EMBEDDING A STOCHASTIC DIFFERENCE EQUATION INTO A CONTINUOUS-TIME PROCESS

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A concept of divisibility is introduced for stochastic difference equations. Infinite divisibility then leads to a continuous time process in which a nested sequence of divisible stochastic difference equations can be embedded.

Stochastic difference equation * divisibility * infinite divisibility * embedding * continuous time

1. Introduction

Let the random sequence $\{Y_n\}_{n=0}^{\infty}$ be defined recursively as follows

$$Y_n = A_n Y_{n-1} + B_n \quad (n = 1, 2, \ldots). \tag{1.1}$$

The relation (1.1) is called a stochastic difference equation (SDE). In this paper we shall consider sequences $\{Y_n\}$ satisfying (1.1) with

$$\{A_n, B_n\}_{n=1}^{\infty} \text{ is an i.i.d. sequence,}$$
$$A_1 > 0 \quad \text{with probability one.} \tag{1.2}$$

An example is a stock of material checked at regular time intervals. A_n is the intrinsic decay or increase of the stock and B_n the quantity added or taken away just before time n. More examples are given in Vervaat (1979). Obviously for $k, n \geq 0$,

$$Y_{n+k} = \prod_{j=1}^{k} A_{n+j} Y_n + \sum_{r=1}^{k} \prod_{j=r+1}^{k} A_{n+j} B_{n+r}$$

(with $\prod_{j=n+1}^{n} := 1$), i.e.

$$Y_{n+k} = A^n_{n+k} Y_n + B^n_{n+k}, \tag{1.3}$$

where

$$(A^n_{n+k}, B^n_{n+k}) := \left(\prod_{j=1}^{k} A_{n+j}, \sum_{r=1}^{k} \prod_{j=r+1}^{k} A_{n+j} B_{n+r} \right). \tag{1.4}$$

In particular

$$Y_{2n} = A^{2(n-1)}_{2n} Y_{2(n-1)} + B^{2(n-1)}_{2n}$$

i.e. the process $\{Y_{2n}\}_{n=0}^{\infty}$ satisfies relations of the type (1.1) and (1.2). We say that the sequence $\{Y_n\}$ is divisible if there exists a sequence $\{Y^*_n\}$ satisfying relations of the type (1.1) and (1.2) such that

$$Y_n = Y^*_{2n} \quad \text{for } n = 0, 1, 2, \ldots.$$

Clearly not every sequence $\{Y_n\}$ is divisible. On the other hand such a property is attractive since e.g. in the case of the savings account restriction to a fixed time interval is unnatural.

If we require that the divisibility property continues to hold when one goes to smaller and smaller time intervals and if we pass from rationals to reals, we arrive—via (1.3) and (1.4)—at the following requirements for a continuous-time process $\{X_t\}_{t \geq 0}$ which is such that $\{X_{nh}\}_{n=0}^{\infty}$ satisfies relations of the type (1.1) and (1.2) for every $h > 0$.

$$X_t = A^s_t X_s + B^s_t \quad \text{for } 0 \leq s \leq t, \tag{1.5}$$

where $A^s_t > 0$ and B^s_t are random functionals satisfying:

(i) for $0 \leq s \leq u \leq t$ almost surely

$$A^s_t = A^s_u A^u_t, \qquad B^s_t = A^u_t B^s_u + B^u_t;$$

(ii) for $0 \leq a \leq b \leq c \leq d$ the families of random variables

$$\{(A^s_t, B^s_t) | a \leq s \leq t \leq b\} \quad \text{and} \quad \{(A^s_t, B^s_t) | c \leq s \leq t \leq d\} \tag{1.6}$$

are independent;

(iii) the distribution of

$$\{(A^{s+h}_{t+h}, B^{s+h}_{t+h})\}_{s,t}$$

does not depend on h.

