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COHERENT MIXTURES OF SRSWOR SAMPLING SCHEMES FOR BOUNDED RISK ESTIMATION OF A FINITE POPULATION MEAN

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ABSTRACT

In a recent study(Mukhopadhyay et al (1988), Sinha (1989)) a problem on characterization of coherent sequential sampling schemes arose in the context of bounded risk estimation of a finite population mean. In this paper we provide some solutions to the problem after properly formulating the same.

1. INTRODUCTION

In Mukhopadhyay et al (1988) the problem of bounded risk estimation of a finite population mean was considered and a comparison of relative performance of three popular strategies based on simple random with replacement (SRSWR) and simple random without replacement (SRSWOR) sampling procedures was made. Roughly speaking, it was demonstrated that the strategy (SRSWR, sample mean of distinct units)

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compares favourably well with the strategy (SRSWOR, sample mean) for bounded risk estimation of a finite population mean (with known variance) under squared error loss. In Sinha (1989) the above comparison was re-examined in terms of sampling strategies based on some classes of balanced sampling schemes (Vide Roy and Chakraborti (1960), Sinha (1976)) and an interesting characterization problem was posed. In this paper we provide some results regarding this characterization problem.

Below we briefly describe the background and clearly state the problem. In section 2 we display the main results and make some concluding remarks.

Considered is a finite labelled population of N units and a study variable Y with the value Y_i on unit i, $1 \leq i \leq N$. As usual $\bar{Y}$ and σ_Y^2 denote the population mean and variance respectively. It is desired to estimate the population mean $\bar{Y}$ with the risk, arising out of the loss function $A(\hat{\bar{Y}} - \bar{Y})^2$, bounded above by a pre-assigned known number $W > 0$. We assume $A(> 0)$ and σ_Y^2 to be known.

Under SRSWOR sampling, using the sample mean as the estimator, the sample size necessary for bounded risk estimation of Y is given by n_{oo} which satisfies the inequality

$$n_{oo}^{-1} \leq N^{-1}\{1 + W(N - 1)A^{-1} \sigma^2\} < (n_{oo} - 1)^{-1} \qquad (1.1)$$

In the above we have conveniently replaced σ_Y^2 by σ^2 and we will do so for the rest of the paper. Under SRSWR sampling, using the sample mean of distinct units as the estimator, the sample size necessary for bounded risk estimation of $\bar{Y}$ is given by n_o which satisfies a similar inequality

$$E(v_{n_o}^{-1}) \leq N^{-1} \{1 + W(N - 1)A^{-1} \sigma^{-2}\} < E(v_{n_o-1}^{-1}) \qquad (1.2)$$

where

v_n = number of distinct units in SRSWR(N,n) sampling.

It is known (Vide, for example, Basu (1958); Raj and Khamis (1958))

(a) $P(v_n = k) = \binom{N}{k} \Delta^k 0^n / N^n$, $k = 1,2, \ldots, \min(n, N)$;

(b) $E(v_n) = n\,[1 - (N-1)^n N^{-n}]$;

(c) $E(v_n^{-1}) = \sum_{k=1}^{\min(n,N)} k^{-1}\binom{N}{k}\Delta^k 0^n/N^n = \sum_{j=1}^{N} j^{n-1}/N^n$. (1.3)

In Mukhopadhyay et al. (1988) a comparison of SRSWR and SRSWOR sampling for bounded risk estimation of $\bar{Y}$ was made and the following inequality was established.

Theorem 1 : Uniformly in A, W, N and σ^2 ,

$$-1 < E(v_{n_o}) - n_{oo} < 2$$

where n_o and n_{oo} satisfy (1.1) and (1.2).

For a sample s, let $v_i(s) = 1$ if $i \in s$; $= 0$ if $i \notin s$, $1 \le i \le N$ so that $v(s) = \sum_{i=1}^{N} v_i(s)$ = number of distinct units in s (also termed effective size of the sample s). A balanced sampling scheme is characterized by the conditions (Vide Sinha (1976))

(i) $E\{v_i(s)/v(s)\} = N^{-1}$, $1 \le i \le N$

(ii) $E\{v_i(s)/v^2(s)\} = N^{-1}E\{v^{-1}(s)\}$, $1 \le i \le N$

(iii) $E\{v_i(s)v_j(s)/v^2(s)\} = N^{-1}(N-1)^{-1}\{1 - E(v^{-1}(s)\}$, $1 \le i \ne j \le N$. (1.4)

If a sampling scheme is balanced, then the estimator (termed the best regular estimator)

$$e_o(s,Y) = \sum_{i \in s} v_i(s)Y_i/v(s) = \text{sample mean of distinct units} \quad (1.5)$$

possesses least variance (equal to $(N-1)^{-1}\{NE(v^{-1}(s)) - 1\}\sigma^2$) among all homogeneous linear unbiased estimators with variance proportional to σ^2 (such estimators being termed regular estimators). Both SRSWR and SRSWOR sampling schemes are examples of balanced sampling schemes.

