Sankhyā : The Indian Journal of Statistics
1989, Volume 51, Series A, Pt. 2, pp. 144–167.

SOME FURTHER RESULTS ON NONUNIFORM RATES OF CONVERGENCE TO NORMALITY

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SUMMARY. For row sums of independent random variables in a triangular array $\{X_{ni} : 1 \leqslant i \leqslant n, n \geqslant 1\}$; under certain conditions on the random variables which ensure that all the moments of X_{ni} exist but the moment generating function of the random variables do not necessarily exist, non uniform rates of convergence to normality are studied. Included as special cases are rates of standardised sums of i.i.d. random variables. As applications of these non uniform rates, probabilities of large deviations are found. Necessity of the assumptions made are also proved. The rates are further utilised to deal large deviations of the type $Cn^{1/2}$, $c > 0$ in limiting form, to prove certain moment type convergences and in deriving non uniform L_p versions of the Berry-Esseen theorem. The results are extended to general non-linear statistics. Applications are made in the case of L statistics.

1. Introduction

Consider a double sequence $\{X_{ni} : 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ of random variables where variables within each row are independently distributed and satisfy

$$E\, X_{ni} = 0,\ \sup_{n \geqslant 1} \max_{1 \leqslant i \leqslant n} E\, X_{ni}^2\, g(X_{ni}) < \infty \qquad \ldots \quad (1.1)$$

$$\left(\text{or that} \sup_{n \geqslant 1} n^{-1} \sum_{i=1}^{n} EX_{ni}^2\, g(X_{ni}) < \infty\right)$$

where $g(x)$ is a non negative even function. We further assume that

$$\inf_{n \geqslant 1} n^{-1} s_n^2 > 0 \text{ where } s_n^2 = \sum_{i=1}^{n} EX_{ni}^2. \qquad \ldots \quad (1.2)$$

Denote $F_n(t) = P(s_n^{-1} S_n \leqslant t)$, t real, $S_n = \sum_{i=1}^{n} X_{ni}$. Under the above assumptions it is known that, in the i.i.d. case with $g(x) \equiv 1$,

$$\lim_{n \geqslant 1} \sup_t |F_n(t) - \Phi(t)| = 0$$

where $\Phi(t) = \int_{-\infty}^{t} (2\pi)^{-1/2} \exp(-x^2/2)dx$ is the standard normal distribution function. The uniform rate of convergence of $|F_n(t) - \Phi(t)|$ to zero was studied by Berry and Esseen and later was extended by Katz (1963).

AMS (1980) *subject classification* : 60F99.

Key words and phrases : Non-uniform rates; Non linear statistics; Moment convergence

The non uniform rates of convergence of the above are of great interest with applications to the probabilities of deviations and certain types of moment convergences. With $g(x) = |x|^c$; $c > 0$ Michel (1976) obtained non uniform bounds in i.i.d. case which later were extended by Ghosh and Dasgupta (1978), for $g(x) = |x|^c u(x)$, $c \geqslant 0$, $u(x) < |x|^\epsilon + L$, $\forall\ \epsilon > 0$ and some $L > 0$ for independent r.v.s. in a triangular array. Ghosh and Dasgupta also extended the results for general non-linear statistics. Under different set up Statulevicius (1966), Petrov (1972), Nagaev (1979) etc. have results on deviations. See also Linnik (1961, 1962).

This paper studies the non uniform rates of convergence to normality under (1.1) and (1.2) with

$$K(c)|x|^c + L(c) < g(x) \leqslant \exp(s|x|),\ \forall\ c > 0 \qquad \ldots \quad (1.3)$$

and some $s > 0$ where $K(c)$ and $L(c)$ are constants depending only on c and $x^{-1} \log g(x)$ is non increasing for $x > x_0$ ($\geqslant 0$). This may be relaxed by the weaker condition, $\inf\limits_{x \in (x_0, x_n)} x^{-1} \log g(x) \geqslant x_n^{-1} \log g(x_n)\,(1+o(1))$ for a sequence $x_n \to \infty$ and some x_0 fixed. This includes functions g for which $x^{-1} \log g(x)$ is non increasing with small oscillations.

In other words the cases when all the finite moments exist but the moment generating function of the random variables do not necessarily exist is the subject of study. Examples of such functions are $g(x) = \exp\,(\log^m(1+|x|))$, $m > 1$; $g(x) = \exp(\log(1+|x|)\ \log\ \log(e+|x|))$, $g(x) = \exp(|x|^\alpha)\ 0 \leqslant \alpha \leqslant 1$ etc. Nagaev (1979) assumed g to be differentiable to prove deviation results. Such assumptions are not required in the present paper.

As applications of these nonuniform bounds the range of values of t_n where $1 - F_n(t_n) \sim \Phi(-t_n) \sim F_n(-t_n)$, $t_n \to \infty$ is found. This gives a clear picture about the variation of the normal approximation zone depending on the functional form of g (Theorems 2.3, 2.15). Immediately after theorem 2.15 it is shown that the zone computed here is larger than those obtained by previous authors (see e.g. Nagaev (1979)).

That the assumptions are necessary are also shown in Theorem 2.9—2.11. The nonuniform bounds are further utilised to obtain more stronger form of the L_p versions of the Berry Esseen theorem compared to those obtained by Erickson (1973) and to prove certain moment type convergences. Apart from large deviation it is also shown that large deviation of the form $c\,n^{1/2}$, $c > 0$, (see Bahadur, 1960) can be obtained in limiting sense (Theorem 2.6).

Results on triangular array are proved in Section 2. In Section 3 extension of the above results are made to general nonlinear statistics. As an example we include the L-statistic in Section 4.

2. The results on row sums of random variables in a triangular array

The following theorem states the rates of convergence of $F_n(t)$ to $\Phi(t)$ depending on n and t when t is in a neighbourhood of the origin.

Theorem 2.1 : *Let* (1.1), (1.2) *with* (1.3) *hold. Then for*

$$1 \leqslant t^2 \leqslant 2\ (\log|t| + \log\ g(r\, s_n t)) \qquad \ldots \quad (2.1)$$

with $|t| < \epsilon_2 \sqrt{n}$, *where* $\epsilon_2\ (> 0)$ *is small, there exists a constant* $b > 0$ *depending on* r, $0 < r < 1/2$ *such that*

$$|F_n(t) - \Phi(t)| \leqslant b\ exp(-t^2/2)|t|^{-1}|\ exp(O(n^{-1/2}|t|^3)) - 1|$$
$$+ b\ exp(-t^2/2 + O(n^{-1/2}|t|^3))\ n^{-1/2}$$
$$+ \sum_{i=1}^{n} P(|X_{ni}| > r\, s_n\ |t|). \qquad \ldots \quad (2.2)$$

Remark 2.1 : The 2nd term in the r.h.s. of (2.2) can very well be dropped but it is written in conformity with Theorem 2.3. For $t^2 \leqslant 1$, one may use unform bound $O(n^{-1/2})$ since all the moments exist ; this comment holds for Theorem 2.4 also. In (2.1) we take $t^2 \geqslant 1$ so that log $|t|$ appearing therein is bounded below.

The proof of the theorem follows the lines of arguments in Ghosh and Dasgupta (1978) on observing that for $0 < \epsilon, t > 1$ and $\max\ (3, x_0) < x < r\, s_n t$, the inequality $(1+\epsilon)t \leqslant s_n\ |x|^{-1} \log\ (x^3 g(x))$ holds.

Remark 2.2 : The 2nd part of the condition (1.1) may be relaxed. From the proof of the theorem it follows that instead of (1.1) it is sufficient to assume a weaker condition

$$Ex_{ni} = 0,\ \sup_{n \geqslant 1} \frac{1}{n} \sum_{i=1}^{n} E\, X_{ni}^2\, g(X_{ni}) < \infty. \qquad \ldots \quad (2.3)$$

Remark 2.3 : In some special case of g, ϵ while selecting r can be made zero i.e. $r = 1/2$ in those cases e.g. if $g(x) = \exp\ (|x|)$ then for $t > 0$

$$|x|^p \exp(t|x|/s_n) < x^2 g(x), \quad 0 \leqslant x_0 < x < r s_n t$$

if $t|x|/s_n < |x| - (p-2) \log |x|$ i.e., if $t < s_n \left(1 - \frac{p-2}{|x|} \log |x|\right)$ which is always true when

$$t \leqslant s_n \left(1 - \frac{p-2}{|x_0|} \log |x_0|\right).$$

It may further be noted that for general g the region (2.1) may be made to $1 \leqslant t^2 \leqslant M(\log |t| + \log g(rs_n t)), M > 0$ large by a small choice of r.

