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ON PROBABILITIES OF MODERATE DEVIATIONS FOR DEPENDENT PROCESSES

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SUMMARY. Probabilities of moderate deviations of the sample mean are obtained for linear processes. The same of sample quantiles and of suitable L-estimators are obtained for mixing processes.

1. Introduction

Recently, there has been some interest in developing the theory for probabilities of moderate deviations (PMD) for dependent processes (See e.g., Ghosh and Babu (1977), Babu, Ghosh and Singh (1978) and Babu and Singh (1978)*. In the present paper we obtain expressions for PMD of sample mean for a statistically important process, namely linear process, which is not covered by the previous results. Also we utilise the PMD results known for sample mean to obtain the same for sample quantiles and L-estimators making use of the idea of asymptotic representation of quantiles.

Let $\{X_n\}$ be a sequence of r.v.'s defined on a probability space (Ω, A, P). Let M_a^b denote the σ-field generated by $X_i (a \leqslant i \leqslant b)$. $\{X_n\}$ is called ϕ-mixing if

$$\sup_{k \geqslant 1} \sup_{A \in M_{-\infty}^k} \sup_{B \in M_{k+n}^\infty} |P(B|A)-P(B)| \leqslant \phi(n), P(A) > 0 \qquad \ldots \quad (1.1)$$

for some non-increasing sequence of real numbers $\{\phi(i)\}$, with limit zero. The process is called strong-mixing if for some sequence of positive numbers $\{\alpha(n)\}$ with limit zero,

$$\sup_{k \geqslant 1} \sup_{A \in M_{-\infty}^k} \sup_{B \in M_{k+n}^\infty} |P(A \cap B)-P(A)\,P(B)| \leqslant \alpha(n) \qquad \ldots \quad (1.2)$$

and it is called a linear process if for a sequence of real numbers $\{a_i\}$ with $\Sigma a_i^2 < \infty$ and $\{\xi_n\}$, some pure white noise process,

$$X_n = \sum_{i=1}^{\infty} a_i \, \xi_{n-i+1} \qquad \ldots \quad (1.3)$$

or

$$X_n = \sum_{i=1}^{\infty} a_i \, \xi_{n+i-1} \qquad \ldots \quad (1.4)$$

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* Published in this issue, see pp. 38–43.

In Ghosh and Babu (1977) and Babu, Ghosh and Singh (1978), the expressions for PMD of sample mean are obtained for ϕ-mixing processes with

$$\Sigma\phi^{\frac{1}{2}}(i) < \infty \quad \ldots \quad (1.5)$$

and in Babu and Singh (1978) the result is established for strong-mixing processes with $\{\alpha(n)\}$ satisfying the condition

$$\alpha(n) = O(\exp(-\lambda n)), \ \lambda > 0. \quad \ldots \quad (1.6)$$

It is known that linear processes are not ϕ-mixing in general even for exponentially decaying a_i's (see Chanda, 1976). However, under some restrictions, linear processes are strong-mixing with $\alpha(i)$ depending on a_i's (see Chanda, 1976 and Goredtski, 1977). But unfortunately, according to the existing knowledge, the condition (1.6) is met only for exponentially decaying a_i's. In the present paper, we exploit the particular structure of linear processes to get exact asymptotic expressions for PMD assuming only

$$\sum_i i\ |a_i| < \infty. \quad \ldots \quad (1.7)$$

Section 3, deals with the PMD results of quantiles and L-estimators for the mixing processes described above. The L-estimators are defined in Section 3. The result does not appear to be known in literature even for the independent case.

2. PMD for linear processes

In this section we prove the following theorem

Theorem 2.1 : *Let $\{X_n\}$ be a linear process with a_i satisfying* (1.7). *If for some $c > 0$, $E\ |\xi_1|^{c^2+2} < \infty$ and $\sum_1^{\infty} a_i = z \neq 0$, one has, for $S_n = \sum_1^n X_i$ and $a^2 = V(\xi_1)$,*

$$P(S_n - E(S_n) > c|az|(n \log n)^{\frac{1}{2}}) \sim (2\pi c^2 \log n)^{-\frac{1}{2}}\, n^{-c^2/2} \quad \ldots \quad (2.1)$$

$$P(|S_n - E(S_n)| > c|az|(n \log n)^{\frac{1}{2}}) \sim 2(2\pi c^2 \log n)^{-\frac{1}{2}} n^{-c^2/2} \quad \ldots \quad (2.2)$$

In the proof of the above theorem, we shall make use of the following result, which follows from Theorem 5 of Babu, Ghosh and Singh (1978).

