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OPERATOR METHODS IN ORDER STATISTICS

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Summary

This paper presents operator methods, based on difference and differential operators, for establishing identities satisfied by distributions of order statistics. In addition to deducing many known identities, these operator methods may also be used to derive several new identities for distributions of order statistics.

Key words: Order statistics; identities; difference and differential operators; combinatorial identities.

1. Introduction

In the last thirty years or so, many identities and recurrence relations for distributions of order statistics have been established by several authors including Srikantan (1962), Govindarajulu (1963), Joshi (1973), Arnold (1977), Joshi & Balakrishnan (1982), and Balakrishnan & Malik (1985); interested readers may also refer to David (1981) and Arnold & Balakrishnan (1989) for a more comprehensive list of references. The proofs for the identities of order statistics established by these and other authors hinge upon some combinatorial techniques, some easy and some involved.

In this paper we derive operator equalities (based on both difference and differential operators) which when applied on suitably chosen functions generate identities for distributions of order statistics. Many known identities are deduced easily by this approach, and, in addition, many new identities are also generated. Furthermore, unlike the direct combinatorial approach, the operator method presented in this paper may easily be extended to identities involving joint distributions of order statistics. This is illustrated for joint distributions of two order statistics and many new identities are derived in this case.

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2. Identities for Single Order Statistics by Operator Methods

Let $X_1, \ldots, X_n$ be a random sample of size n from a population with cumulative distribution function $F(x)$ and probability density function $f(x)$. Let $X_{1:n} \le \cdots \le X_{n:n}$ be the order statistics obtained from the above sample. Then, the density function of $X_{r:n}$ $(1 \le r \le n)$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}\big(F(x)\big)^{r-1}\big(1-F(x)\big)^{n-r} f(x) \quad (-\infty < x < \infty). \tag{2.1}$$

Further, let Δ and E be the difference and the shift operators with common difference 1 acting on functions of y (where y is independent of x). Then, we have the following operator equalities satisfied by distributions of order statistics.

Theorem 1. *For $n \ge 2$,*

$$\sum_{r=1}^{n} f_{r:n}(x) E^{n-r} = \sum_{r=1}^{n} \binom{n}{r} f_{1:r}(x) \Delta^{r-1} \tag{2.2}$$

and

$$\sum_{r=1}^{n} f_{r:n}(x) E^{r-1} = \sum_{r=1}^{n} \binom{n}{r} f_{r:r}(x) \Delta^{r-1}. \tag{2.3}$$

Proof. We show below the equality in (2.2); (2.3) can be proved similarly. We start from

$$\begin{aligned}
&\sum_{r=1}^{n} (1+t)^{n-r} f_{r:n}(x) \\
&= \sum_{r=1}^{n} \frac{n!}{(r-1)!\,(n-r)!}\big(F(x)\big)^{r-1}\big\{(1+t)\big(1-F(x)\big)\big\}^{n-r} f(x) \\
&= n\big\{1 + t\big(1-F(x)\big)\big\}^{n-1} f(x) = n \sum_{r=0}^{n-1} \binom{n-1}{r} \big(1-F(x)\big)^{r} f(x) t^{r}.
\end{aligned}$$

Then

$$\sum_{r=1}^{n} (1+t)^{n-r} f_{r:n}(x) = \sum_{r=1}^{n} \binom{n}{r} f_{1:r}(x) t^{r-1}. \tag{2.4}$$

Equation (2.2) follows from (2.4) on setting $t = \Delta$.

Example 1. For the function $T(y) = 1/(n-y)$,

$$E^{n-r} T(y) = \frac{1}{r-y} \quad \text{and} \quad \Delta^{r-1} T(y) = \frac{(r-1)!}{(n-y)^{(r)}},$$

where $m^{(k)} = m(m-1)\dots(m-k+1)$ for $k \geq 1$ and $= 1$ for $k = 0$. Setting $y = 0$, (2.2) now gives

$$\sum_{r=1}^{n} \frac{1}{r} f_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{r} f_{1:r}(x).$$

Similarly, (2.3) gives

$$\sum_{r=1}^{n} \frac{1}{n-r+1} f_{r:n}(x) = \sum_{r=1}^{n} \frac{1}{r} f_{r:r}(x).$$

These identities are due to Joshi (1973).

