ in Lemma 3.1, it follows that for any $a > 0$, if $0 \leq r < (2a+2)^{-1}$, then there exists a $k = k(a, r)$ s.t.*

$$P(E_r \geq kn^{-\frac{1}{2}}l_n^{\frac{1}{2}}) \ll n^{-a}. \quad \dots \quad (3.5)$$

We require only this special case of Lemma 3.1 for proving Theorem 3. An important converse question here is whether the bound $r < (2r+2)^{-1}$ is the best possible for a moderate deviation bound like (3.5). We offer a partial answer to this question in the remark that follows. By $a_n \gg b_n$ we mean $b_n \ll a_n$.

Remark 3.1 : If $r > (2a+2)^{-1}$, there exists a $\gamma > 0$ such that for any $k > 0$,

$$P(E_r \geq kn^{-\frac{1}{2}}l_n^{\frac{1}{2}}) \gg n^{-a+\gamma},$$

Proof: Since E_r is a non-decreasing function of r, we assume w.l.g. that $r < 1/2$. Let us choose $\gamma > 0$ such that $\gamma < (1+a)-(2r)^{-1}$. We actually show that for any $k > 0$,

$$P(|V_n(t_n)-t_n|t_n^{-r} > 2kn^{-\frac{1}{2}}t_n^{1/2}) \gg n^{-a+\gamma}$$

where $t_n = n^{-1-a+\gamma}$. To this end, we first note that $t_n^{1-r} \ll n^{-\frac{1}{2}-\epsilon}$ for a positive ϵ, and hence it suffices to see that

$$P(V_n(t_n)t_n^{-r} > kn^{-\frac{1}{2}}t_n^{1/2}) \gg n^{-a+\gamma}.$$

Since $n^{-1}t_n^{-r} = n^{-\frac{1}{2}+\epsilon_1}$ for a positive ϵ_1, we have for all n large enough

$$P(V_n(t_n)t_n^{-r} > kn^{-\frac{1}{2}}t_n^{1/2}) \geqslant P(V_n(t_n) \geqslant n^{-1})$$

$$= P(U_{1(n)} \leqslant t_n) = 1-(1-t_n)^n \sim n^{-a+\gamma}$$

where $U_{1(n)}$ in the above expression denotes minimum of $\{U_1, \ldots, U_n\}$.

In the boundary case, i.e. when $r = (2a+2)^{-1}$, we are only able to say that for a k large enough

$$P(E_{(2a+2)^{-1}} \geqslant kn^{-\frac{1}{2}}t_n^{1/2}) \ll n^{-a}t_n^{1+a}$$

and for any $k > 0$,

$$P(E_{(2a+2)^{-1}} \geqslant kn^{-\frac{1}{2}}t_n^{1/2}) \gg n^{-a}t_n^{-1-a}.$$

Lemma 3.2: *If* $E_* = \sup_{0 \leqslant t \leqslant 1} |V_n^{-1}(t)-t|$, *then* $E_* \leqslant E_0$.

Proof: For any $\gamma > 1$,

$$t-\gamma E_0 < V_n(t) < t+\gamma E_0 \qquad \text{for all } 0 \leqslant t \leqslant 1,$$

$$\Longrightarrow V_n^{-1}(t-\gamma E_0) \leqslant t \qquad \text{for all } t \epsilon[\gamma E_0, 1+\gamma E_0]$$

$$\Longrightarrow V_n^{-1}(s) \geqslant s+\gamma E_0 \qquad \text{for all } 0 \leqslant s \leqslant 1.$$

Similarly $V_n^{-1}(s) \geqslant s-\gamma E_0$ for all $0 \leqslant s \leqslant 1$. The lemma follows from these conclusions, since $\gamma > 1$ is arbitrary.

Lemma 3.3 : *Let ξ, η be two r.v.'s (in general dependent).*

(a) *For any $\epsilon > 0$,*

$$\|P(\xi+\eta \leqslant x)-\Phi(x)\|_\infty \leqslant \|P(\xi \leqslant x)-\Phi(x)\|_\infty+\epsilon(2\pi)^{-1/2}+P(|\eta| > \epsilon$$

(b) *For any $\epsilon > 0$ and x real such that $\epsilon < |x|$ we have*

$$|P(\xi+\eta \leqslant x)-\Phi(x)|$$

$$\leqslant \max_{s=x\pm\epsilon} |P(\xi \leqslant s)-\Phi(s)|+P(|\eta| > \epsilon)+\epsilon(2\pi)^{-\frac{1}{2}} exp(-(|x|-\epsilon)^2/2).$$

Proof of this lemma is trivial.

We now prove the two theorems of this section.

Proof of Theorem 3 : We present the proof in two parts; in Part I we get a deviation bound for R_n and in Part II we complete the proof using the bound obtained in Part I.