An example (cf. Wolfe, 1982) is the Ornstein–Uhlenbeck process

$$X_t := e^{-\rho t} X_0 + \int_0^t e^{-\rho(t-u)} \, dW_u$$

with ρ a positive constant and W_t Brownian motion. This process satisfies (1.5) and (1.6) with $A^s_t = e^{-\rho(t-s)}$ and $B^s_t = \int_s^t e^{-\rho(t-u)} \, dW_u$.

In Section 2 we characterize the process $\{X_t\}$ satisfying (1.5) and (1.6) and obtain a representation of $\{X_t\}$ in terms of a two-dimensional process with stationary and independent increments. As a consequence we can answer the question under what conditions on the distribution of (A_1, B_1) from (1.1) and (1.2) one can embed the sequence $\{Y_n\}$ into a continuous-time process with the stated properties. It seems natural, even when one deals with a discrete-time problem, to use a pair (A_1, B_1) compatible with a continuous-time model.

In Section 3 we obtain a stationary solution of (1.5)—extended to the entire real line—under appropriate conditions.

The existence of a stationary distribution for the related stochastic differential equation (2.3) below has been studied by Zabczyk (1983) and Jacod (1985) for the special case $dR_t = dt$.

2. The continuous time process

All the processes we consider are defined on a fixed probability space $(\Omega, \mathscr{A}, P)$.

Let us fix $\{A_t^s, B_t^s\}$ satisfying (1.6). It is easy to see that then $\{X_t\}$ given by (1.5) is a Markov process with stationary transition probabilities. Of course, $X_0, \{A_t^s, B_t^s\}$ uniquely determine $\{X_t\}$. We are now going to obtain a representation of $\{A_t^s, B_t^s\}$ and hence $\{X_t\}$ in terms of an $\mathbb{R}^2$-valued process with stationary independent increments.

We will assume, in addition, that

$$A_t := A_t^0 \to 1 \quad \text{and} \quad B_t := B_t^0 \to 0 \qquad \text{in probability as } t \downarrow 0, \tag{2.1}$$

$$A_t^0 > 0 \quad \text{a.s.} \quad \text{for every } t. \tag{2.2}$$

Our first result is on path properties of $\{A_t\}, \{B_t\}$.

Lemma 2.1. *Let $\{A_t^s, B_t^s\}$ satisfy* (1.6), (2.1) *and* (2.2). *Then $\{A_t\}, \{B_t\}$ admit RCLL (right continuous with left-hand limits) modifications.*

Proof. (1.6) and (2.1) together imply that $\{A_t\}, \{B_t\}$ are continuous in probability. Further, $\{M_t\} := \{-\log A_t\}$ is a process with stationary independent increments.

This yields existence of a RCLL modification of $\{A_t\}$. See e.g. Theorem 14.20 in Breiman (1968). We now prove the result for $\{B_t\}$ by modifying the arguments in the proof of the Theorem referred to above. Fix T. For a continuous strictly increasing function ψ such that $\lim_{x \to -\infty} \psi(x) = -\alpha$, $\lim_{x \to \infty} \psi(x) = \alpha$, $0 < \alpha < \infty$, define

$$Y_t = E[\psi(B_T)|(A_s, B_s), s \leq t].$$

Then, using (1.6), it follows that

$$Y_t = \theta(T - t, B_t)$$

where

$$\theta(s, x) = E[\psi(A_s x + B_s)].$$

Using continuity in probability of $\{A_s, B_s\}$, it follows that $\theta(s, x)$ is continuous in s for each x. The rest of the arguments are exactly as in the proof of Theorem 14.20 in Breiman (1968). □

From now on A_t, B_t will refer to this RCLL version. This and (1.6) yield the existence of a RCLL version of (A_t^s, B_t^s) as well. This also shows that X_t defined by (1.5) is RCLL.

For $a < b$ let $\mathcal{F}_b^a$ denote the smallest σ-field which contains all P-null sets and with respect to which $\{(A_t^s, B_t^s): a \le s \le t \le b\}$ is measurable.

