Sankhyā : The Indian Journal of Statistics
1978, Volume 40, Series A, Pt. 4, pp. 313-318.

ON A QUESTION OF D. MAHARAM CONCERNING TAIL FIELDS

By M. G. NADKARNI and D. RAMACHANDRAN
Indian Statistical Institute

SUMMARY. The purpose of this note is to answer a question of Dorothy Maharam concerning tail fields by generalizing her method in Maharam (1977).

1. Definitions and notation

We use the notation $(\mathcal{A}, m)$ to denote a measure space $(X, \mathcal{A}, m)$. Let $\mathcal{A}_1 \subseteq \mathcal{A}$ be a sub σ-algebra of $\mathcal{A}$. A measure space $(\mathcal{B}, \mu)$ (on X) is said to be a natural complement of $(\mathcal{A}_1, m)$ in $(\mathcal{A}, m)$ if

(i) $\mathcal{B} \vee \mathcal{A}_1 = \mathcal{A}$ where $\mathcal{B} \vee \mathcal{A}_1$ denotes the σ-algebra generated by $\mathcal{B}$ and $\mathcal{A}_1$ and

(ii) $m(B \cap A) = \mu(B) \cdot m(A)$ for every $B \in \mathcal{B}$ and $A \in \mathcal{A}_1$.

If $(\mathcal{B}, \mu)$ is a natural complement of $(\mathcal{A}_1, m)$ in $(\mathcal{A}, m)$ then we write $(\mathcal{B}, \mu) \vee (\mathcal{A}_1, m) = (\mathcal{A}, m)$. If $\{(\mathcal{B}_n, \mu_n)\}_{n \geqslant 1}$ is a sequence of measure spaces then by $\bigvee_{n=1}^{\infty} (\mathcal{B}_n, \mu_n)$ we denote the measure space $\left(\bigvee_{n=1}^{\infty} \mathcal{B}_n, \mu\right)$ where $\bigvee_{n=1}^{\infty} \mathcal{B}_n$ is the σ-algebra generated by $\left(\bigcup_{n=1}^{\infty} \mathcal{B}_n\right)$ and μ on $\bigvee_{n=1}^{\infty} \mathcal{B}_n$ is a measure, whenever it exists, such that if $B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2, \ldots B_n \in \mathcal{B}_n$ then $\mu(B_1 \cap B_2 \cap \ldots \cap B_n) = \mu_1(B_1) \cdot \mu_2(B_2) \ldots \mu_n(B_n)$. (In general, such a μ may not exist but in our set-up it does. Sufficient conditions for the existence of such a μ can be found in Theorem 3 of Marczewski, 1951. We however do not use this result.)

All measures considered in this note are σ-finite. For the definition and examples of perfect and nonperfect probabilities we refer to Sazonov (1962).

2. Preliminaries

Consider the measure space $(\mathcal{A}, m)$ and let $\{\mathcal{A}_n\}_{n=0}^{\infty}$ be a sequence of sub σ-algebras of $\mathcal{A}$ such that $\mathcal{A}_0 = \mathcal{A}$ and for every $n \geqslant 0$, $\mathcal{A}_{n+1} \subseteq \mathcal{A}_n$. Let $\{(\mathcal{B}_n, \mu_n)\}_{n=1}^{\infty}$ be a sequence of measure spaces on X such that for each $n \geqslant 1$, $(\mathcal{B}_n, \mu_n)$ is a natural complement of $(\mathcal{A}_n, m)$ in $(\mathcal{A}_{n-1}, m)$, i.e., $\mathcal{B}_n \vee \mathcal{A}_n = \mathcal{A}_{n-1}$ and $m(B \cap A) = \mu_n(B) \cdot m(A)$ for every $B \in \mathcal{B}_n$, $A \in \mathcal{A}_n$. We then have for each $n = 1, 2, \ldots$

$$(\mathcal{A}, m) = (\mathcal{B}_1, \mu_1) \vee (\mathcal{B}_2, \mu_2) \vee \ldots \vee (\mathcal{B}_n, \mu_n) \vee (\mathcal{A}_n, m).$$