Part I. We show here that the under the set-up of Theorem 3, there exists a constant b_6 such that

$$P(R_n \geqslant b_6 n^{-1}l_n) \ll n^{-c/2-\delta_1} \quad \dots \quad (3.6)$$

for a $\delta_1 > 0$. To prove this, let us fix $b_7 > 0$ using Corollary 3.1 such that $P(\Omega_n^c) \ll n^{-c}$ where $\Omega_n \equiv \{E_0 \leqslant b_7 n^{-\frac{1}{2}}l_n^{1/2}\}$. The proof supplied for Theorem 1 reveals that on Ω_n,

$$R_n \ll (E_0^2+n^{-1}) \; |\int G(u)dV_n(u)|+E_0^2+E_0 n^{-\frac{1}{2}}l_n^{1/2}+E_*^2$$

$$+E_r^2 \int (u(1-u))^{2r}dG(u)+E_{**} \quad \dots \quad (3.7)$$

where r is some non-negative number and $E_{**} = \sup\limits_{0\leqslant t\leqslant 1} |V_n(V_n^{-1}(t))-t|$. As a consequence, (3.6) follows using Lemmas 2.2 (iii), 2.3, 3.2, and Corollary 3.1, if we show further that

$$P((E_0^2+n^{-1})|\int G(u)dV_n(u)| \geqslant b_8 n^{-1}l_n) \ll n^{-c/2-\delta_1}$$

for a $\delta_1 > 0$. But since on Ω_n, $E_0^2 \leqslant b_7^2 n^{-1}l_n$, we essentially have to show that

$$P(n^{-1}\left|\sum_{i=1}^{n}\right| G(U_i)|-E|G(U_1)|| \geqslant b_9) \ll \text{r.h.s. of (3.6)} \quad \dots \quad (3.8)$$

We conclude (3.8) using Theorem 3 of Sen (1970) in the case $0 < c < 2$ and Theorem 5 of Michel (1976) in the case $c \geqslant 2$.

Part II. For $|x| \leqslant 1$; we use Lemma 3.3(a) with $\xi = n^{-\frac{1}{2}} \sum_{1}^{n} Z_i$, $\eta = n^{\frac{1}{2}} R_n$ and $\epsilon = n^{-\frac{1}{2}} l_n$. In this case the conclusion follows using (3.6) and Katz-Petrov theorem (see Katz (1963)). In the case $|x| \geqslant 1$, we apply Lemma 3.3(b) with the same choice of ξ, η and ϵ. If $|t| \geqslant 1$,

$$(2\pi)^{-\frac{1}{2}}\epsilon \exp(-(|t|-\epsilon)^2/2) \leqslant b_{10} n^{-c^*/2} l_n \cdot \exp(-(1-\bar{c})t^2/2). \quad \ldots \quad (3.9)$$

For $y = x+\epsilon$ or $x-\epsilon$, if $y^2 \geqslant (c+1)l_n$ we obtain by an appeal to Theorem 2 of Michel (1976) that

$$\left| P\left(n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_i \leqslant \sigma y\right) - \Phi(y) \right| \leqslant n^{-c/2}|y|^{-2(2+c)} + nP(|Z| > b_{11} n^{\frac{1}{2}} |y|)$$

$$\leqslant n^{-c/2}|x|^{-2(2+c)} + nP(|Z| > b_{11} n^{1/2}|x|) \quad \ldots \quad (3.10)$$

and in the other case, i.e. when $y^2 \leqslant (c+1)l_n$ we apply Theorem 1 of Michel (1976) and find that

$$\text{l.h.s. of (3.10)} \quad \leqslant n^{-c^*} \exp((1-\bar{c})y^2/2) + nP(|Z| > b_{12} n^{\frac{1}{2}} |y|)$$

$$\leqslant n^{-c^*} \exp((1-\bar{c})x^2/2) + nP(|Z| > 2b_{12} n^{\frac{1}{2}} |x|). \quad \ldots \quad (3.11)$$

Now, (3.7), (3.9) and (3.11) along with Lemma 3.3(b) complete the proof of Theorem 3.

It is obvious from the above proof that l_n in r.h.s. of (3.1) is unnecessary when $0 < c < 1$.