Our first result is the following.

Theorem 2.2. *Suppose* (A_t^s, B_t^s) *satisfy* (1.6), (2.1), (2.2). *Then,*

(i) $M_t = -\log A_t$ *is a process with stationary independent increments w.r.t.* $\{\mathcal{F}_t\}$ *(by this we mean M_t is $\mathcal{F}_t$ adapted, $M_t - M_s$ is independent of $\mathcal{F}_s$ and M_t has stationary increments). As a consequence, M_t, A_t are $\mathcal{F}_t$-semimartingales.*

(ii) *Let* $R_t = \int_0^t A_{s-}^{-1} \, dA_s$. *Then R_t is a process with stationary independent increments w.r.t.* $\{\mathcal{F}_t\}$.

(iii) $N_t := B_t - \int_0^t B_{s-}(A_{s-})^{-1} \, dA_s$ *is a process with stationary independent increments w.r.t.* $\{\mathcal{F}_t\}$.

Here and in the sequel, $\int_s^t f_u \, dZ_u$ stands for the stochastic integral $\int 1_{(s,t]}(u) f_u \, dZ_u$, where Z_u is a semimartingale and f_u is a predictable process.

Proof. (i) is a direct consequence of (1.6). The semimartingale property of M_t (and hence A_t) follows from Jacod (1979, p. 63).

(ii) follows from (1.6) and the fact that for $s < t$, $R_t - R_s$ is limit in probability of

$$\sum_{j=0}^{k-1} (A_{t_j})^{-1}[A_{t_{j+1}} - A_{t_j}] = \sum_{j=0}^{k-1} [A_{t_{j+1}}^{t_j} - 1],$$

where $s = t_0 < \cdots < t_k = t$ and limit is taken as $\sup_j |t_{j+1} - t_j| \to 0$.

For (iii) note that

$$
\begin{aligned}
N_t - N_s &= B_t - B_s - \int_s^t B_{u-}(A_{u-})^{-1} \, dA_u \\
&= B_t^0 - B_s^0 - \int_s^t B_{u-}^0 (A_{u-})^{-1} \, dA_u \\
&= B_s^0(A_t^s - 1) + B_t^s - \int_s^t (B_s^0 A_{u-}^s + B_{u-}^s)(A_{u-})^{-1} \, dA_u \\
&= B_s^0 \left\{ A_t^s - 1 - \int_s^t (A_s^0)^{-1} \, dA_u \right\} + B_t^s - \int_s^t B_{u-}^s (A_{u-})^{-1} \, dA_u \\
&= B_s^0 \{ A_t^s - 1 - (A_s^0)^{-1}(A_t - A_s) \} + B_t^s - \int_s^t B_{u-}^s (A_{u-})^{-1} \, dA_u \\
&= B_t^s - \int_s^t B_{u-}^s (A_{u-})^{-1} \, dA_u.
\end{aligned}
$$

For s fixed, $\{A_t^s\}_{t \geq s}$ is a $\mathcal{F}_t^s$-semimartingale and it is easily checked (using Riemann sum approximation of the integral) that

$$\int_s^t B_u^s \, (A_{u-})^{-1} \, dA_u = \int_s^t B_{u-}^s \cdot (A_{u-}^s)^{-1} \, dA_u^s.$$

Hence

$$N_t - N_s = B_t^s - \int_s^t B_{u-}^s \cdot (A_{u-}^s)^{-1} \, dA_u^s.$$

Thus, $N_t - N_s$ is $\mathcal{F}_t^s$-measurable, and hence independent of $\mathcal{F}_s$. Stationarity of the increments of N follows from (1.6) (iii). This completes the proof. $\square$

Note that $A_0 = 1$, $B_0 = 0$ implies that $M_0 = 0 = N_0$.