Define

$$d_k \equiv \text{SRSWOR}(N, k), \quad 1 \le k \le N \quad (1.6)$$

$$D(p) = \sum_{k=1}^{N} p_k d_k , \; 0 \le p_k \le 1 , \; \Sigma p_k = 1 . \quad (1.7)$$

It is known that all mixture sampling designs of the type D(p) are examples of balanced sampling schemes (thought not all balanced sampling schemes do possess this structure; Vide Sinha (1976)). The following choice, for a fixed integer n ,

$$p_k = p_{k,n} = \binom{N}{k} \Delta^k 0^n/N^n , \quad k = 1,2, \ldots , \tilde{n} , \tilde{n} = \min(n,N) \quad (1.8)$$
$$= 0 \qquad \text{otherwise}$$

leads to SRSWR (N,n) sampling design (Vide Basu (1958), for example).

Based on certain classes of mixture sampling designs of the type (1.7), the best regular estimator (1.5) compares favourably well with the strategy

$$(\text{SRSWOR}(N,n_{oo}), \text{ sample mean of } n_{oo} \text{ units}) \qquad (1.9)$$

in the sense that a meaningful version of Theorem 1 holds. This is illustrated below.

Following Sinha (1989), an upper triangular matrix or array of non-negative numbers p_{ij}'s

$$\mathbf{P} = ((p_{ij})), \quad 1 \leq i \leq j \leq N \qquad (1.10)$$

will be said to generate a <u>coherent</u> <u>sequential</u> <u>structure</u> of the mixture designs $D_1, D_2, \ldots , D_N$ formed as

$$D_j = \sum_{i \leq j} p_{ij} d_i , \; 1 \leq j \leq N , \; \sum_{i \leq j} p_{ij} = 1, \; 1 \leq j \leq N , \; p_{11} = 1 \qquad (1.11)$$

provided the following requirements are satisfied:

(a) $\phi_j = E(v(s)| D_j) = \sum_{i \leq j} i p_{ij} \nearrow$ in j

(b) $\psi_j = (E(v^{-1}(s)| D_j) = \sum_{i \leq j} i^{-1} p_{ij} \searrow$ in j (1.12)

(c) $\psi_j \phi_{j-1} = E(v^{-1}(s)| D_j)E(v(s)|D_{j-1}) < 1$ for $2 \leq j \leq N$

(d) Implementation of D_{j+1} can be achieved by implementation of D_j and following it up with implementation of one single-phase sampling so that the conditional probability P (implementation of D_{j+1} leads to selection of $d_{i'}$ | implementation of D_j led to selection of d_i)

$$= \alpha_{ij} \qquad \text{if } i' = i$$
$$= \beta_{ij} \qquad \text{if } i' = i + 1$$
$$= 0 \qquad \text{otherwise}$$

for some

$$0 \le \alpha_{ij}, \beta_{ij} \le 1, \ \alpha_{ij} + \beta_{ij} = 1 , \ 1 \le i \le j \le N. \tag{1.13}$$

The rational behind 1.12(d) is that in the set-up of sequential sampling, after having chosen a sample s involving v(s) distinct units, if it is decided to continue sampling and draw one additional unit, then the number of distinct units in the resulting sample will be either v(s) or v(s) + 1. Suppose at the j-th stage, the sample s has been drawn using the sampling design d_i so that v(s) = i. If it is now decided to draw one additional unit from the population, then v(s) will either remain unaltered with assigned probability α_{ij} or will increase by unity with assigned probability $\beta_{ij} = 1 - \alpha_{ij}$ (upon selecting one unit at random from the complement of s).