Remark 2.4 : The value of ϵ_2 in theorem 2.1 is immaterial when $g(x) = o(\exp(s|x|)) \forall s > 0$, for in that case (2.1) asserts $t = o(n^{1/2})$. But when $g(x) = \exp(s|x|)$ $s > 0$, (2.1) asserts $t < ss_n$; hence the value of ϵ_2 matters in the case. Proof of the above theorem leads us to conclude that ϵ_2 is basically determined through the counstant b in the relation $\epsilon_1 = b\,\epsilon_2$ to estimate (2.11) of Ghosh and Dasgupta (1978) and b can be taken to be

$$(c/6)\, n^{-1} \sum_{i=1}^{n} E|X_{ni}|^3 \exp(s|x_{ni}|) \text{ if } t < s' s_n,\ s' < s$$

(see expansion of $f_i(t)$)

where
$$c^{-1} = \inf_{n \geqslant 1} (s_n/\sqrt{n})^3.$$

The ultimate value of ϵ_2, constraining the value of ϵ_1 turns out to be

$$\left[c\, n^{-1} \sum_{i=1}^{n} E|X_{ni}|^3 \exp(s'|X_{ni}|)\right]^{-1} \text{such that } |B_n(t)| \leqslant bn^{-1}t^3 \leqslant \epsilon_1 t.$$

Thus Theorem 2.1 is valid for

$$t < \left[\left\{c\, n^{-1} \sum_{i=1}^{n} E(|X_{ni}|^3 \exp(s'|X_{ni}|)\right\}^{-1} \Lambda\, s' c^{-1/3}\right] n^{1/2},\ s' < s$$

when $g(x) = \exp(s|x|)$.

Similarly the order of 1st and 2nd terms of the r.h.s. of (2.2) i.e. $\exp(O(n^{-1/2}|t|^3)) = \exp(Kn^{-1/2}|t|^3)$ with

$$K = c\, n^{-1} \sum_{i=1}^{n} E|X_{ni}|^3 \exp(s'\ |X_{ni}|)$$

For Theorem 2.2 similarly we have, $K = c\, n^{-1} \sum_{i=1}^{n} EX_{ni}^4 \exp(s'|X_{ni}|)$, $c^{-1} = \inf_{n \geqslant 1} (s_n/\sqrt{n})^4$.

Noting that moment generating function (m.g.f) of a r.v. X exists around a neighbourhood of the origin implies $E(\exp(s|x|)) < \infty$ for some $s > 0$, a few observations which are immediate from theorem 2.1 are listed below.

Corollary 2.1 : *If the m.g.f. of* $\{X_{ni},\ n \geq 1,\ 1 \leq i \leq n\}$ *exist and the mean of the m.g.f's is uniformly bounded around a fixed nbhd of the origin then under* (1.2)

$$1-F_n(t_n) \sim \Phi(-t_n) \sim Fn(-t_n) \text{ if } t_n = o(n^{1/6}),\ t_n \to \infty.$$

Remark 2.5 : When X_{ni} 's have identical distribution the above reduces to a theorem of Cramer (1938). Subsequently we shall show that even in the case of triangular array the conditions of the Corrollary 2.1 can be substancially relaxed to obtain the same conclusion (see Theorem 2.4).

Proof of the Corollary 2.1 : In view of the well known result $\Phi(-x) \sim (2\pi)^{-1/2} x^{-1} \exp(-x^2/2),\ x \to \infty$, it suffices to show that

$$t_n \exp(t_n^2/2)(1-F(t_n)-\Phi(-t_n)) = o(1).$$

This follows from Theorem 2.1 alongwith remark 2.2, as $n^{-1/2} t_n^3 = o(1)$ and

$$\sum_{i=1}^{n} P(|X_{ni}| > rs_n t) \leq bt^{-2}(g(rs_n t))^{-1} s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2 g(X_{ni}) I(|X_{ni}| > rs_n t)$$

$$= O(t^{-2} \exp(-rs_n t))$$

$$= o(t^{-1} \exp(-t^2/2) \text{ as } t = o(n^{1/6}). \qquad \ldots \quad (2.4)$$

Remark 2.6 : From the proof of the theorem 2.1 it follows that the truncation of the random variables is not necessary when the m.g.f. exist. Hence the calculations (2.4) may be omitted in the case.

Remark 2.7 : The normal approximation zone can be extended to

$$o(n^{1/4}) \text{ when } \qquad EX_{ni}^3 = 0,\ n \geq 1,\ 1 \leq i \leq n \qquad \ldots \quad (2.5)$$

Then we have the following

Theorem 2.2 : *Under the assumptions of Theorem* 2.1 *and* (2.5) *for* $1 \leq t^2 \leq 2\ (\log|t| + \log g(rs_n t))$ *with* $|t| \leq \epsilon_2 n^{1/2}$, *where* $\epsilon_2\ (>0)$ *is small, there exist a constant b depending on r,* $0 < r < 1/2$ *such that*

$$|F_n(t)-\Phi(t)| \leq b \exp(-t^2/2))|t|^{-1}|\exp(O(n^{-1}t^4)) - 1|$$
$$+ b \exp(-t^2/2 + O(n^{-1}t^4))\, n^{-1/2}$$
$$+ \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|). \qquad \ldots \quad (2.6)$$

Remark 2.8 : Say that the 2nd term on the r.h.s. ensures that the overall order of n, $(-\infty < t < \infty)$ cannot be less than $n^{-1/2}$.

The proof of the Theorem 2.2 essentially follows the same lines as that of Theorem 2.1.

Using Theorem 2.2 and following the lines of proof of Corollary 2.1 when the mean of the m.g.f. of X_{ni}'s are uniformly bounded around a fixed *nbhd* of the origin one proves remark 2.7.

As a consequence of Theorems 2.1 and 2.2 we may obtain normal approximation zones for general g which will be helpful to obtain normal approximation results known so far under weaker assumptions.

Theorem 2.3 : *Under the assumptions* (2.3), (1.2) *with* (1.3) *for a sequence* $\{t_n\}$ *satisfying*

(i) $t_n = o(n^{1/6})$

(ii) $t_n^2 - 2(\log\ t_n + \log\ g(r\ s_n\ t_n) \to -\infty,\ 0 < r < 1/2$ *the following holds*

$$1 - F_n(t_n) \sim \Phi(-t_n) \sim F_n(-t_n) \text{ as } t_n \to \infty.$$

Further if (2.5) is satisfied, (i) may be replaced by

$$t_n = o(n^{1/4}).$$

r may be taken to be 1/2 in some cases according to Remark 2.3.

Proof : The proof is immediate from Theorems 2.1 and 2.2 along the lines of Corollary 2.1 with the following consideration

$$\sum_{i=1}^{n} P(|X_{ni}| > rs_n t) \leqslant bt^{-2} (g(rs_n t))^{-1} \sup_{n \geqslant 1} s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2 g(X_{ni})$$

$$\times I(|X_{ni}| > rs_n t)$$

$$= O(t^{-2} (g(rs_n t))^{-1}) \qquad \ldots \quad (2.7)$$

$$= o(t^{-1} \exp(-t^2/2)) \qquad \ldots \quad (2.8)$$

as $t = t_n$ satisfies (ii).

Remark 2.9 : If the sequence $\{X_{ni}^2\ g(X_{ni})\}$ is uniformly integrable then the conclusion of Theorem 2.3 holds even if l.h.s. of (ii) is bounded above, since

$$\sum_{i=1}^{n} P(|X_{ni}| > rs_n t) = o(t^{-2}(g(rs_n t))^{-1}) \qquad (2.9)$$

in that case.

Let us calculate the normal approximation zone when $g(x) = \exp(s|x|^{\alpha})$, $s > 0$, $0 < \alpha \leqslant 1$. Letting $t = t_n \to \infty$ from (ii) we have

$$t^2 \leqslant 2\, s\, (rs_n\, t)^{\alpha},\ t > 0$$

i.e., $t^{2-\alpha} \leqslant 2sr^{\alpha}\ n^{\alpha/2}\, \lambda^2$ where $\lambda^2 = \inf\,(s_n^2/n)$ i.e., $t \leqslant (2sr^{\alpha}\,\lambda^{\alpha})^{1/(2-\alpha)}\ n^{\alpha/2(2-\alpha)}$, $r = \frac{1}{2}$.