Corollary 2.3. (i) A_t, B_t, M_t, N_t are $\mathcal{F}_t$-semimartingales.

(ii) *Suppose X_0 is independent of $\mathcal{F}_\infty^0$. Let $\mathcal{G}_t$ be $\sigma(X_0, \mathcal{F}_t)$. Let X_t be defined by* (1.5). *Then X_t, A_t, B_t, M_t, N_t are $\mathcal{G}_t$-semimartingales.* $\square$

R_t and M_t are related via (see Jacod, 1979, p. 190),

$$R_t = -M_t + \tfrac{1}{2}\langle M^c, M^c \rangle_t + \sum_{s \leq t} [\Delta M_s + \{\exp(-\Delta M_s) - 1\}].$$

We now show that the process X_t satisfies an SDE driven by $\{R_t, N_t\}$.

Theorem 2.4. *Let X_t be given by* (1.5) *where (A_t^s, B_t^s) satisfy* (1.6), (2.1), (2.2). *Then X_t is the unique solution to the stochastic differential equation*

$$dX_t = X_{t-} \, dR_t + dN_t \tag{2.3}$$

where R_t, N_t are as in Theorem 2.2.

Proof.

$$dX_t = X_0 \, dA_t + dB_t = X_0 A_{t-} \, dR_t + dN_t + B_{t-} A_{t-}^{-1} \, dA_t$$

$$= (X_0 A_{t-} + B_{t-}) \, dR_t + dN_t = X_{t-} \, dR_t + dN_t.$$

Uniqueness of the solution follows from standard results on SDE. $\square$

We will now obtain explicit expressions for X_t, A_t^s, B_t^s in terms of processes $\{R_t, N_t\}$.

Theorem 2.5. (i) *Let $U_t := A_t^{-1}$ and $V_t := \int_0^t U_{s-}^{-1} \, dU_s$. Then V_t is a process with stationary independent increments w.r.t. $\{\mathcal{F}_t\}$ and*

$$V_t = M_t + \tfrac{1}{2}\langle M^c, M^c \rangle_t + \sum_{s \leq t} [-\Delta M_s + \{\exp(\Delta M_s) - 1\}].$$

(ii) *Let* $S_t := N_t + [V, N]_t$. *Then* S_t *is a process with stationary independent increments w.r.t.* $\{\mathcal{F}_t\}$.

(iii) $N_t = S_t + [R, S]_t$.

(iv) $B_t = A_t \cdot \int_0^t (A_{s-})^{-1} \, dS_s = \exp(-M_t) \int_0^t \exp(M_{s-}) \, dS_s$.

Proof. (i) follows by arguments similar to those in the proof of Theorem 2.2. For (ii) we need to prove that $[V, N]_t$ is a process with stationary independent increments w.r.t. $\{\mathcal{F}_t\}$, given that V_t, N_t have the same property. This follows from the fact that $[V, N]_t - [V, N]_s$ is limit in probability of

$$\sum_{j=0}^{k-1} (V_{t_{j+1}} + N_{t_{j+1}} - V_{t_j} - N_{t_j})^2 - (V_{t_{j+1}} - N_{t_{j+1}} - V_{t_j} + N_{t_j})^2,$$

where $s = t_0 < \cdots < t_k = t$ and limit is taken as $\sup_j |t_{j+1} - t_j| \to 0$ (see Jacod, 1979, p. 34, 37). (iii) can be deduced from the relations expressing R_t, V_t in terms of M_t. From the definition of R, N, S, it follows that

$$B_t = \int_0^t B_{s-} \, dR_s + S_t + [R, S]_t.$$

This is an integral equation expressing B_t in terms of (R_t, S_t). It is well known that this equation (or the corresponding stochastic differential equation) admits a unique solution. It is easy to check, via Itô's formula or integration by parts formula that

$$A_t \cdot \int_0^t (A_{s-})^{-1} \, dS_s$$

satisfies the equation and hence

$$B_t = A_t \cdot \int_0^t (A_{s-})^{-1} \, dS_s.$$

This completes the proof.   □

These two results together yield the following result.