The requirements 1.12(a) and 1.12(b) are motivated by the sequential nature of the sampling designs D_1, D_2..., D_N and it has been pointed out in Sinha (1989) that 1.12(d) $\Longrightarrow$ 1.12(a) and 1.12(b). As a matter of fact, we may state the following relations without proof.

$$\phi_{n+1} = \phi_n + \sum_{k=1}^{n} p_{k,n} \beta_{kn} \tag{1.14}$$

$$\psi_{n+1} = \psi_n - \sum_{k=1}^{n} (p_{k,n} \beta_{kn} / k(k + 1)). \tag{1.15}$$

It should be noted, however, that 1.12(d) $\not\Longrightarrow$ 1.12(c) for arbitrary choices of α_{ij}'s and β_{ij}'s, as illustrated by the following example.

Example 1. Consider the choice $\beta_{ij} = \frac{i}{j+1}$, $\alpha_{ij} = 1 - \frac{i}{j+1}$. This corresponds to the coherent sequential structure for which $P = ((p_{ij} = \frac{1}{j}))$, $1 \le i \le j \le N$. We then have

$$\phi_n = \frac{n+1}{2} \quad \text{and} \quad \psi_n = \frac{1}{n} \sum_{k=1}^{n} 1/k .$$

It is easy to verify that $\psi_n \phi_{n-1} > 1$ for $n \geq 4$.

The requirement 1.12(c) is crucial in arriving at the following result essentially proved in Sinha (1989).

Theorem 2. Suppose there exists a coherent sequential structure of the designs $D_1, D_2, \ldots, D_N$ such that for bounded risk estimation of $\bar{Y}$, using the best regular estimator $e_o(s, Y)$ in (1.5), uniformly in A, W, N and σ^2 , we have, for some j, $1 \leq j \leq N$,

$$E(v^{-1}(s)|\, D_j) \leq N^{-1}\{1 + W(N-1)A^{-1}\sigma^{-2}\} < E(v^{-1}(s)|\, D_{j-1}). \quad (1.16)$$

Then.

$$-1 < E(v(s)|\, D_j) - n_{oo} < 2 \quad (1.17)$$

where n_{oo} is defined in (1.1).

The inequality in (1.17) illustrates that a large variety of sampling strategies do indeed perform favourably well when these are compared with the strategy in (1.9) which corresponds to the coherent sequential structure

$$\{D_n \equiv d_n ,\ 1 \leq n \leq N\}$$

with

$$\mathbf{P} = \mathbf{P}_{oo} = \mathbf{I}_N \text{ (identity matrix of order N).}$$

In terms of the parameters α_{ij}'s and β_{ij}'s in (1.13), we have

$$\alpha_{ij} = 0,\ 1 \leq i \leq j \leq N \ \text{ corresponding to } \mathbf{P}_{oo} .$$

The main contribution of the present paper is to exhibit some general forms of coherent sequential structures. We conclude this section by giving a formal representation of p_{ij}, in terms of α_{ij}'s and β_{ij}'s, using the condition 1.12(d) imposed on the mixture designs derived from the array **P** .

Theorem 3. According to the condition 1.12(d) imposed on the mixture designs, for $1 \leq n \leq N$,

$$p_{k,n} = \sum_{1 \le i_1 < \ldots < i_{k-1} \le n-1} \beta_{1i_1} \beta_{2i_2} \cdots \beta_{k-1\, i_{k-1}} \left(\prod_{1}^{i_1-1} \alpha_{1r_1}\right)\left(\prod_{i_1+1}^{i_2-1} \alpha_{2r_2}\right) \cdots$$

$$\left(\prod_{i_{k-2}+1}^{i_{k-1}-1} \alpha_{k-1r_{k-1}}\right)\left(\prod_{i_{k-1}+1}^{n-1} \alpha_{kr_k}\right) \quad \text{for } 2 \le k \le n-1$$

$$p_{1,n} = \alpha_{11}\alpha_{12} \cdots \alpha_{1n-1}$$

$$p_{n.n} = \beta_{11}\beta_{22} \cdots \beta_{n-1\ n-1} \tag{1.18}$$

Proof. (By induction on n.) According to (d), for $2 \le k \le n-1$,

$$p_{k,n} = p_{k,n-1}\alpha_{k,n-1} + p_{k-1,\, n-1}\, \beta_{k-1,\, n-1}$$

$$= \alpha_{k,n-1}\Big[\sum_{1 \le i_1 < \ldots < i_{k-1} \le n-2} \beta_{1i_1}\beta_{2i_2}\cdots\beta_{k-1i_{k-1}} \prod_{1}^{i_1-1} \alpha_{1r_1} \prod_{i_1+1}^{i_2-1} \alpha_{2r_2} \cdots$$