Note that $\dfrac{\alpha}{2(2-\alpha)} = \dfrac{1}{6}$, $\dfrac{1}{4}$ if $\alpha = \dfrac{1}{2}$, $\dfrac{2}{3}$ respectively. Therefore in view of Theorem 2.3 we have the following

Theorem 2.4 : *The conclusion of Corollary* 2.1 *remains valid* (*i.e.* $1-F_n(t_n) \sim \Phi(-t_n) \sim F_n(-t_n)$, $t_n = o(n^{1/6})$, $t_n \to \infty$) *under the relaxed condition*

$$\sup_{n \geqslant 1} \frac{1}{n} \sum_{i=1}^{n} E[X_{ni}^2\ \exp(s\,|X_{ni}|^{1/2})] < \infty \text{ for some } s > 0. \qquad \dots \quad (2.10)$$

Similarly, under (2.5) the conclusion of Remark 2.7 holds (i.e. $1-F_n(t_n) \sim \Phi(-t_n) \sim F_n(-t_n)$, $t_n = o(n^{1/4})$, $t_n \to \infty$) even if

$$\sup_{n \geqslant 1} \frac{1}{n} \sum_{i=1}^{n} E[X_{ni}^2 \exp(s\,|X_{ni}|^{2/3})] < \infty \text{ for some } s > 0 \qquad \dots \quad (2.11)$$

(We assume $EX_{ni} = 0\ \forall\ n \geqslant 1$, $1 \leqslant i \leqslant n$, and $\underline{\lim}\ n^{-1}\ s_n^2 > 0$ alongwith (2.10) and (2.11)).

Remark 2.10 : Since g has growth more than any power bound, it is immaterial whether we consider $x^2\ g(x)$ or $g(x)$. We preferred to consider $x^2\ g(x)$ rather than $g(x)$ because of following two reasons. Firstly it is known that the conclusion on the rates of convergence cannot be achieved unless we assume a bit more than the existance of the 2nd moment (see e.g. Katz, 1963). Therefore we wanted to base our conclusion on rates solely on the excess of x^2 viz. $g(x)$. Also the estimates in computation take nice form if we consider $x^2\ g(x)$.

The following theorem says that deviations of the form $a\sqrt{n}$, $a > 0$, can be tackled in the limiting form.

Corollary 2.2. *Under the assumption of Theorem* 2.2 *for* $t_n = \epsilon\sqrt{n}$, $\epsilon > 0$ *small*

$$1-F_n(t_n) \leqslant b\, t_n^{-1} \exp\left[-\frac{t_n^2}{2}\,(1-K\,\epsilon)\right] \qquad \dots \quad (2.12)$$

under additional assumption (2.5)

$$1-F_n(t_n) \leqslant bt^{-1} \exp\left[-\frac{t_n^2}{2}(1-K\epsilon^3)\right] \quad \ldots \quad (2.13)$$

Further in special case of i.i.d. random variables, one has

$$\exp\left[-\frac{t_n^2}{2}(1+O(\epsilon))\right] \leqslant 1-F_n(t_n) \leqslant b\, t_n^{-1} \exp\left[-\frac{t_n^2}{2}(1-K\epsilon)\right] \ldots \quad (2.14)$$

and under (2.5)

$$\exp\left[-\frac{t_n^2}{2}(1+O(\epsilon^3))\right] \leqslant 1-F_n(t_n) \leqslant b\, t_n^{-1} \exp\left[-\frac{t_n^2}{2}(1-K\epsilon^3)\right] \ldots \quad (2.15)$$

Also for $t_n = o(b^{1/2}) = \epsilon_n \sqrt{n}$ where $\epsilon_n \to 0$ as $n \to \infty$

$$1-F_n(t_n) \leqslant b\, t_n^{-1} \exp\left[-\frac{t_n^2}{2}(1-K\epsilon_n)\right] \quad \ldots \quad (2.16)$$

under additional assumption (2.5)

$$1-F_n(t_n) \leqslant b\, t_n^{-1} \exp\left[-\frac{t_n^2}{2}(1-K\epsilon_n^2)\right] \quad \ldots \quad (2.17)$$

when X_{ni} are i.i.d. r.v. one may further have

$$1-F_n(t_n) \geqslant b \exp\left[-\frac{t_n^2}{2}(1+O(1))\right] \quad \ldots \quad (2.18)$$

where K is defined in Remark 2.4.

Proof: Since the m.g.f. of X_{ni} exist, the third term of r.h.s. of (2.2) is absent (see Remark 2.6). Hence from Theorem 2.1.

$$1-F_n(t_n) \leqslant \Phi(-t_n)+b\, t_n^{-1} \exp\left[-\frac{t_n^2}{2}+Kn^{-1/2}t_n^3\right]$$

$$= b\, t_n^{-1} \exp\left[-\frac{t_n^2}{2}(1-K\epsilon)\right]$$

Similarly (2.13) follows from Theorem 2.3.

For lower class inequality in special case of i.i.d. random variables, w.o.l.g. assuming $EX_1^2 = 1$. We have from Chernoff's theorem,

$$\lim_{n\to\infty} \frac{2}{\epsilon^2 n} \log P(\bar{X}_n > \epsilon) = \frac{2}{\epsilon^2} \log \inf_{t \geqslant 0} E\, e^{t(X_1-\epsilon)}$$

Now

$$E\, e^{t(X_1-\epsilon)} \geqslant E\left[1+t(X_1-\epsilon)+\frac{t^2}{2}(X_1-\epsilon)^2+\frac{t^3}{6}(X_1-\epsilon)^3\right]$$

$$=1-\epsilon t+\frac{t^2}{2}(1+\epsilon^2)+\frac{t^3}{6}(\mu_3-3\epsilon-\epsilon^3). \quad \ldots \quad (2.19)$$

Minimise the r.h.s. w.r.t t and see that the minimum is attained at $t=\epsilon+o(\epsilon)$ if $\mu_3 \neq 0$ and $t=\epsilon+o(\epsilon^2)$ if $\mu_3=0$. Putting these approximate solutions to the r.h.s. of (2.19), (2.14) and (2.15) follows.

Proof of (2.16), (2.17) are similar to those of (2.12) and (2.13) ; we only prove (2.18).

Note that for large n, $\epsilon_n = n^{1/2}\, t_n = o(1) < \epsilon\ (>0)$ fixed. So

$$\liminf_{n\to\infty} \frac{2}{\epsilon_n^2\, n} \log P(\bar{X}_n > \epsilon_n) \geqslant \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{2}{\epsilon^2 n} \log P(\bar{X}_n > \epsilon) = -1$$

Hence $1-F_n(t_n) \geqslant b \exp\left[-\frac{t_n^2}{2}(1+o(1)\right]$

The next theorem provides the nonuniform rates of convergence in the complementary zone of Theorem 2.1.

Theorem 2.5 : *For* $t^2 \geqslant 2(\log|t|+\log g(rs_n\, t))$ *with* $x^{-1} \log g(x) \to 0$ *as* $x\to\infty$, *we have*

$$|F_n(t)-\Phi(t)| = O(|t|\, g(rs_n t))^{-1+\epsilon_{n,t}} + \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|) \quad \ldots \quad (2.20)$$

where $\qquad \epsilon_{n,t} = O(t^{-1}\, s_n^{-1} \log (tg(rts_n))) \to 0$ *as* $n\to\infty$.

Proof: The proof follows by the Bernstien inequality

$$P(s_n^{-1}S_n' > t) \leqslant \prod_{i=1}^{n} \beta_i \exp(-hs_n\, t) \quad \ldots \quad (2.21)$$

where $\beta_i = E(\exp(hY_i))$ and $h = 2t^{-1}\, s_n^{-1} \log (tg(rs_n\, t))$.

Remark 2.11 : In the case $x^{-1} \log g(x) \to s(>0)$ as $x\to\infty$ e.g. when $g(x) = \exp(s|x|)$ then for $t=o(n^{1/6})$ we may use Theorem 2.1 or Theorem 2.2 and for

$$t^2 \geqslant 2[\log|t|+\log g(rs_n t\, K_n)]\ (\text{i.e. } |t| \geqslant s_n K_n = o(n^{1/6}))$$

we have

$$|F_n(t)-\Phi(t)| = O(|t|g(rs_n t\, K_n))^{-1+O(K_n)} + \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|) \quad \ldots \quad (2.22)$$

where K_n is any sequence $\to o$ as $n \to \infty$.