Theorem 2.6. *Suppose* $\{A_t^s, B_t^s\}$ *satisfy* (1.6), (2.1), (2.2). *Then there exist processes* $\{M_t\}, \{S_t\}$ *with stationary independent increments w.r.t.* $\{\mathcal{F}_t\}$ *such that*

$$A_t^s = \exp(-(M_t - M_s)). \tag{2.4}$$

$$B_t^s = \exp(-M_t) \int_s^t \exp(M_{u-}) \, dS_u. \tag{2.5}$$

and then the process X_t *defined by* (1.5) *can be expressed as*

$$X_t = X_0 \exp(-M_t) + \exp(-M_t) \int_0^t \exp(M_{u-}) \, dS_u. \tag{2.6}$$

Note that $A_0 = 1$, $B_0 = 0$ *and, by definition,* $M_0 = S_0 = R_0 = N_0 = 0$.

Conversely, suppose that the $\mathbb{R}^2$-*valued process* $\{M_t, S_t\}$ *has stationary independent increments. Then* $\{A_t^s, B_t^s\}$ *defined by* (2.4), (2.5) *satisfy* (1.6) *and* X_t *defined by* (2.6) *satisfies* (1.5) *for this choice.* $\quad\square$

Example. Let $\{M_t\}$ be a homogeneous Poisson process and $S_t :=$ $Y_1 + \cdots + Y_{M_t}$ $(t \geqslant 0)$ with $Y_1, Y_2, \ldots$, i.i.d. standard normal and independent of $\{M_t\}$. Then the process $\{X_t\}$ is constant except for the jump epochs of $\{M_t\}$. Moreover the process

$$Y_n := X_{N_n}$$

$(n = 1, 2, \ldots)$ with $\{N_n\}$ the jump epochs of $\{M_t\}$ satisfies

$$Y_{n+1} = \rho Y_n + U_n$$

with $0 < \rho < 1$ and $U_1, U_2, \ldots$, i.i.d. standard normal. Clearly a stationary distribution exists and is normal as well.

Remark. One sees that the functionals $\{A_t^s\}$ and $\{B_t^s\}$ cannot be independent except in trivial cases. Note that in contrast the case of independent A_n and B_n (in (1.1)) plays an important role in Vervaat's (1979) paper.

Remark. The process M, R, V, N, S introduced earlier in this section can be described directly in terms of (A_t^s, B_t^s) as follows. In the statements given below, $0 = t_0 < t_1 < \cdots < t_j = T$ is an arbitrary partition of $[0, T]$, the limit is to be understood as limit in probability uniformly in t and the limit is taken as the maximum width $\sup_i (t_{i+1} - t_i)$ goes to zero.

(i) $\quad \sum\limits_{i: t_i < t} -\log(A_{t_{i+1}}^{t_i}) \to M_t.$

(ii) $\quad \sum\limits_{i: t_i < t} (A_{t_{i+1}}^{t_i} - 1) \to R_t.$

(iii) $\quad \sum\limits_{i: t_i < t} ((A_{t_{i+1}}^{t_i})^{-1} - 1) \to V_t.$

(iv) $\quad \sum\limits_{i: t_i < t} B_{t_{i+1}}^{t_i} \to N_t.$

(v) $\quad \sum\limits_{i: t_i < t} (A_{t_{i+1}}^{t_i})^{-1} B_{t_{i+1}}^{t_i} \to S_t.$

These statements can be proved using the representation of (A_t^s, B_t^s) obtained in the previous theorem and results in Emery (1978). It may be noted that (iv) cannot be used as definition of N_t because we do not have a direct proof that $\sum B_{t_{i+1}}^{t_i}$ converges.

