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A NOTE ON COMPUTATION OF FACTOR FOR TOLERANCE LIMITS FOR A NORMAL DISTRIBUTION

By D. T. GHOSH
Indian Statistical Institute

SUMMARY. The computation of tolerance factor (λ) as proposed by Wald and Wolfowitz involves a tedious iterative procedure. The approximate formula developed by Bowker was found to be good only for large samples. An approximate formula is developed here which can be used for both small and large samples.

1. Introduction

In Industry it is often required to obtain a range for a quality characteristic, which is usually found to be distributed normally, such that a certain percentage of the product from a manufacturing process have the quality characteristic falling between the limits. The problem is then to construct two limits L_1 and L_2 called tolerance limits on the basis of the sample such that the probability that the limits L_1 and L_2 will include at least a given proportion γ of the population is equal to preassigned value β. For meaningful tolerance limits both β and γ should be very high.

It was shown by Wald and Wolfowitz (1946) that the tolerance limits for a normal variable with unknown mean and variance are given by

$$\bar{x} \pm \lambda s$$

where

$$\lambda = r\sqrt{\frac{n}{\chi^2_{n \cdot \beta}}} \qquad \dots \quad (1.1)$$

and N, $\bar{x}$ and s are respectively sample size, sample mean and sample standard deviation, $n = N-1$, $\chi^2_{n,\beta}$ is the value such that $P[\chi^2_n > \chi^2_{n \cdot \beta}] = \beta$ and r is the root of the equation

$$\frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sqrt{N}}-r}^{\frac{1}{\sqrt{N}}+r} e^{-t^2/2}\, dt = \gamma. \qquad \dots \quad (1.2)$$

The determination of λ, tolerance factor, is computationally tedious largely because finding r as the root of equation (1.2) involves an iterative procedure and is quite time consuming even for large N. Further $\chi^2_{n,\beta}$ is not readily available for all values of N particularly when N is large. The values of λ for selected values of N, β and γ were tabulated by Bowker and are given in the book by Statistical Research Group, Columbia University. However, the table may not be readily available for Quality Control practitioners, or even if available the interest may be on values of N, β and γ outside the range of tabulation.

It is therefore useful to have an approximate formula for λ which will be satisfactory for practical purposes.

2. Bowker's approximate formula for λ for large N

Bowker (1946) proposed the following formula for λ when N is large,

$$\lambda \simeq r_\infty \left(1 - \frac{x_{1-\beta}}{\sqrt{2N}} + \frac{5x_{1-\beta}^2+10}{12N}\right) \qquad \dots \quad (2.1)$$

where

$$\frac{1}{\sqrt{2\pi}} \int_{-r_\infty}^{r_\infty} e^{-t^2/2}\,dt = \gamma$$

and $x_{1-\beta}$ is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_{1-\beta}} e^{-t^2/2}\,dt = 1-\beta.$$

This has the advantage that in computing λ, a table for normal distribution alone is required. But for smaller values of N where usually lies our interest the error is considerable.

It will be noted from Bowker (1946) that his formula (2.1) is derived in two stages :

(i) r is expressed in term of $\frac{1}{\sqrt{N}}$ as $r \simeq r_\infty \left(1 + \frac{1}{2N}\right)$;

(ii) χ^2 is expanded in normal form using (Goldberg and Levine, 1946) which gives

$$\frac{\chi^2_{n,\beta}}{n} \simeq 1 + \frac{\sqrt{2}\,x_{1-\beta}}{\sqrt{n}} + \frac{2}{3}\,\frac{x_{1-\beta}^2-1}{n}$$

after neglecting terms containing higher power of $1/N$.

3. An approximate formula for λ for all N

We show here that

$$\lambda \simeq r_\infty \sqrt{\frac{N}{\chi^2_{n,\beta}}} \qquad \dots \quad (3.1)$$

we have

$$\sqrt{n} = (N-1)^{\frac{1}{2}} = N^{\frac{1}{2}}\left(1-\frac{1}{N}\right)^{\frac{1}{2}}$$

$$\simeq N^{\frac{1}{2}}\left(1-\frac{1}{2N}\right) \text{ after neglecting terms containing } O\left(\frac{1}{N^2}\right).$$

Using Bowker's result we have

$$r\sqrt{n} \simeq r_\infty\left(1+\frac{1}{2N}\right)N^{\frac{1}{2}}\left(1-\frac{1}{2N}\right)$$

$$\simeq r_\infty \sqrt{N}.$$

Hence follows (3.1).

Keeping the first four terms of the formula given by Goldberg and Levine (1946) for χ^2 in terms of a standard normal deviate we have

$$\chi^2_{n,\beta} \simeq n+\sqrt{2n}\, x_{1-\beta}+\frac{2}{3}\left(x^2_{1-\beta}-1\right)+\frac{x^3_{1-\beta}-7x_{1-\beta}}{9\sqrt{2n}}. \qquad \dots \quad (3.2)$$

[We note that in order to facilitate expansion of $\left(\frac{n}{\chi^2_{n,\beta}}\right)^{\frac{1}{2}}$ Bowker used the first 3 terms only.]

i.e.,
$$\chi^2_{n,\beta} \simeq n+G_1(x)\sqrt{n}+G_2(x)+\frac{G_3(x)}{\sqrt{n}} = D \text{ (say)}. \qquad \dots \quad (3.3)$$

The values of $G_1(x)$, $G_2(x)$ and $G_3(x)$ can be tabulated for different values of β once for all. Thus we have

$$\lambda \simeq r_\infty \sqrt{\frac{N}{D}} \qquad \dots \quad (3.4)$$

and let us call this formula F. F is expected to work satisfactorily for both small and large N.

4. Comparison of F and Bowker's formula

In order to study the performance of these two formulas for large as well as small N the approximate values of λ were computed by both formulas for different N for all combinations of (β, γ) with 0·75, 0·95 and 0·99 for values of β and 0·75, 0·95 and 0·999 for values of γ.

These are compared with the exact values of λ given in the table of the book by Statistical Research Group, Columbia University and the maximum and minimum difference over all combinations of (β, γ) in the above range for a given N are shown in Table 1.

TABLE 1. COMPARATIVE VALUES OF EXACT AND APPROXIMATE λ

N	F error		Bowker error	
	min	max	min	max
10	0.007	0.063	0.051	1.112
15	0.003	0.020	0.015	0.511
20	0.001	0.010	0.015	0.304
25	0.001	0.008	0.010	0.206
30	0.001	0.004	0.007	0.151
50	0.000	0.001	0.003	0.063
100	0.000	0.000	0.001	0.020
160	0.000	0.000	0.001	0.010
500	0.000	0.000	0.000	0.002
800	0.000	0.000	0.000	0.000

It is seen that F performs better than Bowker's formula for all values of N.

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