Proof of the above remark follows the same lines as that of Theorem 2.5 with

$$h = 2t^{-1}\, s_n^{-1} \log (tg(rs_n\, tK_n)).$$

As a consequence of Theorems 2.1, 2.5/Remark 2.11 we may obtain following nonuniform bound over the entire range of t, $-\infty < t < \infty$.

Theorem 2.6 : *Let* (2.3), (1.2) *and* (1.3) *hold. Also let for some* $\lambda_1, \lambda_2, \lambda_3$, *positive constants*, $|g(rs_n\, t)]^{-1+\epsilon} \leqslant \lambda_1 n^{-1/2}\, [g(\lambda_2\, t)]^{-1}$ *for all sufficiently large* n *when* x^{-1} *log* $g(x) \to 0$ *as* $x \to \infty$ *with* t *satisfying*

$$t^2 \geqslant 2(\log\, |t| + \log g(rs_n\, t))[g(rs_n\, t\, K_n)]^{-1+\epsilon}$$
$$\leqslant \lambda_3\, n^{-1/2}\, [g(\lambda_2\, t)]^{-1} \quad \ldots \quad (2.23)$$

for all sufficiently large n *when* x^{-1} *log* $g(x) \to s(>0)$ *as* $x \to \infty$ *with* t *satisfying*

$$t^2 \geqslant 2\, (\log\, |t| + \log g(rs_n\, t\, K_n))$$

where K_n *is some sequence converging to zero. Then* ... (2.24)

$$|F_n(t)-\Phi(t)| \leqslant bn^{-1/2}\, [g(\lambda_2\, t)]^{-1} + \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|). \quad \ldots \quad (2.25)$$

Further if

$$[t^3 g(rs_n\, t)]^{-1} \leqslant bn^{-1/2}\, [g(\lambda_2\, t)]^{-1} \text{ for all } t > t_0(\geqslant 0) \quad \ldots \quad (2.26)$$

then

$$|F_n(t)-\Phi(t)| \leqslant bn^{-1/2}\, [g(\lambda_2\, t)]^{-1}. \quad \ldots \quad (2.27)$$

Proof: The 1st and and terms of the r.h.s. (2.2), are of the order $\exp\, (-pt^2/2)$ with $0 < p < 1$ ($p < 1$ letting $t = o(n^{1/2})$ therein) and

$$\exp \left(-\frac{p}{2}\, t^2\right) \leqslant n^{-1/2} \exp\, (-at^2),\ 0 < a < p/2 \text{ if } t^2 \geqslant (p-2a)^{-1} \log n.$$

Since $\exp\, (-at^2) \leqslant b\, [g(\lambda_2\, t)]^{-1}$ from (1.3), (2.25) in the case $t^2 \geqslant (p-2a)^{-1}$ $\log n$ follows from Theorems 2.1, 2.5/Remark 2.11 with $K_n = o(\log n)^{-1}$ say, along with assumption (2.23)/(2.24).

For $t^2 \leqslant (p-2a)^{-1} \log n$, note that $|\exp\, (O(|t|^3\, n^{-1/2}))-1| = O(n^{-1/2}|t|^3)$ Hence

$$|F_n(t)-\Phi(t)| \leqslant bn^{-1/2}\, e^{-r_1 t^2} + \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|)$$

for some $r_1 > 0$, for $t_0^2 \leqslant t^2 \leqslant (p-2a)^{-1} \log n$ with some $t_0^2 > 0$. For $t^2 < t_0^2$ the assertion follows from the uniform bound $O(n^{-1/2})$ of $|F_n(t)-\Phi(t)|$.

Finally (2.27) follows from (2.25) and (2.26).

Remark 2.12. Assumptions (2.23), (2.24) and (2.26) intutively follows from the fact that g has growth more than any power bound. All these conditions are satisfied for $g(x) = |x| \exp(\log^m(1+|x|))$, $m > 1$, $g(x) = |x| \exp(|x|^\nu)$, $0 < \nu \leqslant 1$ etc, where $\lambda_2 > 0$ can be made arbitrarily large and $\epsilon > 0$ arbitrarily small.

From (2.27) it is easy to obtain the following non-uniform L_p version of the Berry-Esseen theorem.

Theorem 2.7 : *Under the assumptions of Theorems 2.6*

$$\|g(\lambda_2 t)(1+|t|)^{-q/p}(F_n(t)-\Phi(t)\|_p = O(n^{-1/2}) \qquad \dots \quad (2.28)$$

for any $p > 1$ and $q > 1$.

Theorem 2.6 may further be utilised to find the rate of convergence of expectations of some functions based on $Y_n = |s_n^{-1} S_n|$ to that of $T = |N(0, 1)|$. Related results are due to Von Bahr (1965), Michel (1976) and Ghosh and Dasgupta (1978).

Theorem 2.8 : *Under the assumptions of Theorem 2.6 and* $\frac{d}{dx}[x^2 g(x)] \leqslant \lambda_1 g(\lambda_2 x)(1+x)^{-q}+\lambda_3$ $\forall\, x \geqslant 0$ *and some* $\lambda_1, \lambda_3 > 0$, $q > 1$, λ_2 *same as that of Theorem 2.6, one has*

$$|E(Y_n^2 g(Y_n))-E(T^2 g(T))| = O(n^{-1/2}) \qquad \dots \quad (2.29)$$

Proof : Let $h(x) = x^2 g(x)$, $x \geqslant 0$ with the representation

$$|Eh(Y_n)-Eh(T)| \leqslant \int_0^\infty h'(t)|P(|s_n^{-1}S_n| \leqslant t)-P(|N(0,1)| \leqslant t)|dt$$

the theorem follows from (2.27).

We now proceed to show the necessity of the assumptions made to prove the earlier results. Let X_{ni}^s be the symmetrised random variables obtained from X_{ni} i.e. $X_{ni}^s = X_{ni}-X'_{ni}$ where X_{ni} and X'_{ni} are i.i.d. Let $S_n^s = \sum_{i=1}^n X_{ni}^s$ be the sum of symmetrized random variables.

Let $$EX_{ni} = 0 \;\forall\; n \geqslant 1,\; 1 \leqslant i \leqslant n. \qquad \dots \quad (2.30)$$

By weak symmetrization inequalities (see Loève, 244-245) and (5.9), (5.11) of Feller (p.147) we obtain

$$\sum_{i=1}^{n} P(|X_{ni}| > (1+\epsilon)y) \leqslant \sum_{i=1}^{n} P(|X_{ni}^{s}| > y) \leqslant P(s_n^{-1}|S_n| > y/2s_n) \qquad (2.31)$$

Further for $t_n = y/2\ s_n$ satisfying (i) and (ii) of Theorem 2.3 we have $P(s_n^{-1}|S_n| > y/2sn) \ll t_n^{-1}\exp(-t_n^2/2) \ll t_n^{-1}(g(r^*s_nt_n))^{-1}$ whenever $0 < r^* < r$, g is such that

$$g(x)/g(kx) \to \infty \text{ as } x \to \infty,\ 0 < k < 1 \qquad \dots \quad (2.32)$$

and (ii) is more stringent than (i). ... (2.33)

Then,

$$s_n^{-2} \sum_{i=1}^{n} P(|X_{ni}| > y) \ll y^{-2}(g(r^*y/2(1+\epsilon)))^{-1} \ll y^{-2}(g(r^*y/2))^{-1} \quad \dots \quad (2.34)$$

Hence by noting that $0 < r^* < r < 1/2$, we have for $0 < k < 1$ and for all sufficiently large n, say $n \geqslant n_0$, $s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2 g(r^*k\ X_{ni}/2)$

$$\leqslant 1 + \sum_{m=1}^{\infty} (m+1)^2\, g(r^*k(m+1)/2) m^{-2} (g(r^*m/2))^{-1} \qquad (2.35)$$

$$\ll 1.$$

under the condition

$$g(k(x+1)) \ll g(x)\, x^{-1-\delta} \text{ for large } x,\ \delta > 0 \text{ and } 0 < k < 1 \quad \dots \quad (2.36)$$

which is stronger than (2.32).

Since the above is true for any r^*, $0 < r^* < r$ and $0 < k < 1$ is arbitrary we have

$$\sup_{n \geqslant n_0} s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2 g(r^* X_{ni}/2) < \infty,\ 0 < r^* < r < 1/2 \qquad \dots \quad (2.37)$$

for some $n_0 \geqslant 1$.