Example 2. By choosing the function $T(y) = 1/(N-y)^{(k)}$, we obtain from (2.2) and (2.3) (with $y = 0$) the following identities

$$\sum_{r=1}^{n} \frac{1}{(N-n+r)^{(k)}} f_{r:n}(x) = \sum_{r=1}^{n} \binom{n}{r} \frac{(k+r-2)^{(r-1)}}{N^{(k+r-1)}} f_{1:r}(x), \tag{2.5}$$

$$\sum_{r=1}^{n} \frac{1}{(N-r+1)^{(k)}} f_{r:n}(x) = \sum_{r=1}^{n} \binom{n}{r} \frac{(k+r-2)^{(r-1)}}{N^{(k+r-1)}} f_{r:r}(x). \tag{2.6}$$

By setting $N = n + k - 1$ in these two identities, we get the identities due to Balakrishnan & Malik (1985) and, therefore, (2.5) and (2.6) can be regarded as generalizations of their results.

Alternatively, by setting $t = D$ in (2.4) where D is the differentiation operator acting on functions of y (with y being independent of x), we have the following theorem.

Theorem 2. *For $n \geq 2$,*

$$\sum_{r=1}^{n} f_{r:n}(x)(1+D)^{n-r} = \sum_{r=1}^{n} \binom{n}{r} f_{1:r}(x) D^{r-1} \tag{2.7}$$

and

$$\sum_{r=1}^{n} f_{r:n}(x)(1+D)^{r-1} = \sum_{r=1}^{n} \binom{n}{r} f_{r:r}(x) D^{r-1}. \tag{2.8}$$

Example 3. By choosing the function $T(y) = y^k$, we obtain from (2.7) the identity

$$\sum_{r=1}^{n} f_{r:n}(x) \sum_{s=0}^{n-r} \binom{n-r}{s} k^{(s)} y^{k-s} = \sum_{r=1}^{n} \binom{n}{r} f_{1:r}(x) k^{(r-1)} y^{k-r+1}. \tag{2.9}$$

Comparing the coefficients of y^m on both sides of (2.9) yields the identity

$$\sum_{r=1}^{n} \binom{n-r}{k-m} f_{r:n}(x) = \binom{n}{k+1-m} f_{1:k+1-m}(x),$$

a result due to Downton (1966).

3. Identities for Two Order Statistics by Operator Methods

The joint density function $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ is given by

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!} \times \left(F(x)\right)^{r-1}\left(F(y)-F(x)\right)^{s-r-1} \times \left(1-F(y)\right)^{n-s} f(x)f(y), \qquad (-\infty < x < y < \infty). \tag{3.1}$$

Let Δ_1 and E_1 be the difference and the shift operators with common difference 1 acting on functions of w, and similarly Δ_2 and E_2 act on functions of z (where w and z are independent each being independent of both x and y). Then, we have the following operator equalities satisfied by joint distributions of two order statistics.

Theorem 3. *For* $n \geq 3$,

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} f_{r,s:n}(x,y) E_1^{r-1} E_2^{n-s} = \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \binom{n}{s} f_{r,r+1:s}(x,y) \Delta_1^{r-1} \Delta_2^{s-r-1}, \tag{3.2}$$

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} f_{r,s:n}(x,y) E_1^{r-1} E_2^{s-r-1} = \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \binom{n}{s} f_{r,s:s}(x,y) \Delta_1^{r-1} \Delta_2^{s-r-1}, \tag{3.3}$$

and

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} f_{r,s:n}(x,y) E_1^{s-r-1} E_2^{n-s} = \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \binom{n}{s} f_{1,r+1:s}(x,y) \Delta_1^{r-1} \Delta_2^{s-r-1}. \tag{3.4}$$