We are concerned with the question of equality of

$$(\mathcal{A}, m) = \bigvee_{n=1}^{\infty} (\mathcal{B}_n, \mu_n) \qquad \ldots \quad (*)$$

given that $\left(\bigcap_{n=0}^{\infty} \mathcal{A}_n, m\right)$ is trivial in the sense that for every $A \in \bigcap_{n=0}^{\infty} \mathcal{A}_n$ either $m(A) = 0$ or $m(A^c) = 0$. This question presents itself in a certain formulation of non-linear prediction theory as pointed out in Masani (1966, pp. 90-94).

If equality in (*) of $(\mathcal{A}, m)$ with $\bigvee_{n=1}^{\infty} (\mathcal{B}_n, \mu_n) = \left(\bigvee_{n=1}^{\infty} \mathcal{B}_n, \mu\right)$ is interpreted to mean $\mathcal{A} = \bigvee_{n=1}^{\infty} \mathcal{B}_n$ and $m = \mu$ then examples are known (see Nadkarni, Ramachandran and Bhaskara Rao, 1975) where equality fails to hold. In this note we interpret equality in (*) in the following two senses :

(i) there exists a σ-isomorphism ϕ between $\mathcal{A}$ and $\bigvee_{n=1}^{\infty} \mathcal{B}_n$ such that $m\phi^{-1} = \mu$;

(ii) there exists a measure algebra isomorphism U between the measure algebras $\mathcal{A}/m$ and $\left(\bigvee_{n=1}^{\infty} \mathcal{B}_n\right)/\mu$ such that $mU^{-1} = \mu$.

We give examples to show that probability space $(\mathcal{A}, m)$ need not be equal to $\left(\bigvee_{n=1}^{\infty} \mathcal{B}_n, \mu\right)$ where $(\mathcal{A}, m)$ is interpreted in any of the above senses.

3. A general construction and the examples

Let $(Y, \mathcal{C}, \lambda)$ be a σ-finite measure space and let T be a $1-1$ bimeasurable, measure preserving, ergodic transformation on $(Y, \mathcal{C}, \lambda)$. Let $(\{-1, 1\}^N, \mathcal{D}, P)$, where N is the set of natural numbers, stand for the unilateral product of the discrete two element space $\{-1, 1\}$ with the measure giving mass $\frac{1}{2}$ to $\{-1\}$ and $\{1\}$.

Let

$$X = Y \times \{-1, 1\}^N,$$

$$\mathcal{A} = \mathcal{C} \times \mathcal{D},$$

and

$$m = \lambda \times P.$$

For each $k = 1, 2, \ldots$, we define a $1-1$ bimeasurable measure preserving trasformation $T_k : X \to X$, of *period* 2, by

$$T_k(y, p_1, p_2, \ldots, p_k, p_{k+1}, \ldots) = (T^{p_k} y, p_1, p_2, \ldots, -p_k, p_{k+1}, \ldots)$$

where each $p_i = \pm 1$. Let G_k be the *abelian group of order* 2^k generated by $T_1, \ldots, T_k$.

Let for each $k \geqslant 1$,

$$\mathscr{A}_k = \{A \in \mathscr{A} : \overline{T}A = A \text{ for every } \overline{T} \in G_k\}.$$

Remark : Suppose $(p_{i_1}, p_{i_2}, \ldots, p_{i_k})$ is permutation of $(p_1, p_2, \ldots, p_k)$ where each $p_j = \pm 1$ $(j = 1, 2, \ldots, k)$ then since $\sum_{j=1}^{k} p_j = \sum_{j=1}^{k} p_{i_j}$ it can be checked that the number of coordinates at which $p_j = -1$ and $p_{i_j} = -1$ is equal to the number of coordinates at which $p_j = -1$ and $p_{i_j} = +1$. Hence it follows that

$$(y, p_1, p_2, \ldots, p_k, p_{k+1}, \ldots) \in A \in \mathscr{A}_k$$
$$\Longrightarrow (y, p_{i_1}, p_{i_2}, \ldots, p_{i_k}, p_{k+1}, \ldots) \in A \in \mathscr{A}_k.$$