Summarising the above we have the following theorem :

Theorem 2.9 : *under* (2.30) *and* (2.36), *condition* (2.37) *is necessary to obtain the conclusion of Theorem* 2.3 *for normal approximation of both lower and upper tail probabilities provided (ii) is more stringent than (i) therein. This assertion remains valid when (i) is replaced by* $t_n = o(n^{1/4})$ *under* (2.5) *as in Theorem* 2.3.

Note that by the above theorem to obtain $1-F_n(t_n) \sim \Phi(-t_n) \sim F_n(-t_n)$ $t_n \leqslant an^{1/8}$ $(= o(n^{1/4}))$ it is necessary to have

$\sup_{n \geqslant n_0} s_n^{-2} \sum_{t=1}^{n} EX_{nt}^2 e^{s|X_{nt}|^{2/3}} < \infty$ for some $s = s(a) > 0$ which implies for any finite $p \geqslant 2$

$$\sup_{n \geqslant n} s^{-2} \sum_{t=1}^{n} E|X_{nt}|^p = f(p) < \infty. \qquad \dots \quad (2.38)$$

This fact is to be used later.

Now consider the case $g(x) = \exp(s|x|^{1/2})$ or $g(x) = \exp(s|x|^{2/3})$ for some $s > 0$. In this case (ii) in Theorem 2.3 assets that $t \leqslant (2s\sigma^{-2}\lambda^{s})^{1/(2-a)}$ $n^{a/2\,(2-a)}$, $\alpha = 1/2,\ 2/3$ (see the calculations in between (2.9)—(2.10)) i.e. $t \leqslant s^*\ n^{1/6}$ or $t \leqslant s^*\ n^{1/4}$ for some $s^* > 0$, whereas (i) asserts $t = o(n^{1/6})$ or $t = o(n^{1/4})$. That is unlike previous case (i) is more stringent than (ii). In this case under moderate assumptions, we propose to show that the conditions of Theorem 2.4 are essential too.

Consider $g(x) = \exp(s|x|^{1/2})$, $s > 0$, the other case follows similarly. If

$$\sup_{n \geqslant n_0} s_n^{-2} \sum_{t=1}^{n} EX_{nt}^2 e^{s|X_{nt}|^{1/2}} = \infty, \ \forall\ s \text{ fixed} > 0 \qquad \dots \quad (2.39)$$

then there exist a sequence $c_n \to 0$, $c_n \neq 0\ \forall\ n$ such that

$$\sup_{n \geqslant n_0} s_n^{-2} \sum_{t=1}^{n} EX_{nt}^2 e^{c_n|X_{nt}|^{1/2}} = \infty \qquad \dots \quad (2.40)$$

proof of the above follows from the following general result.

Lemma 2.1. *If $h(n, s)$ be a function such that $\sup_{n \geqslant 1} h(n, s) = \infty\ \forall\ s > 0$ fixed, then there exists a sequence $c_n \to 0$, $c_n \neq 0\ \forall\ n$ such that $\sup_{n \geqslant 1} h(n, c_n) = \infty$.*

Proof: We construct such a sequence c_n. If supremum of $h(n, s)$ is attained for any fixed s at a finite $n' = n'(s)$ then the proof is trivial, consider a sequence converging to zero with first term as s then

$$\sup_n h(n, c_n) \geqslant h(n', s) = \infty.$$

The remaining case is lim sup $h(n, s) = \infty$, $\forall\ s > 0$ fixed. In this case for every fixed $K > 0$ get n_K such that $h\left(n_K, \frac{1}{K}\right) > K$ w.o.l.g let $n_K \uparrow \infty$.

Define $c_{n_K} = \frac{1}{K}$ and for $n_K \leqslant n < n_{K+1}$, $c_n = \frac{1}{K}$. Clearly $c_n \to 0$ and $\sup_K h(n, c_n)$

$\geqslant \sup_K h(n_K, c_{n_K}) = \sup_K h\left(n_K, \frac{1}{K}\right) = \geqslant \sup_K K = \infty.$

We next show that calculation (2.35) can be extended for the function of the type

$$g(x) = g_n(x) = e^{c_n|x|^{1/2}},\ c_n \to 0.$$

Consider

$$\begin{aligned}
& s_n^{-2} \sum_{i=1}^{n} Eg(r^* k\, X_{ni}/2) \\
& \leqslant 1 + \sum_{m=1}^{\infty} g_n(r^* k(m+1)/2)\,(g_n\,(r^* m/2))^{-1}\, m^{-2} \\
& = 1 + \sum_{m=1}^{\infty} e^{c_n(|r^* k(m+1)/2|^{1/2} - |r^* m/2|^{1/2})} . m^{-2} \\
& \leqslant 1 + e^{c_n(r^* k/2)^{1/2}} \sum_{m=1}^{\infty} e^{c_n(r^* m/2)^{1/2}(k^{1/2}-1)} . m^{-2} \\
& \leqslant \infty \text{ if } 0 < k < 1. \qquad \ldots \quad (2.41)
\end{aligned}$$

Therefore if we have $1 - F_n(t_n) \sim \Phi(-t_n), \sim F_n(-t_n),\ t_n \to \infty,\ t_n = o(n^{1/6})$ even when (2.39) holds, then we must have, taking $g(x) = e^{2c_n|x|^{1/2}}$ in (2.41) and considering the case

$$t_n \leqslant c^{2/3}\, n^{1/6} = o(n^{1/6}),\ c_n \to 0;\ \sup_{n \geqslant n_0}\ s_n^{-2} \sum_{i=1}^{n} Ee^{2c_n|X_{ni}|^{1/2}} < \infty \quad \ldots \quad (2.42)$$

Now,

$$\begin{aligned}
\sum_{i=1}^{n} EX_{ni}^2 e^{c_n|X_{ni}|^{1/2}} & \leqslant \sum_{i=1}^{n} (E^{1/2}\, X_{ni}^4)\,(E^{1/2}\, e^{2c_n|X_{ni}|^{1/2}}) \\
& \leqslant \left(\sum_{i=1}^{n} EX_{ni}^4 \right)^{1/2} \left(\sum_{i=1}^{n} E\, e^{2c_n|X_{ni}|^{1/2}} \right)^{1/2}
\end{aligned}$$

by Cauchy—Schwartz inequality, i.e.

$$s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2 e^{c_n|X_{ni}|^{1/2}} \leqslant \left(s_n^{-2} \sum_{i=1}^{n} EX_{ni}^4 \right)^{1/2} \left(s_n^{-2} \sum_{i=1}^{n} E\, e^{2c_n|X_{ni}|^{1/2}} \right)^{1/2}$$

So

$$\sup_{n \geqslant n_0} s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2\, e^{c_n|X_{ni}|^{1/2}} < \infty \text{ from (2.38) and (2.42)}$$

contradicting (2.40), hence we have the following theorem.

Theorem 2.10 : *Under* (2.30) *to obtain* $1-F_n(t_n) \sim \Phi(-t_n)$, $F(-t_n) \sim \Phi(-t_n)$, $t_n = o(n^{1/6})$, $t_n \to \infty$, *the following assumption* $\sup_{n \geq n_0} s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2 e^{s|X_{ni}|^{1/3}} < \infty$ *for some* $s > 0$ *and some* $n_0 \geqslant 1$ *is necessary. Similary under* (2.5) *and* (2.30) *to obtain* $1-F_n(t_n) \sim \Phi(-t_n)$, $F(t_n) \sim \Phi(-t_n)$, $t_n = o(n^{1/4})$, $t_n \to \infty$, *the following condition* $\sup_{n \geq n_0} s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2 e^{s|X_{ni}|^{2/3}} < \infty$ *for some* $s > 0$ *and some* $n_0 \geqslant 1$ *is necessary, where* $s_n^2 = \sum_{i=1}^{n} EX_{ni}^2$.

Similarly Under (2.30), (1.3) and (2.36), the following assumption

$$\sup_{n \geq 1} s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2\, g(r\, X_{ni}/2) < \infty,\ 0 < r < 1/2 \qquad \ldots \quad (2.43)$$

is necessary to obtain (2.2)/(2.6) for $1 < t^2 < M(\log|t|+\log\ g(rs_n\ t))$, $0 < r < 1/2$, $M < r^{-1}$.

One may note that under $s_n^2 \ll n$ necessary and sufficient conditions are the same, vide Theorem 2.4 except for first few arrays. This is satisfied e.g. in the i.i.d. case.