Theorem 2.2 : *Let $\{Y_n\}$ be a (possibly non-stationary) sequence of independent r.v.'s. Define $S_n^* = \sum_1^n Y_i$ and $b_n^2 = \frac{1}{n} V(S_n^*)$. If $\liminf_{n\to\infty} b_n > 0$ and $\sup_{n\geq 1} E|Y_n|^{c^2+2} \leqslant W < \infty$ for some $c > 0$, one has for any $\delta > 0$,*

$$P(S_n^* - E(S_n^*) > c(b_n \pm n^{-\delta})(n \log n)^{\frac{1}{2}}) \sim (2\pi c^2 \log n)^{-\frac{1}{2}} n^{-c^2/2}.$$

Proof of Theorem 2.1 : We shall prove the theorem for the process $\{X_n\}$ satisfying (1.3) and the proof for the second case is similar. Clearly we need to prove only (2.1) since (2.2) follows from (2.1) by symmetry. From now on, it is assumed without any loss of generality that $E\xi_1^2 = 1$ and $E\xi_1 = 0$ which, in view of (1.7) and the fact that $z \neq 0$, is equivalent to assuming $EX_1 = 0$.

We shall prove first that

$$P(S_n > c|z_n|(n \log n)^{\frac{1}{2}}) \sim (2\pi c^2 \log n)^{-\frac{1}{2}} n^{-c^2/2} \qquad \dots \quad (2.3)$$

where $z_n^2 = \frac{1}{n} \sum_{i=1}^{n} t_i^2$ and $t_n = \sum_{j=1}^{n} a_j$, and then show that (2.3) holds with z_n replaced by z.

Towards this end, we define

$$X_{m,n} = \sum_{i=1}^{m} a_i \xi_{n-i+1}, \quad m \geqslant 1$$

and decompose S_n as

$$S_n = \sum_{i=1}^{n} X_{i,i} + \sum_{i=1}^{n} (X_i - X_{i,i}). \qquad \dots \quad (2.4)$$

We shall show that, for any $\delta > 0$

$$P\Big(\sum_{i=1}^{n} X_{i,i} > c|z_n|(1 \pm n^{-\delta})(n \log n)^{\frac{1}{2}} \Big) \sim (2\pi c^2 \log n)^{-\frac{1}{2}} n^{-c^2/2} \quad \dots \quad (2.5)$$

and for sufficiently small $\varepsilon > 0$

$$P\Big(\Big| \sum_{i=1}^{n} (X_i - X_{i,i}) \Big| > c|z_n| n^{-\varepsilon} (n \log n)^{\frac{1}{2}} \Big) = o((\log n)^{-\frac{1}{2}} n^{-c^2/2}) \quad \dots \quad (2.6)$$

It is easy to combine (2.5) and (2.6) to conclude (2.3).

To prove (2.5), let us observe that

$$S_n' = \sum_{i=1}^{n} X_{i,i} = \sum_{i=1}^{n} t_{n-i+1} \xi_i. \qquad \dots \quad (2.7)$$

But the r.h.s. of (2.7) has the same distribution as $\sum_{i=1}^{n} t_i \xi_i$, using the symmetry of the distribution of $(\xi_1, \xi_2, \dots, \xi_n)$. Also $V(S_n') = \sum_{1}^{n} t_i^2$. Thus by Cesaro mean convergence theorem $\lim_{n \to \infty} n^{-1} V(S_n') = z^2 > 0$. Now, (2.5) follows by taking $Y_i = t_i X_i$ in Theorem 2.2.

Coming to (2.6), it is easily seen, taking $\varepsilon = 1/2(c^2+2)$ and using Chebychev's inequality and Minkowsky's inequality, that for some absolute constant K_1,

$$\text{l.h.s. of (2.6)} \leqslant K_1 n^{\frac{1}{2}}(n \log n)^{-\frac{1}{2}(c^2+2)} E\left|\sum_{i=1}^{n}(X_i - X_{i,i})\right|^{c^2+2}$$

$$\leqslant K_1(\log n)^{-\frac{1}{2}(c^2+2)}\, n^{-c/2-\frac{1}{2}}\left(\sum_{i=1}^{\infty} i|a_i|\right)^{c^2+2} E|\xi_1|^{c^2+2}$$

$$= O(n^{-\frac{1}{2}c^2-\frac{1}{2}}). \qquad \dots \quad (2.8)$$

The last statement is due to (1.7).