Proof. Consider the identity

$$\begin{aligned}
\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} &(1+u)^{r-1}(1+v)^{s-r-1}(1+w)^{n-s} f_{r,s:n}(x,y) \\
&= \sum_{r=1}^{n-1} \frac{n!}{(r-1)!\,(n-r-1)!}\{(1+u)F(x)\}^{r-1} \\
&\quad \times \sum_{s=0}^{n-r-1}\binom{n-r-1}{s}\{(1+v)\big(F(y)-F(x)\big)\}^{s} \\
&\quad \times \{(1+w)\big(1-F(y)\big)\}^{n-r-1-s} f(x)f(y) \\
&= n(n-1)\sum_{r=0}^{n-1}\binom{n-2}{r}\{(1+u)F(x)\}^{r} \\
&\quad \times \{(1+v)\big(F(y)-F(x)\big)+(1+w)(1-F(y))\}^{n-r-2} f(x)f(y) \\
&= n(n-1)\{uF(x)+v\big(F(y)-F(x)\big)+w\big(1-F(y)\big)+1\}^{n-2} f(x)f(y). \quad (3.5)
\end{aligned}$$

By setting $v = 0$ and expanding the term on the right-hand side of (3.5) binomially, we get

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} f_{r,s:n}(x,y)(1+u)^{r-1}(1+w)^{n-s} = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s} f_{r,s:s}(x,y)u^{r-1}w^{s-r-1}$$

from which the identity in (3.2) follows immediately by taking $u = \Delta_1$ and $w = \Delta_2$. By setting $w = 0$ ($u = 0$) in (3.5) and proceeding on similar lines, we derive the identity in (3.3) (equation (3.4)).

Example 4. Apply (3.2)–(3.4) to the function $T(w,z) = 1/(n-w-z-1)$. Then setting $w = z = 0$ leads to the identities

$$\frac{1}{n}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{s-r} f_{r,s:n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{s(s-1)} f_{r,r+1:s}(x,y),$$

$$\frac{1}{n}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{n-s+1} f_{r,s:n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{s(s-1)} f_{r,s:s}(x,y),$$

and

$$\frac{1}{n}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{r} f_{r,s:n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{s(s-1)} f_{1,r+1:s}(x,y),$$

respectively. These identities are due to Balakrishnan, Bendre & Malik (1992).

Example 5. Apply (3.2)–(3.4) to the function $T(w,z) = 1/(N-w-z-1)^{(k)}$. Setting $w = z = 0$ leads to

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{(N-n+s-r)^{(k)}}f_{r,s:n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s}\frac{(k+s-3)^{(s-2)}}{(N-1)^{(k+s-2)}}f_{r,r+1:s}(x,y),$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{(N-s+1)^{(k)}}f_{r,s:n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s}\frac{(k+s-3)^{(s-2)}}{(N-1)^{(k+s-2)}}f_{r,s:s}(x,y),$$

and

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\frac{1}{(N-n+r)^{(k)}}f_{r,s:n}(x,y) = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s}\frac{(k+s-3)^{(s-2)}}{(N-1)^{(k+s-2)}}f_{1,r+1:s}(x,y).$$

These three identities are generalizations of identities due to Balakrishnan, Bendre & Malik (1992) whose results correspond to the case $k = 1$ and $N = n$.

Here is an equivalent form of Theorem 3 in terms of differentiation operators.

Theorem 4. *For $n \geq 3$,*

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} f_{r,s:n}(x,y)(1+D_1)^{r-1}(1+D_2)^{n-s} = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s} f_{r,r+1:s}(x,y)D_1^{r-1}D_2^{s-r-1}, \tag{3.6}$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} f_{r,s:n}(x,y)(1+D_1)^{r-1}(1+D_2)^{s-r-1} = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s} f_{r,s:s}(x,y)D_1^{r-1}D_2^{s-r-1}, \tag{3.7}$$

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} f_{r,s:n}(x,y)(1+D_1)^{s-r-1}(1+D_2)^{n-s} = \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s} f_{1,r+1:s}(x,y)D_1^{r-1}D_2^{s-r-1}, \tag{3.8}$$

where D_1 and D_2 are differentiation operators acting on functions of w and z respectively.