$$\prod_{i_{k-1}+1}^{n-2} \alpha_{kr_k}\Big] + \beta_{k-1,n-1}\Big[\sum_{1 \le i_1 < \ldots < i_{k-2} \le n-2} \beta_{1i_1}\beta_{2i_2} \cdots \beta_{k-2i_{k-2}}$$

$$\prod_{1}^{i_1-1} \alpha_{1r_1} \prod_{i_1+1}^{i_2-1} \alpha_{2r_2} \cdots \prod_{i_{k-2}+1}^{n-2} \alpha_{k-1r_{k-1}}\Big]$$

$$= \sum_{1 \le i_1 < \ldots < i_{k-1} \le n-2} \beta_{1i_1}\beta_{2i_2}\cdots\beta_{k-1i_{k-1}} \prod_{1}^{i_1-1}\alpha_{1r_1} \prod_{i_1+1}^{i_2-1} \alpha_{2r_2} \cdots \prod_{i_{k-1}+1}^{n-1} \alpha_{kr_k} +$$

$$\sum_{1 \le i_1 < \ldots < i_{k-2} \le n-2} \beta_{1i_1}\beta_{2i_2}\cdots\beta_{k-2i_{k-2}}\beta_{k-1n-1} \prod_{1}^{i_1-1} \alpha_{1r_1} \prod_{i_1+1}^{i_2-1} \alpha_{2r_2} \cdots \prod_{i_{k-2}+1}^{n-2} \alpha_{k-1r_{k-1}}$$

$$= \sum_{1 \le i_1 < \ldots < i_{k-1} \le n-1} \beta_{1i_1}\beta_{2i_2}\cdots\beta_{k-1\, i_{k-1}} \prod_{1}^{i_1-1} \alpha_{1r_1} \prod_{i_1+1}^{i_2-1} \alpha_{2r_2} \cdots \prod_{i_{k-1}+1}^{n-1} \alpha_{kr_k} .$$

Further, it is readily seen that $p_{1,n} = \prod_{1}^{n-1} \alpha_{1r}$ and $p_{n,n} = \prod_{1}^{n-1} \beta_{rr}$. Hence the theorem.

Corollary 1. For $\alpha_{ij} = 0$, $1 \le i \le j \le N$, we have $p_{k,n} = 0$ if $k < n$ and $p_{n,n} = 1$, $1 \le n \le N$.

Corollary 2. For $\alpha_{ij} = \frac{i}{N}$, $1 \le i \le j \le N$, we have $p_{k,n} = \binom{N}{k} \Delta^k O^n / N^n$, $1 \le k \le n \le N$.

The proofs are not difficult and, hence, omitted.

2. SOME CLASSES OF COHERENT SEQUENTIAL PROCEDURES

We will exhibit various choices of α_{ij}'s and β_{ij}'s and arrive at coherent structures of sampling schemes $D_1, D_2, \dots, D_N$.

(1) $\alpha_{ij} = \alpha$, $\beta_{ij} = 1 - \alpha$, $1 \le i \le j \le N$, $0 \le \alpha < 1$.

Referring to (1.18), we obtain

$$p_{k,n} = \beta^{k-1}\alpha^{n-k} \sum_{1 \le i_1 < \dots < i_{k-1} \le n-1} \cdots \sum 1 = \binom{n-1}{k-1} \beta^{k-1}\alpha^{n-k}, \quad 1 \le k \le n \le N. \tag{2.1}$$

From the definitions of ϕ and ψ in (1.12), we obtain

$$\phi_n = 1 + (n-1)\beta, \quad \psi_n = \frac{1-\alpha^n}{n\beta}. \tag{2.2}$$

As expected, (1.12)(a) and (1.12)(b) are trivially satisfied. We then need to satisfy (1.12)(c) and this leads, for a fixed $n \le N$, to

$$\begin{aligned} \phi_{n-1}\psi_n < 1 &\iff \{1 + (n-2)\beta\}(1-\alpha^n) < n\beta \\ &\iff \{(n-1) - (n-2)\alpha\}(1-\alpha^n) < n(1-\alpha) \\ &\iff 1 - 2\alpha + (n-1)\alpha^n - (n-2)\alpha^{n+1} > 0 \end{aligned} \tag{2.3}$$