When $R_t = \rho t$ and $\{N_t\}$ is Brownian motion, $\{X_t\}$ is the well known Ornstein–Uhlenbeck process. In view of the similarity of the SDE (2.3) and the solution with Ornstein–Uhlenbeck process, a process $\{X_t\}$ which is a solution to (2.3) or equivalently given by (2.6) can be called a generalized Ornstein–Uhlenbeck process.

3. Stationarity

In this section we examine as to when does X_t converge in distribution (as $t \to \infty$). If it does, and the limit does not depend on X_0, then it follows that the limit distribution is a stationary initial distribution for the Markov process $\{X_t\}$.

Our first result is:

Theorem 3.1. *Let $\{A_t^s, B_t^s\}$ satisfy* (1.6), (2.1), (2.2). *Further suppose that*

$$E[\log A_1^0] < 0 \tag{3.1}$$

and

$$E[\log^+ |B_1^0|] < \infty. \tag{3.2}$$

Then for all X_0, X_t defined by (1.5) *converges in distribution to a probability measure μ (not depending on X_0). Further, μ is a stationary initial distribution for the Markov process $\{X_t\}$.*

Proof. Since $X_{n+1} = X_n \cdot A_{n+1}^n + B_{n+1}^n$; $n \geq 1$ and $\{A_{i+1}^i, B_{i+1}^i\}$ is an i.i.d. sequence, $X_n \xrightarrow{d} \mu$, where μ does not depend on X_0. This follows from results in Vervaat (1979), and can be proved directly using strong law of large numbers and Kolmogorov's three series theorem. Condition (3.1) implies $A_t^0 \xrightarrow{P} 0$ and hence it follows that $B_n^0 \xrightarrow{d} \mu$. Let $t_i \in \mathbb{R}^+$, $t_i \to \infty$ be arbitrary. We need to prove that $X_{t_i} \xrightarrow{d} \mu$. Let n_i be integers such that $n_i \leq t_i < n_i + 1$, and $u_i = t_i - n_i$. Then $n_i \to \infty$ as well. Now

$$B_{t_i}^0 - B_{t_i}^{u_i} = B_{u_i}^0 A_{t_i}^{u_i} \tag{3.3}$$

where $B_{t_i}^0$ and $A_{t_i}^{u_i}$ are independent. As noted earlier, $A_n^0 \xrightarrow{P} 0$ and hence $A_{t_i}^{u_i} \overset{d}{=} A_{n_i}^0 \xrightarrow{P} 0$. On the other hand $|B_{u_i}^0| \leq \sup_{0 \leq u \leq 1} |B_u^0| < \infty$ as paths of B_u^0 are RCLL. Hence

$$B_{t_i}^0 - B_{t_i}^{u_i} \xrightarrow{P} 0. \tag{3.4}$$

But $B_{t_i}^{u_i} \overset{d}{=} B_{n_i}^0$ and hence (3.4) gives $B_{t_i}^0 \xrightarrow{d} \mu$.

This along with $A_{t_i}^0 \xrightarrow{P} 0$ yields

$$X_{t_i} \xrightarrow{d} \mu.$$

That μ is a stationary initial distribution follows by standard arguments. $\square$

The previous result gives sufficient conditions on $\{A_t^s, B_t^s\}$ for existence of a stationary solution to (1.5). It would be good to obtain conditions on $\{M_t, S_t\}$ instead, in view of the representation (2.5).

The first condition (3.1) easily translates as

$$EM_1 > 0. \tag{3.1}'$$

It seems reasonable to expect that in presence of (3.1)',

$$E \log^+ |S_1| < \infty \tag{3.5}$$

would imply (3.2). However we are unable to prove this. We will prove that (3.5) and $E|M_1| < \infty$ together imply (3.2). This will be done in several steps.

