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GENERATING AN EVENT WITH PROBABILITY p^α, $\alpha > 0$

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SUMMARY. Given a coin whose probability $p(0 < p < 1)$ of showing up a head is unknown, we have generated an event A whose probability is $p^\alpha, \alpha > 0$. This answers a question raised by D. Basu (1975).

1. Introduction

Consider a coin whose probability p $(0 < p < 1)$ of showing up a head is unknown. Repeated tosses of this coin will produce i.i.d. r.v.'s $X_1, X_2, \ldots$ with $P(X_i = 1) = p$, $P(X_i = 0) = 1-p$, $i = 1, 2, \ldots$. Basu (1975) raised the question whether one could perform an experiment with the coin to generate an event A such that $P(A) = \sqrt{p}$. In fact, the general problem, as mentioned by him, would be to investigate the possibility of generating a r.v. U, on the basis of $X_1, X_2, \ldots$, such that

$$P[U = 1] = f(p), \qquad P[U = 0] = 1-f(p) \qquad \ldots \quad (1.1)$$

where $0 < f(p) < 1$ is a known function of p or equivalently to characterize the class of all such functions $f(p)$ satisfying (1.1). In this paper we deal with the case $f(p) = p^\alpha$, $\alpha > 0$ and generate a r.v. U satisfying (1.1). (However see, in this connection Remark 1.) The case when $f(p) = a/(a+b)$ with a and b positive integers was earlier studied by Dwass (1972) in a slightly different context. It may be mentioned that a related but somewhat different problem of unbiased estimation of various functions of p has been attempted by several authors including, among others, Singh (1964), Sinha and Sinha (1975).

2. The procedure

We first consider the case $0 < \alpha \leqslant 1$ and write $\alpha = 1-\beta$, $0 \leqslant \beta < 1$. Then p^{α} can be expressed as

$$p^{\alpha} = p\left\{1+\frac{\beta^{(1)}}{1!}q+\frac{(\beta+1)^{(2)}}{2!}q^{2}+\ldots+\frac{(\beta+r-1)^{(r)}}{r!}q^{r}+\ldots\right\} \quad \ldots \quad (2.1)$$

where

$$(\beta+j-1)^{(j)} = (\beta+j-1)(\beta+j-2)\ldots\beta, \quad j = 1, 2, \ldots$$

Motivated by (2.1), we define the event A as follows :

$$A = \{H\} \bigcup \left(\bigcup_{r=1}^{\infty} A_r E_r \right) \quad \ldots \quad (2.2)$$

with

$$\begin{cases} A_1 = \{TH\}, \\ A_2 = (A_1 \bar{E}_1 T) \bigcup \{TTH\}, \\ A_3 = (A_2 \bar{E}_2 T) \bigcup \{TTTH\}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_r = (A_{r-1} \bar{E}_{r-1} T) \bigcup \{\underbrace{T\ldots T}_{r \text{ times}} H\}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \quad \ldots \quad (2.3)$$

where $\{\underbrace{T\ldots T}_{j \text{ times}} H\}$ denotes the event that the first j tosses of the coin show all tails and the $(j+1)$-st toss results in a head, $j = 0, 1, \ldots$; $E_1, E_2, \ldots$ are mutually independent events (also independent of the X_i's) satisfying

$$\begin{cases} P(E_1) = \beta^{(1)}, \\ P(E_2) = \dfrac{(\beta+1)P(E_1)\theta_0}{2(1+\theta_0 P(E_1))}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P(E_{r+1}) = \dfrac{(\beta+r)P(E_r)\theta_{r-1}}{(r+1)(1+\theta_{r-1}P(E_r))}. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \quad \ldots \quad (2.4)$$

Here $\bar{E}_j$ is the event complementary to E_j, $j = 1, 2, \ldots$ and $\{\theta_i\}$ is a sequence of positive real numbers defined as

$$\begin{cases} \theta_0 = 1, \\ \theta_1 = 1+P(\bar{E}_1), \\ \theta_2 = \theta_1 P(\bar{E}_2)+1, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \theta_r = \theta_{r-1} P(\bar{E}_r)+1, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \quad \ldots \quad (2.5)$$

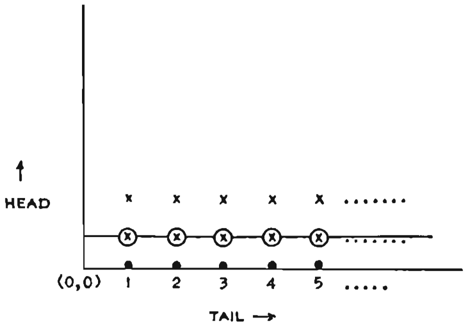
It is easy to check using (2.4) and (2.5) that $0 < P(E_r) < 1$, $r = 2, 3, \ldots$. Defining U as the indicator function of A and noting that the component events comprising A are all mutually exclusive, one gets the desired result.

Remark 1 : Note that when the event A occurs we can find this out after observing a finite number of the X_i's. However to decide that A has not occurred we must have knowledge of the whole sequence $\{X_i\}$! So the experiment, say E, whose outcomes are A or A^c may not be "performable" in the sense of Basu.

Remark 2 : Any set of arbitrary mutually independent events $E_1, E_2, \ldots$ (also independent of the X_i's) whose probabilities satisfy (2.4) and (2.5) can be taken for defining the event A.

Remark 3 : By Dwass (1972), the given coin can be used to generate the E_i's provided α is rational.

Remark 4 : The following figure illustrates the idea behind defining A (2.2).



× : Stopping points ;
● : Continuation points ;
⊗ : Conditional stopping points.

Whenever the conditional stopping point $(j, 1)$ corresponding to the event A_j is reached, we stop (i.e., A occurs) or continue to toss the coin according as whether the event E_j occurs or fails to occur, $j = 1, 2, \ldots$. It is clear that a sort of post randomization is involved at every such stage.

Remark 5 : If $\alpha > 1$, then α can be written as $\alpha = n+(1-\beta)$ where n is a positive integer and $0 \leqslant \beta < 1$. Define A^* as $A^* = A \bigcap B$ where A is defined in (2.2) and B is the event (independent of A) defined by $\{\underbrace{H \ldots H}_{n \text{ times}}\}$.

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