Proof of Theorem 4 : W.l.o.g. we assume $x > 0$. We break the zone $x^2 \geqslant (c+1)l_n$ into two parts namely J_{5n} and J_{6n} where

$$J_{5n} = \{(c+1)l_n \leqslant x^2 \leqslant n^{1/4}\}$$

and

$$J_{6n} = \{x^2 \geqslant n^{1/4}\}.$$

(This division of the zone may be avoided if w has no jump points). We have different arguments for the two parts. For $x \epsilon J_{5n}$, we write

$$P(L_n - L \geqslant n^{-\frac{1}{2}} x)$$

$$= P(\bar{Z}_n - E(Z) \geqslant (n^{-\frac{1}{2}} \pm n^{-3/4})\sigma x) \pm P(|R_n| > n^{-3/4} x).$$

Now, we follow the proof of Theorem 3 to see that both the terms in the right hand expression above are

$$\leqslant n^{-c/2}(x)^{-2(2+c)}+nP(|Z| > b_{13}x).$$

The condition $x^2 \leqslant n^{1/4}$ is needed in choosing $\Omega_n = \{E_0 \leqslant b_{14}n^{-\frac{1}{2}}l_n^{\frac{1}{2}}\}$ such that $P(\Omega_n^c) \leqslant n^{-c/2}x^{-2(2+c)}$. If $x^2 > n^{1/4}$, we use the bound

$$P(L_n - L \geqslant n^{-\frac{1}{2}}x) \leqslant P(\bar{Z}_n - E(Z) \geqslant n^{-\frac{1}{2}}\sigma x/2) + P(|R_n| > n^{-\frac{1}{2}}\sigma x/2).$$

Since in J_{6n}, $x/2 \geqslant (c+1)l_n$ for all n large enough, it follows from Theorem 2 of Michel (1976) that $P(\bar{Z}_n - E(Z)) \geqslant n^{-\frac{1}{2}}x/2)$ is of the desired order. To estimate $P(|R_n| \geqslant n^{-\frac{1}{2}}\sigma x/2)$ for x in J_{6n} we use a crude bound for R_n. Obviously, if w is continuous at its jump points, then $I_{n2} \leqslant E_0\bar{Z}_n$ and hence the arguments of step 2 of Theorem 3 show that at $r_0 = 1/(2+c)$

$$|R_n| \leqslant (E_0 + n^{-1})|Z_n| + E_0^2 + E_*^2 + E_0E_* + E_{**} + E_{r_0}^2 \int (u(1-u))^{2r_0} dG(u)$$

$$\leqslant (E_0 + n^{-1})|Z_n| + E_{r_0}^2 + n^{-1},$$

due to the facts that $E_* \leqslant E_0$, $E_r \geqslant E_0$ for any $r > 0$, $E_{**} \leqslant n^{-1}$ a.s. and $E|X|^{2+c} < \infty$.

Therefore

$$P(|R_n| > n^{-\frac{1}{2}}\sigma x/2)$$

$$\leqslant P(E_0 \geqslant \theta n^{-\frac{1}{4}} x^{c'}) + P(|\bar{Z}_n| > \theta x^{1-c'}) + P(E_{1/(2+c)}^2 > n^{-\frac{1}{2}}\theta x)$$

where θ is some positive number and $c' = \varepsilon(2(2+c+\varepsilon))^{-1}$. The proof is concluded now using Lemma 3.1 and Theorem 2 of Michel (1976).

Remark 3.2 : If L_n is a trimmed type L-statistic, i.e. $w(u) = 0$ if $w \notin [\alpha, \beta]$ for some $0 < \alpha < \beta < 1$, then Z_i's are bounded r.v.'s; moreover $\int (u(1-u))^r |w^*(u)| dG(u) < \infty$ for all $r \geqslant 0$. Consequently, Theorems 1 and 2 do not require any moment condition on Z and X; also Theorems 3 and 4 hold for all $c > 0$ without requiring any moment restriction.

A few interesting corrolaries derived easily from Theorems 3 and 4 follow now.

Corollary 3.2 : *Let t_n be a sequence with $t_n \to \infty$ such that $t_n^2 - cl_n - (c+1)ll_n$ is boundeed above. Then, under the restrictions of Theorem 3,*

$$P(L_n - L \geqslant \sigma n^{-\frac{1}{2}} t_n) \sim (2\pi)^{-\frac{1}{2}} t_n^{-1} \exp(-t_n^2/2).$$

This is an analogue of Theorem 4 of Michel (1976) for L_n.

Corollary 3.3 : *If Theorems 3 and 4 are valid, it is immediate that*

$$\|P(n^{\frac{1}{2}}(L_n - L) \leqslant \sigma x) - \phi(x)\|_p \ll n^{-c^*/2} l_n$$

for any $p \geqslant 1$. *This is of course a* L_p *version of Berry-Esseen theorem for* L_n.

Corollary 3.4 : *Let* g *be a symmetric continuous loss function such that* $g(x)$ *is non-decreasing in* $x \geqslant 0$ *and* $\sup_{x>0} g(x)(1+x)^{-(2+\epsilon)} < \infty$. *If Theorems 3 and 4 hold, then*

$$|Eg(\sigma^{-1} n^{\frac{1}{2}}(L_n - L)) - E_\phi g(x)| \ll n^{-c^*} l_n.$$

In particular the corollary yields convergence of moments of $(L_n - L)$. The result is obtained using essentially the idea of Theorem 6 of Michel (1976). Details are easy.

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