Proposition 3.2. *Let* $\{Y_t\}$ *be a local* L^2-*martingale with* $Y_0 = 0$. *Suppose that* $E[\log'\langle Y\rangle_t] < \infty$. *Then*

$$E\left[\log'\left(\sup_{s>t}|Y_s|\right)\right] < \infty. \tag{3.6}$$

Here, $\{\langle Y\rangle_t\}$ *denotes the unique predictable increasing process with* $\langle Y\rangle_0 = 0$ *such that* $\{Y_t^2 - \langle Y\rangle_t\}$ *is a local martingale.*

Proof. Follows from Yor (1979) by noting that $\langle Y\rangle_t$ dominates $|Y|_t^2$ in the sense

$$E|Y_T|^2 \le E\langle Y\rangle_T$$

for all stop times T and that if $\Phi(x) = \log^+ x$, then

$$F(x) = x\int_x^\cdot \frac{1}{u}\,\mathrm{d}\Phi(u) \le 1. \quad \square$$

Proposition 3.3. *Let* $\{S_t\}$ *be a process with stationary independent increments, with RCLL paths. Let*

$$S_t' := \sum_{u \le t} \Delta S_u \cdot 1_{\{\Delta S_u > 1\}}, \qquad S_t'' := -\sum_{u \le t} \Delta S_u \cdot 1_{\{\Delta S_u < -1\}}$$

and

$$\tilde{S}_t = S_t - S_t' + S_t''.$$

(Here, $\Delta S_u = S_u - S_{u-} = S_u - \lim_{t \to u, t < u} S_t$.*) Then we have:*

(i) $\{S_t'\}, \{S_t''\}, \{\tilde{S}_t\}$ *are processes with stationary independent increments and the three processes are independent of each other.*

(ii) $E\tilde{S}_t^2 < \infty$.

(iii) $E\tilde{S}_t = at$ *and* $E(\tilde{S}_t - at)^2 = bt$ *for some* a, b.

(iv) $\hat{S}_t := \tilde{S}_t - at$ *is a square integrable martingale with* $\langle \hat{S}\rangle_t = bt$.

(v) *If* $E\log^+|S_t| < \infty$, *then* $E\log^+|S_t'| < \infty$, $E\log^+|S_t''| < \infty$.

Proof. (i) is an easy consequence of the Levy-Khincin formula for the characteristic function of $\{S_t\}$. (ii) follows from the observation that jumps of $\tilde{S}_t$ are bounded by 1, see Ramachandran (1969). (iii) and (iv) are easy consequences of (i) and (ii). (v) follows from independence of S_t', S_t'' and $\tilde{S}_t$ and the inequality

$$\log^+|x + y| \le 1 + \log^+|x| + \log^+|y|. \quad \square$$

Remark. Suppose M_t is a process with stationary independent increments, with $E|M_t| < \infty$. Then decomposing M_t into M_t', M_t'' and $\tilde{M}_t$ as above, one gets $EM_t' < \infty$, $EM_t'' < \infty$. It is easy to see from (iv) that $E\sup_{s \le t}|\tilde{M}_s| < \infty$. This and the fact that M_t', M_t'' are increasing processes yields

$$E\sup_{s \le t}|M_s| \le E\sup_{s \le t}|\tilde{M}_s| + E|M_t'| + E|M_t''| < \infty. \tag{3.7}$$

Our main result follows.

Theorem 3.4. *Let (A_t^s, B_t^s) be given by (2.3), (2.4) where $\{M_t, S_t\}$ is a process with stationary independent increments. Suppose*

$$E|M_1| < \infty \quad and \quad EM_1 > 0, \tag{3.8}$$

$$E \log^+ |S_1| < \infty. \tag{3.9}$$

Then conditions (3.1), (3.2) of Theorem 3.1 hold.