The next two theorems generalize the results of Linnik (1961 and 1962).

Theorem 2.13 : *Let* $[X_{ni} : 1 \leqslant i \leqslant n,\ n \geqslant 1]$ *be a sequence of independent random variables in a triangular array. Let* (1.2), (1.3) *and* (2.3) *hold. Let* t^* *be the largest value of* $|t|$ *satisfying* $1 \leqslant t^2 \leqslant 2(\log|t|+\log\ g(rs_n\ t))$. *Define* $\psi_k = \psi_k(ni)$, *the kth order semi-invariant of* $Y_i = Y_{ni} = X_{ni}\ I(|X_{ni}| \leqslant rs_n t^*)$, $0 < r < 1/2$ *and* $\psi_k^* = \psi_k^*(n) = s_n^{-2} \sum_{i=1}^{n} \psi_k(ni)$.

Then for the Zone (2.1) i.e. $1 \leqslant t^2 \leqslant 2(\log|t|+\log g(rs_n\ t))$ the following holds.

$$|F_n(t)-\Phi(t)| \leqslant I_1+I_2+I_3+\sum_{i=1}^{n} P(|X_{ni}| > rs_n\ t^*)$$

$$I_1 \leqslant bn^{-1/2} \exp\left(-t^2/2+\sum_{k=3}^{\infty} \frac{1}{k!} \frac{t^k}{s_n^{k-2}}\ \psi_k^*+O(t^{*3}g^{-1}(rs_n\ t^*))\right) \qquad \ldots \quad (2.44)$$

$$I_2 \leqslant bt^{-1} e^{-t^2/2} \left| \exp\left\{ \sum_{k=3}^{\infty} \frac{(k+1)}{k!} \frac{t^k}{s_n^{k-2}}\ \psi_k^* + O(t^{*3}g^{-1}(rs_n\ t^*)) \right\} - 1 \right|$$

$$\exp\left[\frac{n^{-1}}{2c} \left\{ \sum_{k=3}^{\infty} \frac{1}{(k-1)!} \frac{t^{k-1}}{s_n^{k-3}} \psi_k^* \right\}^2 + \sum_{k=3}^{\infty} \frac{1}{(k-1)!} \frac{t^k}{s_n^{k-2}}\ \psi_k^* + O(t^{*3}g^{-1}(rs_n t^*)) \right] \qquad \ldots \quad (2.45)$$

where $\bar{\sigma}_n^2(t) = n^{-1} s_n^2 + \ldots\ldots \geqslant c$

and

$$I_3 \leqslant bt^{-1} e^{-t^2/2} \exp\left\{\frac{1}{2}t^2 \left(\sum_{k=3}^{\infty} \frac{1}{(k-2)!}\left(\frac{t}{s_n}\right)^{k-2} \psi_k^*\right) + O(t^{*2} g^{-1}(rs_n t^*))\right\}$$

$$+bt^{-1} e^{-t^2/2} \left| \exp\left\{\frac{1}{2}t^2 \left(\sum_{k=3}^{\infty} \frac{1}{(k-2)!}\left(\frac{t}{s_n}\right)^{k-2} \psi_k^*\right) + O(t^{*2} g^{-1}(rs_n t^*))\right\} - 1 \right| \quad \ldots \quad (2.46)$$

Note that to obtain the normal approximation zone upto $o(n^{1/6})$, it is necessary that $\sup s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2 \exp(s|X_{ni}|^{1/2}) < \infty$ for some $s > 0$, vide (2.10). Hence the l-th order semi invariant $'\psi_l(ni)$ of X_{ni} are finite.

Theorem 2.15 : *Under the assumption* (1.2), (1.3) *and* (2.3) *alongwith*

$$'\psi_l^*(n) = s_n^{-2} \sum_{i=1}^{n} {}'\psi_l(n_i) = o(s_n^{l-2}|t_n|^{-l}) \text{ for } l = 3, 4, \ldots, k-1 \quad \ldots \quad (2.47)$$

where $'\psi_l(ni)$ *is the l-th order semi-invariant of* X_{ni}, *and for a sequence* $\{t_n\}$ *satisfying*

(*i*) $t_n = o(n^{(k-2)/2k})$

(*ii*) $t_n^2 - 2(\log t_n + \log g(rs_n t_n)) \to -\infty$; $0 < r < 1/2$

we have the following

$$1 - F_n(t_n) \sim \Phi(-t_n),\ F(-t_n) \sim \Phi(-t_n) \text{ as } t_n \to \infty.$$

If (*i*) is more stringent than (ii), e.g. for $g(x) = \exp(s|x|^{(k-2)/(k-1)})$ for some $s > 0$, then (2.47) is equipvalent to $'\psi_l^*(n) = o(n^{-1-l/k})$, $l = 3, 4, \ldots, k-1$.

The normal approximation zone computed so far by different authors, see e.g. Nagaev (1979) are smaller than that presented here, even in the special case of i.i.d.r.v's. With the assumption of the form $EX^2 g(x) < \infty$ vide (1.1) or (2.3), $\phi(n, F)$ of Nagaev (1979) turns out to be the solution of $x^2 = 2\log x + \log g(s_n x)$. The corresponding normal approximation zone is $0 < x < \phi(n, F)$, whereas we have normal approximation zone upto $t^2 \leqslant 2(\log|t| + \log(r s_n t))$ $o < r < \frac{1}{2}$ which is certainly a larger zone using the fact that $x^{-1}\log g(x)$ is non increasing in x (with small oscillation) and therefore $(r s_n t)^{-1}\log g(r\, snt) \geqslant (s_n t)^{-1}\log g(s_n t)$, $o < r < \frac{1}{2}$.

Note that as $k \to \infty$ the numbers $(k-2)/2k$ $\frac{1}{4}, \frac{3}{10}, \frac{1}{3} \ldots \to \frac{1}{2}$.

It is also possible to obtain the necessary conditions for Theorem 2.15 along the lines of Theorems 2.9 and 2.10. The proof virtually remains unchanged with slight modification. In particular we observe from Theorem 2.14 that in order to obtain normal approximation zone upto $o(n^{(k-2)/2k})$ it is necessary to have $\psi_l^*(n) = o(n^{-(1-l/k)})$ for $l = 3, 4, \ldots, k-1$, i.e.

$$\psi_l^*(n) = o(n^{-(1-l/k)}),\ l = 3, 4, \ldots, k-1.$$

The necessary assumptions for Theorem 2.15 to hold are stated below. Since the basic technique of the proof remains the same as that of Theorems 2.9 and 2.10, it is omitted.

Theorem 2.16 : *Under* (2.30) *and* (2.36), *condition* (2.37) *is necessary to obtain the conclusion of Theorem* 2.15 *for normal approximation of upper and lower tail probabilities, provided* (*ii*) *is more stringent than* (*i*) *therein for some* k, $k = 4, 5, 6 \ldots$

Theorem 2.17 : *Under* (2.30),

$$s_n^{-2} \sum_{i=1}^{n} \psi_l(ni) = o(n^{-(1-l/k)}) \text{ for } l = 3, 4, \ldots, k-1$$

and $\sup_{n \geq n_0} s_n^{-2} \sum_{i=1}^{n} EX_{ni}^2 \exp(s|X_{ni}|^{(k-2)/(k-1)}) < \infty$ *for some* $s > 0$ *and some* $n_0 \geq 1$ *are necessary assumptions to obtain* $1 - F_n(t_n) \sim \Phi(-t_n)$ *and* $F(-t_n) \sim \Phi(-t_n)$ *for* $t_n = o(n^{(k-2)/2k})$, $t_n \to \infty$.

Unlike Linnik's (1961) division into classes, Nagaev (1979) considered general g, but the condition assumed on g is more restrictive than that assumed in the present paper. Also the deviation zone for normal approximation by Nagaev (1979) is smaller.

Our results on necessary and sufficient conditions on deviations are sharper than Linnik. We show in i.i.d case the necessary and sufficient assumptions for deviations $o(n^{-(k-2)/2k})$, $k = 3, 4, \ldots$ are same, whereas according to Linnik, these are different as he obtains a Zone while computing necessary condition and a different zone under sufficient condition, viz. $[0, n^{\alpha}\rho(n)]$ for necessary assumptions and $[0, n^{\alpha}/\rho(n)]$ under sufficient assumption, $\rho(n) \to \infty$.