It is clear that (2.3) is satisfied for all $2 \le n \le N$ whenever $0 \le \alpha \le \frac{1}{2}$. However, for $n \ge 4$, (2.3) is not satisfied for all α in the range $1/2 < \alpha < 1$. To carry out the analysis further, let

$$g_n(\alpha) = 1 - 2\alpha + (n-1)\alpha^n - (n-2)\alpha^{n+1}. \tag{2.4}$$

It is evident that $g_n(\alpha) \downarrow$ in n for every fixed α. Hence, the condition $g_n(\alpha) > 0$ for $2 \le n \le N$ is equivalent to $g_N(\alpha) > 0$. We rewrite $g_N(\alpha)$ as

$$g_N(\alpha) = (N - 1)\,\alpha^N - (N - 2)\alpha^{N+1} - (2\alpha - 1) \tag{2.5}$$

and study its behaviour for $\frac{1}{2} \leq \alpha < 1$.

Let $f_N(\alpha) = g_N'(\alpha) = N(N - 1)\,\alpha^{N-1} - (N-2)(N+1)\alpha^N - 2$. Note that $f_N(\frac{1}{2}) < 0 = f_N(1)$. Further, $f_N'(\alpha) = N(N - 1)^2\,\alpha^{N-2} - N(N+1)(N-2)\,\alpha^{N-1}$ and

$$f_N'(\alpha) \gtrless 0 \iff \alpha \lesseqgtr \alpha_o = (N-1)^2/(N+1)(N-2) . \tag{2.6}$$

Hence, $f_N(\alpha) \uparrow$ for $\frac{1}{2} \leq \alpha \leq \alpha_o$ and $\downarrow$ for $\alpha_o < \alpha \leq 1$ and since $f_N(\frac{1}{2}) < 0$, $\exists$ an α_1 , $\frac{1}{2} < \alpha_1 < \alpha_o$, $\ni f_N(\alpha) \gtrless 0$ according as $\alpha \gtrless \alpha_1$. Thus, essentially, $g_N(\alpha) \downarrow$ for $\frac{1}{2} \leq \alpha \leq \alpha_1$ and $\uparrow$ for $\alpha_1 < \alpha \leq 1$. Since $g_N(\frac{1}{2}) > 0 = g_N(1)$, it is evident $g_N(\alpha) > 0$ for $\frac{1}{2} \leq \alpha \leq \alpha_2$ and $g_N(\alpha) < 0$ for $\alpha_2 < \alpha \leq 1$. It is seen that $\alpha_2 \to \frac{1}{2}$ as $N \to \infty$.

We now turn to the inequality in (1.16) and show that for given A, W, N and σ^2 , there exists a range of values of α such that coherent sampling schemes with the parameter α satisfies (1.16) for some j . Utilizing (2.2), we have essentially to show that for a given real number $a > 0$, there exists a pair (n, α_n) such that $\psi_n - N^{-1} \leq a$ for all $\alpha \leq \alpha_n$. Since ψ_n is increasing in α for every fixed n , it is enough to find a pair (n , α_n) for which

$$\left(\frac{1 - \alpha_n^n}{1 - \alpha_n}\right) n^{-1} - N^{-1} \leq a .$$

Let n be an integer $> [N(Na + 1)^{-1}]$ and let $\alpha_n = \{n(aN+1) - N\}/N(n-1)$. Then the above inequality is satisfied. Observe that $n \to N$ as $a \to 0$ and $\alpha_n \to 0$ as $n \to N$.

If $B^{-1} = (a + \frac{1}{N})^{-1}$ is not an integer, it is always possible to find D_j with parameter α_j satisfying (1.16), such that $\phi_j \leq n_{oo}$, n_{oo} being determined by (1.1). This may be ensured by taking $j = n_{oo}$ and $\alpha \leq \alpha_o$, where $\psi_{n_{oo}}(\alpha_o) \leq B$. In Table 1 we present some numerical computations along this line.

Table 1

N	B^{-1}	n_{oo}	α_o	j	ϕ_j
5	2.1	3	.30	3	2.40
	2.4	3	.20	3	2.60
	2.7	3	.10	3	2.80
6	3.2	4	.20	4	3.40
	3.4	4	.15	4	3.55
	3.6	4	.10	4	3.70
	3.8	4	.05	4	3.85

(2) $\alpha_{ij} = bi,\ 1 \le i \le j \le N,\ 0 < b \le N^{-1}$.