Proof. That (3.8) implies (3.1) is obvious. Also one has

$$E \sup_{s \le 1} |M_s| < \infty \tag{3.10}$$

(see Remark above). Since

$$B_1^0 = e^{-M_1} \cdot \int_0^1 e^{M_u-} \, dS_u \quad and \quad \log^+ xy \le \log^+ x + \log^+ y,$$

(3.2) would follow if we show that

$$E \log^+ \left| \int_0^1 e^{M_u-} \, dS_u \right| < \infty. \tag{3.11}$$

Let us write $f_u = e^{M_u-}$ and S_t', S_t'', $\tilde{S}_t$, $\hat{S}_t$, a, b be as in Proposition 3.3. Then the fact that S_t', S_t'' are increasing processes yields

$$\log^+ \left| \int_0^1 f_u \, dS_u' \right| \le 1 + \sup_{s \le 1} |M_s| + \log^+ |S_1'|,$$

$$\log^+ \left| \int_0^1 f_u \, dS_u'' \right| \le 1 + \sup_{s \le 1} |M_s| + \log^+ |S_1''|,$$

$$\log^+ \left| \int_0^1 f_u \, du \right| \le 1 + \sup_{s \le 1} |M_s|.$$

Hence it suffices to show that

$$E \log^+ \left| \int_0^1 f_u \, d\hat{S}_u \right| < \infty. \tag{3.12}$$

Since $\hat{S}_t$ is an L^2-martingale with $\langle \hat{S} \rangle_t = bt$, it follows that $Y_t := \int_0^t f_u \, d\hat{S}_u$ is a local L^2-martingale with $\langle Y \rangle_t = b \cdot \int_0^t f_u^2 \, du$. Hence

$$E \log^+ \langle Y \rangle_1 \le 1 + \log^+ b + E \log^+ \sup_{u \le 1} f_u^2$$

$$\le 1 + \log^+ b + 2E \sup_{u \le 1} |M_u| < \infty.$$

Proposition 3.2 implies (3.12) completing the proof.

Remark. Let ν, γ be the Lévy-measure of the processes M_t, S_t respectively. It is proved in Ramachandran (1969) that

$$E|M| < \infty \quad \text{iff} \quad \int_{|x|>1} |x| \, d\nu(x) < \infty.$$

Thus the condition on M can be easily stated in terms of the characteristic function for M_1. Similarly it is proved in the paper cited above that for $\alpha > 0$

$$E|S_1|^\alpha < \infty \quad \text{iff} \quad \int_{|x|>1} |x|^\alpha \, d\gamma(x) < \infty.$$

Thus, we get that $\int |x|^\alpha \, d\gamma(x) < \infty$ for some $\alpha > 0$ implies $E \log^+ |S_1| < \infty$.

References

L. Breiman, Probability (Addison-Wesley, Reading, MA, 1968).

M. Emery, Stability of solutions of stochastic differential equations from multiplicative stochastic integrals (French), Z. Wahrsch. Verw. Gebiete 41 (1978) 241–262.

J. Jacod, Calcul stochastique et problèmes de martingales, Lecture Notes in Math. 714 (Springer, Berlin, 1979).

J. Jacod, Grossissement de filtration et processus d'Ornstein–Uhlenbeck géneralisés, Lecture Notes in Math. 1118 (Springer, Berlin, 1985) 36–44.

R.L. Karandikar, Stochastic integration w.r.t. continuous local martingales, Stochastic Process. Appl. 15 (1983) 203–209.

B. Ramachandran, On characteristic functions and moments, Sankhyā Ser. A 31 (1969), 1–12.

W. Vervaat, On a stochastic difference equation and a representation of non-negative infinitely divisible random variables, Adv. in Appl. Probab. 11 (1979) 750–783.

S.J. Wolfe, On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$, Stochastic Process Appl. 12 (1982) 301–312.

M. Yor, Les inégalités de sous-martingales, comme conséquences de la relation de domination, Stochastics 3 (1979) 1–15.

J. Zabczyk, Stationary distribution for linear equations driven by general noise, Bull. Polish Acad. Sci. Math. 31 (1983) 197–209.