Example 6. Let the operator identity (3.6) act on $T(w,z) = w^k z^\ell$. Then

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} f_{r,s:n}(x,y)\sum_{i=0}^{r-1}\sum_{j=0}^{n-s}\binom{r-1}{i}\binom{n-s}{j}k^{(i)}\ell^{(j)}w^{k-i}z^{\ell-j}$$
$$=\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{n}{s} f_{r,r+1:s}(x,y)k^{(r-1)}\ell^{(s-r-1)}w^{k-r+1}z^{\ell-s+r+1}. \quad (3.9)$$

Comparing the coefficients of $w^{k-p}z^{\ell-q}$ on both sides of (3.9) gives the identity

$$\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}\binom{r-1}{p}\binom{n-s}{q} f_{r,s:n}(x,y) = \binom{n}{p+q+2} f_{p+1,p+2:p+q+2}(x,y)$$

which is due to Downton (1966).

4. Conclusions

In this paper we have presented a unified method based on operators for establishing identities satisfied by order statistics. We have shown that many well-known identities for order statistics follow as special cases of this method corresponding to different choices of operators and functions. We have also used this method to generate some new interesting identities satisfied by distributions of single as well as two order statistics. In addition, we make the following comments:

(i) In order to illustrate the usefulness of the operator method developed in this paper, we have made a few choices for the function T that the operator acts on. One may be able to generate many more identities for order statistics by making different choices for the function T.

(ii) For purposes of illustration, we have used in this paper only the operators Δ and D. Other choices (e.g. matrices) are possible which could lead to many more new identities for order statistics.

(iii) By writing the identities established in this paper in terms of expectations of functions of order statistics and then by assuming specific distribution for the underlying population and making use of exact explicit expressions for these quantities, one can generate combinatorial identities. see, for example, Joshi & Balakrishnan (1981) and Balasubramanian & Beg (1990).

(iv) The operator method presented in this paper, unlike the direct combinatorial method, lends itself to proving identities involving joint distributions of three or more order statistics in a straight forward manner.

References

ARNOLD, B.C. (1977). Recurrence relations between expectations of functions of order statistics. *Scand. Actuar. J.* **1977**, 169–174.

— & BALAKRISHNAN, N. (1989). *Relations, Bounds and Approximations for Order Statistics*, Lecture Notes in Statistics No. 53. New York: Springer-Verlag.

BALAKRISHNAN, N., BENDRE, S.M., & MALIK, H.J. (1992). General relations and identities for order statistics from non-independent non-identical variables. *Ann. Inst. Statist. Math.* **44**, 177–183.

— & MALIK, H.J. (1985). Some general identities involving order statistics, *Comm. Statist. — Theory Methods* **14**(2), 333–339.

BALASUBRAMANIAN, K. & BEG, M.I. (1990). On expectations of functions of order statistics, *Sankhyā Ser. A* **52**, 103–114.

DAVID, H.A. (1981). *Order Statistics*, Second edition, New York: John Wiley and Sons.

DOWNTON, F. (1966). Linear estimates with polynomial coefficients. *Biometrika* **53**, 129–141.

GOVINDARAJULU, Z. (1963). On moments of order statistics and quasi-ranges from normal populations. *Ann. Math. Statist.* **34**, 633–651.

JOSHI, P.C. (1973). Two identities involving order statistics. *Biometrika* **60**, 428–429.

— & BALAKRISHNAN, N. (1981). Applications of order statistics in combinatorial identities. *J. Combin. Inform. System Sci.* **6**, 271–278.

— & — (1982). Recurrence relations and identities for the product moments of order statistics. *Sankhyā Ser. B* **44**, 39–49.

SRIKANTAN, K.S. (1962). Recurrence relations between the PDFs of order statistics, and some applications. *Ann. Math. Statist.* **33**, 169–177.