Referring to (1.18), we obtain, for $1 \le k \le n \le N$,

$$p_{k,n} = (1-b)(1-2b)\ldots(1-\overline{k-1}b)b^{n-k} \times \left\{ \sum_{1 \le i_1 \le \ldots \le i_{k-1} \le n-1} \ldots \sum 1^{i_1-1}\, 2^{i_2-i_1-1} \ldots (k-1)^{i_{k-1}-i_{k-2}-1}\, k^{n-i_{k-1}-1} \right\}. \tag{2.7}$$

We now simplify the bracketted expression in the above as follows.

Let us consider the choice $b = N^{-1}$. Then according to Corollary 2,

$$p_{k,n} = \binom{N}{k} \Delta^k O^n / N^n, \quad 1 \le k \le n \le N . \tag{2.8}$$

Again, from the representation of $p_{k,n}$ in (2.7), putting $b = N^{-1}$, we obtain

$$p_{k,n} = \binom{N}{k} k!\, N^{-n} \left[\sum_{1 \le i_1 < \ldots < i_{k-1} \le n-1} \sum \ldots \sum 1^{i_1-1}\, 2^{i_2-i_1-1} \ldots (k-1)^{i_{k-1}-i_{k-2}-1}\, k^{n-i_{k-1}-1} \right]. \tag{2.9}$$

Equating (2.8) and (2.9), we obtain

$$\sum_{1 \le i_1 < \ldots i_{k-1} \le n-1} \ldots \sum 1^{i_1-1}\, 2^{i_2-i_1-1} \ldots (k-1)^{i_{k-1}-i_{k-2}-1}\, k^{n-i_{k-1}-1} = \frac{\Delta^k O^n}{k!} \text{ for } k \le n.$$

This yields, in the present context, from (2.7),

$$p_{k,n} = (1 - b)(1 - 2b) \dots (1 - \overline{k-1}b)b^{n-k} \Delta^k 0^n/k!$$

$$= \frac{c(c-1) \dots (c-k+1)}{k!} \Delta^k 0^n/c^n , \; c = b^{-1} . \qquad (2.11)$$

We will henceforth assume that $c(= b^{-1})$ is an integer. (See Remark 1 in this context). We now turn to the computations of ϕ and ψ in (1.12). We have

$$\phi_n = c^{-n+1} [\sum_{k=1}^{n} \{(c-1) \dots (c-k+1)/(k-1)!\} \Delta^k 0^n]$$

$$= c^{-n+1} [\sum_{k=1}^{n} \{ \frac{c(c-1) \dots (c-k+1)}{k!} - \frac{(c-1) \dots (c-k)}{k!} \} \Delta^k 0^n]$$

$$= c^{-n+1} [c^n - (c-1)^n]$$

$$= c[1 - (1-b)^n] \nearrow \text{ in } n, \text{ as expected.} \qquad (2.12)$$

Again,

$$c^n \psi_n = \sum_{k=1}^{n} \frac{c(c-1)\dots(c-k+1)}{k!} \cdot \frac{\Delta^k 0^n}{k} .$$

The above expression can be simplified in a routine manner. We can also utilize (1.3)(c) in this context to derive

$$c^n \psi_n = \sum_{i=1}^{c} i^{n-1} . \qquad (2.13)$$

It is again trivial to check that $\psi_n \downarrow$ in n. We now claim that irrespective of the choice of $c \geq N$ (c an integer), $\phi_{n-1} \psi_n < 1$. Towards this, observe that

$$\phi_{n-1} \psi_n = c\{1 - (1-b)^{n-1}\}b^n(c^{n-1} + (c-1)^{n-1} + \dots + 1) < 1$$

$$\Longleftrightarrow b^{n-1}(1 + 2^{n-1} + \dots + c^{n-1}) < \{1-(1-b)^{n-1}\}^{-1}$$

$$= \sum_{j=o}^{\infty} (1-b)^{(n-1)j} . \qquad (2.14)$$

Since $b < 1$, $(1-b)^j > 1 - jb$ and, hence,

$$\sum_{j=o}^{\infty} (1-b)^{(n-1)j} > \sum_{j=o}^{c} (1-b)^{(n-1)j} > \sum_{j=o}^{c} (1-jb)^{n-1} = b^{n-1} (\sum_{j=o}^{c} j^{n-1})$$