Let $\mathscr{A}_0 = \mathscr{A}$ and consider the sequence $\{\mathscr{A}_n\}_{n=0}^{\infty}$ of sub σ-algebras of $\mathscr{A}$. We have $\mathscr{A}_{n+1} \subseteq \mathscr{A}_n$ for every $n \geqslant 0$. For each $n \geqslant 1$, let

$$B_n^1 = \{(y, p_1, p_2, \ldots) : p_n = 1\}$$

$$B_n^{-1} = \{(y, p_1, p_2, \ldots) : p_n = -1\}$$

and let $(\mathscr{B}_n, \mu_n)$ be the measure space with $\mathscr{B}_n = \{X, B_n^1, B_n^{-1}, \phi\}$ and $\mu_n(B_n^1) = \mu_n(B_n^{-1}) = \frac{1}{2}$. Note that, for each $n \geqslant 1$, $T_n B_n^1 = B_n^{-1}$ and $T_n B_n^{-1} = B_n^1$.

Now, if $A \in \mathscr{A}_{n-1}$ then consider

$$A_1 = (B_n^1 \cap A) \cup (B_n^{-1} \cap T_n A).$$

Since

$$T_n A_1 = A_1, \quad A \in \mathscr{A}_{n-1}$$

and since G_n is abelian we have $A_1 \epsilon \mathscr{A}_n$.

Thus

$$B_n^1 \cap A_1 = B_n^1 \cap A \epsilon \mathscr{B}_n \vee \mathscr{A}_n.$$

Similarly

$$B_n^{-1} \cap A \epsilon \mathscr{B}_n \vee \mathscr{A}_n.$$

Hence

$$A = (B_n^1 \cap A) \cup (B_n^{-1} \cap A) \epsilon \mathscr{B}_n \vee \mathscr{A}_n,$$

or,

$$\mathscr{B}_n \vee \mathscr{A}_n = \mathscr{A}_{n-1}.$$

If $A \epsilon \mathscr{A}_n$, then $T_n A = A$ and so

$$m(B_n^1 \cap A) = m(T_n(B_n^1 \cap A)) = m(T_n B_n^1 \cap T_n A) = m(B_n^{-1} \cap A)$$

and since

$$A = (B_n^1 \cap A) \cup (B_n^{-1} \cap A)$$

we get

$$m(B_n^1 \cap A) = m(B_n^{-1} \cap A) = \frac{1}{2}\, m(A).$$

It follows that $(\mathscr{B}_n, \mu_n)$ is a natural complement of $(\mathscr{A}_n, m)$ in $(\mathscr{A}_{n-1}, m)$.

Claim : $\left(\bigcap_{n=0}^{\infty} \mathscr{A}_n, m\right)$ is trivial, that is, $A \epsilon \bigcap_{n=0}^{\infty} \mathscr{A}_n$ implies $m(A) = 0$ or $m(A^c) = 0$.

Proof : Suppose $A \epsilon \bigcap_{n=0}^{\infty} \mathscr{A}_n$ and $m(A) > 0$. We shall show that $m(A) = m(X)$. For each $y \epsilon Y$, if we let

$$A_y = \{(p_1, p_2, \ldots) : (y, p_1, p_2, \ldots) \epsilon A\}$$

then by the Hewitt-Savage zero-one law, $P(A_y) = 0$ or 1 and moreover $\{y \epsilon Y : P(A_y) = 1\}$ is invariant under T.

Now, let $C = \{y \in Y : P(A_y) > 0\} = \{y \in Y : P(A_y) = 1\}$. Since $m(A) > 0$, by Fubini's theorem, $\lambda(C) > 0$. But $TC = C$ and so by the ergodicity of T we get $\lambda(C) = \lambda(Y)$. Again, by Fubini's theorem, $m(A) = m(X)$.