In general, the gap between these two Zones $[0, \Lambda(n)\rho(n)]$ and $[0, \Lambda(n)/\rho(n)]$ obtained by Linnik are wide as $\rho(n) \to \infty$ (Theorem 1 and 2, paper II), whereas our necessary and sufficient conditions for the same zone, differs by a factor 1/4 inside the function g(see (2.37)).

Under the necessary assumption (2.37) we have the zone (see Theorem 2.3, 2.15).

$$t_n^2 - 2(\log t_n - \log g(rs_n t_n/4)) \to -\infty$$

whereas under sufficient assumption we had

$$t_n^2 - 2(\log t_n - \log g(rs_n t_n)) \to -\infty.$$

In Linnik's notation $x^2 g(x) = e^{h(x)}$. So for i.i.d set up, (ii) of Theorem 2.3 and 2.15 of this paper reduce to

$$\begin{aligned} t^2 &\leq 2(\log t + \log\ grs_n t\ )) \\ &= 2h(rn^{1/2}t) - O(\log n) \text{ as } t \ll n^{1/2} \\ &= 2h(rn^{1/2}\ t)\ (1+o(1)) \text{ as } e^{h(x)} = x^2 g(x) \gg x^k \text{ for } k \text{ arbitrary large.} \end{aligned}$$

Therefore, consider a particular case where (ii) is more stringent than (i) in Theorem 2.3,

$$x^2\ g(x) = e^{h(x)} = e^{|x|^\alpha}\ ;\ 0 < \alpha < 1/2$$

then, from above, the normal zone in positive axis is upto $t \leq (2r^\alpha)^{1/(2-\alpha)}$ $n^{\alpha/2(2-\alpha)}$, $r = \frac{1}{2}$, whereas Linnik gets

$$[0, \Lambda(n)/\rho(n)] = [0, o(n^{\alpha/2(2-\alpha)})],$$

$$\Lambda(n) = n^{\alpha/2(2-\alpha)},\ \rho(n) \to \infty,$$

a smaller zone than the one we obtained.

Our results are for independent random variables in a triangular array, but Linnik's results as are for independent and identically distributed random variables and he considers the normal density function also whereas we deal only normal distribution function.

3. Rates of convergence for general non-linear statistics

In this section we consider non-linear statistics of the form

$$T_n = s_n^{-1}\ S_n + R_n \text{ where } S_n = \sum_{i=1}^{n} X_{ni} \qquad \ldots \quad (3.1)$$

$X_{n1}, X_{n2}, \ldots, X_{nn}$ being independent random variables satisfying (1.1), (1.2) and (1.3).

Under suitable assumptions on the moments of R we shall show results of earlier section may be extended to include T_n.

Suppose that R_n satisfies

$$E(R_n^{2m}) \leqslant c(2m)n^{-m}(\log n)^{hm} \text{ for some } h \geqslant 0,\ m = 1, 2, 3 \ldots \quad \ldots \quad (3.2)$$

where $c(.)$ is constant depending on m, under appropriate restriction on c and h we shall obtain non uniform rates for T_n. w.o.l.g. let $t > 0$. Note that due to representation (3.1)

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant |P(s_n^{-1} S_n \leqslant t\pm a_n(t))-\Phi(t\pm a_n(t))|$$
$$+|\Phi(t\pm a_n(t))-\Phi(t)|+P(|R_n| > a_n(t)) \quad \ldots \quad (3.3)$$

where $a_n(t) > 0$ will be chosen accordingly. Now

$$P(|R_n| > a_n(t)) \leqslant \exp[-A_n a_n(t)]\, E[\exp(A_n|R_n|)] \quad \ldots \quad (3.4)$$

It will be shown that if

$$c(2m) \leqslant (2m)!\, L^{2m} \text{ for same } L > 1 \quad \ldots \quad (3.5)$$

then

$$\sup_n E(\exp(A_n|R_n|)) < \infty \text{ for } A_n = \epsilon\, n^{1/2} (\log n)^{-h/2}, \quad \ldots \quad (3.6)$$

for some $\epsilon > 0$.

First note that

$$\exp(A_n|R_n|)+\exp(-A_n|R_n|) = 2\left[1+\sum_{m=1}^{\infty} \frac{(A_n|R_n|)^{2m}}{(2m)!}\right] \quad \ldots \quad (3.7)$$

Taking expectation both sides and noting that $E \exp(-A_n|R_n|) > 0$. We, have, in view of (3.2) and (3.5)

$$E\exp(A_n|R_n|) \leqslant 2\left[1+\sum_{m=1}^{\infty} (\epsilon L)^{2m}\right] \quad \ldots \quad (3.8)$$

Hence for $\epsilon < L^{-1}$ r.h.s. of (3.8) is a convergent geometric series free from n, therefore we have (3.6) under (3.5).

As a consequence of this result let us obtain normal approximation zone. Let $a_n(t) = n^{-\gamma}$ where $\gamma > 0$ will be chosen later. Then for the first term of the r.h.s. of (3.3) we have from Theorem 2.1 with t satisfying (2.1).

$$\text{1st term of r.h.s. of (3.3)} \leqslant b\exp(-(t\pm n^{-\gamma})^2/2)|t\pm n^{-\gamma}|^{-1}$$

$$\times \exp(O(|t\pm n^{-\gamma}|^3 n^{-1/2}))-1|+\sum_{i=1}^{n} P(|X_i| > rs_n|t\pm n^{-\gamma}|)$$

$$\leqslant b\exp(-t^2/2)|\exp(O(|t|^3 n^{-1/2}))-1|\,|t|^{-1}+\sum_{i=1}^{n} P(|X_i| > r\lambda s_n|t|) \ldots \quad (3.9)$$

for $t = O(n^{\gamma})$, with some $0 < \lambda < 1$

$$\text{2nd term} \leqslant bn^{-\gamma}\exp(-t^2/2) \quad \ldots \quad (3.10)$$

$$\text{3rd term} \leqslant K\exp(-\epsilon n^{1/2-\gamma}(\log n)^{-h/2}) \quad \ldots \quad (3.11)$$

(3.9)—(3.11) with $\gamma = \frac{1}{6}$ implies the following theorem along the lines of Theorem 2.3 and Remark 2.9.

Theorem 3.1 : *Let* (1.1), (1.2), (1.3), (3.1), (3.2) *and* (3.5) *hold. Then for a sequence* $\{t_n\}$ *satisfying*

(i) $t_n = o(n^{1/6})$ *if* $h = 0$

$\leqslant \epsilon' n^{1/6} (\log n)^{-h/4}$ *if* $h > 0$ *for some* $0 < \epsilon' < 1$

and (ii) $t_n^2 - 2 (\log t_n + \log g(r \lambda s_n t_n)) \to -\infty$; $0 < r\lambda < 1/2$,

the following holds

$$1 - P(T_n \leqslant t_n) \sim \Phi(-t_n) \sim P(T_n \leqslant -t_n) \text{ as } t_n \to \infty \quad \ldots \quad (3.12)$$

Further if the sequence $\{X_{ni}^2 g(X_{ni})\}$ *is uniformly integrable then* (3.12) *holds even if l.h.s. of* (i) *is bounded above.*

Now let us have a different choice of $a_n(t)$ viz. $a_n(t) = \alpha t$, $0 < \alpha < 1$. In that case with $t(> 0)$ satisfying (2.1) we have

1st term of r.h.s. of (3.3)

$$\leqslant b \exp\left[-\frac{t^2}{2}(1-\alpha)^2 + Kn^{-1/2}t^3\right]$$

$$+ \sum_{i=1}^{n} P(|X_{ni}| > r(1-\alpha)\, ts_n) \text{ for some } K > 0 \quad \ldots \quad (3.13)$$

$$\text{2nd term} \leqslant n \exp[-(1-\alpha)^2 t^2/2]\, \alpha t \quad \ldots \quad (3.14)$$

$$\text{3rd term} \leqslant b \exp\{-\alpha\, \epsilon\, tn^{1/2} (\log n)^{-h/2}\} \quad \ldots \quad (3.15)$$

Hence we have the following theorem.

Theorem 3.2 : *Suppose* (1.1) *with* $g(x) = \exp(s|x|)$ *for some* $s > 0$, (1.2), (3.1), (3.2) *and* (3.5) *hold. Then*

$$P(T_n > t_n) \leqslant b \exp\left[-\frac{t_n^2}{2}(1+o(1))\right] \quad \ldots \quad (3.16)$$

for $t_n = o(n^{1/2} (\log n)^{-h/2})$, $t_n \to \infty$.