and this establishes (2.14). Clearly, then

$$\phi_{n-1}\,\psi_n < 1 \quad \text{for all} \quad n \le N .$$

Once again, we turn to the inequality in (1.16) and show that for given A, W, N and σ^2, there exists a range of values of b or, equivalently, values of c such that coherent sampling scheme with the parameter c satisfies (1.16) for some j . Utilizing (2.13), we have essentially to show tha for a given real number a > 0, there exists a pair (n, c_n) such that

$$\psi_n(c) - \frac{1}{N} \le a \quad \text{for all} \quad c \ge c_n .$$

Since $\psi_n(c)$ is decreasing in c, it is enough to find a pair (n, c_n) for which

$$\sum_{i=1}^{c_n} i^{n-1}/c_n^n - \frac{1}{N} \le a .$$

Observe that $\sum_{i=1}^{c_n} i^{n-1}/c_n^n < \frac{1}{c_n} + \frac{1}{n}$. Thus, a choice of n satisfying $n > [N(Na+1)^{-1}]$ and subsequently, a choice of c_n satisfying $c_n \ge \frac{Nn}{n(Na+1)-N}$ settles the claim. It is seen that $n \to N$ as $a \to 0$, and, consequently, $c_n \to \infty$.

If $B^{-1} = (a + \frac{1}{N})^{-1}$ is not an integer, it is always possible to find D_j with parameter c, satisfying (1.16), such that $\phi_j < n_{oo}$,n_{oo} being determined by (1.1). This may be ensured by taking $j = n_{oo}$ and $c \ge c_o$, where $\psi_j(c_o) \le B$. In Table 2 we present some numerical computations along this line.

Remark 1. It is true that in the above, we have been able to achieve a large class of coherent sequential structures even with the restriction that c($\ge$ N) is an integer. We believe this restriction is unnecessary and any choice of c($\ge$ N) will do. However, a completely satisfactory proof of this claim has, so far, elluded us.

Table 2

N	B^{-1}	n_{oo}	c_o	j	ϕ_j
4	1.60	2	4	2	1,75
	1.70	2	6	2	1.83
	1.80	2	10	2	1.90
	1.90	2	20	2	1.95
5	2.25	3	5	3	2.44
	2.55	3	10	3	2.71
	2.75	3	20	3	2.85
	2.85	3	30	3	2.90

Remark 2. It is seen that the choice $\alpha_{ij} = \frac{i}{N}$, $1 \leq i \leq j \leq N$ corresponds to the class of SRSWR sampling designs D_1^* , D_2^* , ... D_N^* where D_j^* is the one with sample size j, $1 \leq j \leq N$. The array **P** then takes the form

$$\mathbf{P} \equiv \mathbf{P}_o = ((\frac{\binom{N}{i} \Delta^i O^j}{N^j}))_{1 \leq i \leq j \leq N} .$$

SRSWR designs with arbitrary sample size n(> N) may be made to conform to coherent structures determined by the array

$$\mathbf{P}^* = ((p_{ij}^* = \frac{\binom{N}{i} \Delta^i O^j}{N^j}))_{1 \leq i \leq \min(N,j)}$$

which corresponds to the choice $\alpha_{ij} = \alpha_{ij}^*$ where

$$\alpha_{ij}^* = \begin{cases} \frac{i}{N} & \text{if } 1 \leq i \leq j \leq N \text{ or } 1 \leq i < N \leq j \\ 1 & \text{if } i = N < j . \end{cases}$$

Such an array extends itself horizontally after reaching the column j = N. For any j > N , we use the convention $\alpha_{Nj} = 1$ while still keeping other α_{ij}'s intact.

It is thus possible to generalize the earlier concept of a triangular array to an extended array using the convention $\alpha_{Nj} = 1$ for any j > N while other α_{ij}'s may be kept arbitrary. The general representation for $p_{k,n,}$ for any $n > N$, can be accordingly modified and the characterization results can also be accordingly extended.

Remark 3. We can visualize another family of coherent structures in the following manner. Suppose there are two coherent structures, P_1 and P_2 with α_{ij} as α_{1ij} and α_{2ij} respectively. Define a $(\theta, 1-\theta)$ combination as

$$P(\theta) = ((p_{\theta_{ij}}))$$

corresponding to the choice $\alpha_{ij} = \theta\alpha_{1ij} + (1-\theta)\alpha_{2ij}$. It is possible to spell out a range of values of θ, covering the extreme values 0 and 1, such that $P(\theta)$ admits a coherent structure.

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