We observe that for these $(\mathcal{B}_n, \mu_n)$, $n \geqslant 1$, there exists μ on $\bigvee_{n=1}^{\infty} \mathcal{B}_n$ satisfying the required conditions and that $\left(\bigvee_{n=1}^{\infty} \mathcal{B}_n, \mu\right)$ is σ-isomorphic to $(\mathcal{D}, P)$ in fact, $\bigvee_{n=1}^{\infty} \mathcal{B}_n = Y \times \mathcal{D}$.

We use now the above general construction to get the required examples.

Example 1 : Taking $Y = N$, the set of *all* integers, $\mathcal{C}$ = class of all subsets of N, λ = the counting measure on and T on N defined by $Tn = n+1$ for every $n \in N$, we get the example constructed by Maharam (1977). Here m is an infinite measure and μ is a probability measure and hence equality of $(\mathcal{A}, m)$ with $\left(\bigvee_{n=1}^{\infty} \mathcal{B}_n, \mu\right)$ does not exist in both the senses described in Section 2. Maharam (1977) then raises the question whether a similar example is possible in a space of finite measure. We give below such examples in probability spaces.

Example 2 : Let us take $(Y, \mathcal{C}, \lambda)$ to be the countable bilateral product of a separable, nonperfect probability space $(M, \mathcal{M}, Q)$ with product measure and T to be the shift transformation. Then T is measure preserving and ergodic (see Billingsley, 1965) and our construction holds. Then $(\mathcal{A}, m)$ is a separable, nonperfect probability space which is moreover nonatomic. Hence $\mathcal{A}/m$ and $\left(\bigvee_{n=1}^{\infty} \mathcal{B}_n\right)/\mu$, both being separable, nonatomic measure algebras of total measure one, are isomorphic (see Royden, 1968, p. 321). However, since m is nonperfect and μ is perfect, there is no σ-isomorphism ϕ from $\mathcal{A}$ to $\bigvee_{n=1}^{\infty} \mathcal{B}_n$ such that $m\phi^{-1} = \mu$.

Example 3 : Let us take $(Y, \mathcal{C}, \lambda)$ to be the countable bilateral product of a nonseparable measure space $(M, \mathcal{M}, Q)$ (i.e., $\mathcal{M}/Q$ is not separable) with product measure and T to be the shift transformation. Then $(\mathcal{A}, m)$ is again a nonseparable probability space while $\left(\bigvee_{n=1}^{\infty} \mathcal{B}_n, \mu\right)$ is separable. Thus $\mathcal{A}/m$ is not separable while $\left(\bigvee_{n=1}^{\infty} \mathcal{B}_n\right)/\mu$ is and hence there does not exist a measure algebra isomorphism preserving measure between $\mathcal{A}/m$ and $\left(\bigvee_{n=1}^{\infty} \mathcal{B}_n)/\mu\right)$.

Acknowledgement. The authors thank Professor J. K. Ghosh for the useful discussion they had on Maharam's example.

REFERENCES

BILLINGSLEY, P. (1965): *Ergodic Theory and Information*, Wiley.

MAHARAM, D. (1977): An example on tail fields. Oberwolfach Conference reports. To appear.

MARCZEWSKI, E. (1951): Measures in almost-independent fields. *Fund. Math.*, 38, 217-220.

MASANI, P. (1966): Wiener's contribution to generalised Harmonic analysis, Prediction theory and Filter theory. *Bull. Amer. Math. Soc.*, 72, 73–125.

NADKARNI, M. G., RAMACHANDRAN, D. and BHASKARA RAO, K. P. S. (1975): On sequences of σ-algebras. *Colloq. Math.*, 32, 227-231.

ROYDEN, H. L. (1968): *Real Analysis*, Macmillan.

SAZONOV, V. V. (1962): On perfect measures. *Izv. Akad. Nauk. SSSR Ser. Mat*, 26, 391-414 (in Russian). *Amer. Math. Soc. Transl.*, Series 2, 48, 229-254.

Paper received : March, 1978.
Revised : July, 1978.