Proof of the above theorem is similar to that of Corollary 2.2 letting $\alpha \to 0$.

For $h = 0$, noting that for $t = \epsilon' \sqrt{n}$, $\alpha t n^{1/2} = \alpha t^2/\epsilon'$ letting $\epsilon' \to 0$ along the lines of Corollary 2.2 with the observation that l.h.s. of (3.17) is independent of α (and hence finally letting $\alpha \to 0$) we have the following theorem.

Theorem 3.3 : *Let the conditions of Theorem* 3.2 *hold with* $h=0$, *then for* $t_n = \epsilon' \sqrt{n}$

$$\limsup_{\epsilon' \to 0} \limsup_{n \to \infty} (t_n^2/2)^{-1} \log P(T_n > t_n) \leqslant -1. \qquad \dots \quad (3.17)$$

Results similar to those in Section 2 hold when S_n is replaced by T_n.

4. Rates of convergence to normality for L-statistics

Let $X_{1n} \leqslant X_{2n} \leqslant \dots \leqslant X_{nn}$ denote the order statistics corresponding to n i.i.d.r.v's $X_1 \dots X_n$ having a common distribution function F. Consider linear combinations of the functions of order statistics of the form

$$T_n = \sum_{i=1}^{n} C_{in}\, h(X_{in}) \qquad \dots \quad (4.1)$$

where C_{in}'s are constants and h is some measurable function.

This type of non-linear statistics were considered in section 4 of Ghosh and Dasgupta (1978). It was shown under assumption (i)–(iv) therein that T_n can be split into a main part plus a remainder R_n which is neglibible compared to the main part. Here we propose to find a sharper estimate of R_n with the same notation used therein. For the main part note that m.g.f. of exponential distribution exists ; therefore (1.1) with $g(x) = exp\ (s|x|)$, $s > 0$ and (1.2) are satisfied for the r.v.'s $a_{in}(Z_i - 1)$.

For the remainder R_n calculations therein needs slight modification and correction to obtain a precise bound for ER^{2m}. Immediately before (4.12) in Ghosh and Dasgupta (1978), note that K pairs can be selected out of $2m$ pairs in $2m_{p_K}$ ways $K = 1, \dots, m$, m being the maximum number of distinct pairs. So in total $\sum_{k=1}^{m} 2m_{p_k}$ ways. Now using Stirling's approximation for factorials, we get

$$\sum_{k=1}^{m} 2m_{p_k} \leqslant L^m e^{m \log m} \text{ for some } L > 1.$$

Hence a correct version of (4.13) of Ghosh and Dasgupta (1978) is

$$\text{l.h.s of } (4.11) \leqslant (4m)! \left(\sum_{j=1}^{n} \frac{1}{(n-j+1)} \right)^m L^m e^{m \log m}$$

$$\leqslant L^m e^{5m \log m} \quad (\log n)^m \text{ for some } L > 1.$$

(In between (4.12) and (4.13) of Ghosh and Dasgupta (1978), $\sum_{i=a}^{b} \frac{1}{(n-i+1)^p}$ was wrongly printed).

Hence we obtain

$$ER_n^{2m} \leqslant e^{5m \log m} (\log n)^{2m} n^{-m} L^m \text{ for some } L > 1. \quad \ldots \quad (4.2)$$

This correction, of course, does not effect the results of earlier paper as we donot need the form of the part depending only on m there. In other words (4.2) of this paper satisfies (3.1) of Ghosh and Dasgupta (1978). We now proceed to find an estimate for $P(|R_n| > a_n(t))$. Note that by Markov inequality

$$P(|R_n| > a_n(t)) \leqslant a_n^{-2m} ER_n^{2m} \;;\; a_n = a_n(t)$$

$$\leqslant a_n^{-2m} e^{5m \log m} n^{-m} (\log n)^{2m} L^m = P^* \text{ say}$$

$$\log P^* = -2m \log a_n + 5m \log m - m \log n + 2m \log \log n + m \log L \quad \ldots \quad (4.3)$$

Differentiating w.r.t. m and equating it to zero

$$0 = \frac{d \log P^*}{dm} = -2 \log a_n + 5 + 5 \log m - \log n + 2 \log \log n + \log L \quad \ldots \quad (4.4)$$

with a value of $m = (a_n^2 e^5 n (\log n)^{-2} L)^{1/5}$, we conveniently ignore the fact that m may not be an integer here.

Hence from (4.3) and (4.4) $\log P^* = -5m$ i.e. $p^* = e^{-5m}$

So $P(|R_n| > a_n(t)) = O(\exp(-\epsilon n^{1/2} (\log n)^{-1} a_n(t))^{2/5}))$ for some $\epsilon > 0$.

$$\ldots \quad (4.5)$$

To find out normal approximation zone, letting $a_n(t) = n^{-\gamma} (\log n)^\lambda$; $\lambda, \gamma > 0$ to be chosen later, we have

$$P(|R_n| > a_n(t)) = O(\exp(-(cn^{(1/2)-\gamma}(\log n)^{-1+\lambda})^{2/5}))$$

$$= o(|t|^{-1} \exp(-t^2/2))$$

for $$t = o(n^{(1/2)-\gamma} (\log n)^{-1+\lambda})^{1/5} \quad \ldots \quad (4.6)$$

Also

$$|\Phi(t \pm a_n(t)) - \Phi(t) \leqslant bn^{-\gamma} (\log n)^\lambda \exp(-t^2/2) = o(|t|^{-1} \exp(-t^2/2)$$

$$\text{for } t = o(n^\gamma (\log n)^{-\lambda}) \quad \ldots \quad (4.7)$$

Now equating $n^\gamma(\log n)^{-\lambda} = (n^{(1/2)-\gamma} (\log n)^{-1+\lambda})^{1/5}$ which states $\gamma = 1/12$ and $\lambda = 1/6$, the following theorem follows along calculations (3.9)—(3.11) of Theorem 3.1.

Theorem 4.1 : *Under assumptions I—IV for T_n defined in* (4.1), $1-P(T_n \leqslant t_n) \sim \Phi(-t_n) \sim P(T_n \leqslant -t_n)$ *for*

$$t_n = o(n^{1/12} (\log n)^{-1/6}),\ t_n \to \infty.$$

With a different choice of $a_n(t)$ viz

$a_n(t) = \epsilon\, n^{-1/2} (\log n)^{7/2} |t|$, we have

$$P(|R_n| > a_n(t)) = O(\exp(-(\epsilon^{2/5}|t|^{2/5} \log n)))$$

$$\leqslant bn^{-1/2} \exp(-\lambda |t|^{2/5}) \text{ for } |t| > t_0.$$

Also an uniform bound $O(n^{-1/2} (\log n)^{7/2})$ is available for $\|P(T_n \leqslant t)-\Phi(t)\|$ letting $a_n(t) = n^{-1/2} (\log n)^{7/2}$ and using $\|F(X+Y)-\Phi\| \leqslant \|F(X)-\Phi\| +(2\pi)^{-1/2} a_n+P(|Y| > a_n)$.

Following the technique used to have Theorem 3.4 we have the following non uniform bound.

Theorem 4.2 : *Under the assumptions I–IV for T_n defined in* (4.1), *for any $\lambda > 0$ there exist a constant $b(> 0)$ depending on λ such that*

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant bn^{-1/2} (\log n)^{7/2} \exp(-\lambda |t|^{2/5}) \qquad \ldots \quad (4.8)$$

In view of the above theorem we may have the following two theorems proceeding as in Theorems 2.10, 2.11 and noting that λ in Theorem 4.2 is arbitrary.

Theorem 4.3 : *Under the assumptions of Theorem* 4.2, *for any $\lambda > 0$*

$$|ET_n^2 \exp(\lambda |T_n|^{2/5})-ET^2 \exp(\lambda |T|^{2/5})| = O(n^{-1/2} (\log n)^{7/2},$$

$$T = N(0, 1) \qquad \ldots \quad (4.9)$$

Theorem 4.4 : *Under the assumptions of Theorem* 4.2 *denoting $G_n(t) = P(T_n \leqslant t)$, we have, for any $\lambda > 0$ and $p \geqslant 1$*

$$\|\exp(\lambda |t|)^{2/5}) (G_n(t)-\Phi(t))\|_p = (O)(n^{-1/2} (\log n)^{7/2}). \qquad \ldots \quad (4.10)$$

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Paper received: *October*, 1982.
Revised: *December*, 1987.