To replace $|z_n|$ by $|z|$ in (2.3), we only need to show in view of Theorem 2.2.,

$$|z_n^2 - z^2| = O(n^{-\delta}), \text{ for some } \delta > 0.$$

We prove it as follows

$$|z_n^2 - z^2| \leqslant n^{-1}\sum_{i=1}^{n}|t_i^2 - z^2| \leqslant n^{-1}\sum_{i=1}^{n}|t_i - z|\,|t_i + z|$$

$$\leqslant \left(2\sum_{i=1}^{\infty}|a_i|\right)n^{-1}\left(\sum_{i=1}^{\infty} i|a_i|\right) = O(n^{-1}).$$

The proof of Theorem 2.1 is complete.

Remark 2.1 : It can be verified that (2.5) remains valid under the condition

$$a_i = O\left(i^{\frac{1}{c^2+2}-2-\theta}\right), \theta > 0 \qquad \dots \quad (2.9)$$

for fixed $c > 0$ and hence $a_i = O(i^{-2})$ suffices for all $c > 0$. (2.9) can be taken as an alternative condition to (1.7).

Remark 2.2 : In view of the results of Michel (1974), it appears that the moment condition assumed in Theorem 2.1 is the best possible.

Remark 2.3 : Since Theorem 2.2 remains valid when $\{Y_n\}$ is a ϕ-mixing process, a modified form of Theorem 2.1 can be proved even if $\{\xi_n\}$ is a ϕ-mixing process instead of being pure white noise process.

3. PMD OF QUANTILES AND L-ESTIMATORS

For the process $\{X_n\}$, we define empirical distribution function $F_n(x)$ as

$$F_n(x) = \frac{1}{n}(\# X_i,\ 1 \leqslant i \leqslant n)$$

and for a real number p, $0 < p < 1$, the p-th sample quantile Q_n is defined as

$$Q_n = Q_n(p) = \inf\{x : F_n(x) \geqslant p\}.$$

Throughout this section we assume that the process $\{X_n\}$ is strictly stationary with X_1 having the distribution F. Let Q denote the p-th quantile of F.

We now state our result about quantiles formally.

Theorem 3.1 : *If the process $\{X_n\}$ satisfies either of the conditions*

(i) *$\{X_n\}$ is ϕ-mixing with $\{\phi(i)\}$ satisfying* (1.5)

(ii) *$\{X_n\}$ is strong-mixing with $\{\alpha(i)\}$ satisfying* (1.6)

and F has bounded second derivative in a neighbourhood of Q with $f = F'(Q) > 0$, then one has, for a fixed $c > 0$,

$$P(Q_n - Q > f^{-1}c\sigma(\log n/n)^{1/2}) \sim (2\pi c^2 \log n)^{-1/2} n^{-c^2/2} \quad \ldots \quad (3.1)$$

$$P(|Q_n - Q| > f^{-1}c\sigma(\log n/n)^{1/2}) \sim 2(2\pi c^2 \log n)^{-1/2} n^{-c^2/2} \quad \ldots \quad (3.2)$$

where $\sigma^2 = V(\eta_1) + 2\sum_{i=1}^{\infty} cov(\eta_1, \eta_{1+i}), \quad \eta_i = I(X_i \leqslant Q), \quad i \geqslant 1.$

(It is well known that $\sigma^2 < \infty$ under either of the conditions (i) and (ii)).

Proof : Let us fix $\alpha = 1/8$. Clearly

$$\begin{aligned} & P(p - F_n(Q) > c\sigma(\log n/n)^{1/2} + 7n^{-1/2-\alpha}) \\ & - P((Q_n - Q)f + F_n(Q) - p < -7n^{-1/2-\alpha}) \\ & \leqslant P(Q_n - Q > f^{-1}c\sigma(\log n/n)^{1/2}) \\ & \leqslant P(p - F_n(Q) > c\sigma(\log n/n)^{1/2} - 7n^{-1/2-\alpha}) \\ & + P((Q_n - Q)f + F_n(Q) - p > 7n^{-1/2-\alpha}). \end{aligned}$$

It follows from the results of Ghosh and Babu (1977) and Babu and Singh (1978) that, under (i) as well as under (ii),

$$P(p - F_n(Q) > c\sigma(\log n/n)^{1/2} \pm 7n^{-1/2-\alpha}) \sim (2\pi c^2 \log n)^{1/2} n^{-c^2/2}.$$

Thus we are left to show that

$$P((Q_n - Q)f + F_n(Q) - p > 7n^{-1/2-\alpha}) = o(n^{-c^2/2}(\log n)^{-1/2}) \quad \ldots \quad (3.3)$$

$$P((Q_n - Q)f + F_n(Q) - p < -7n^{-1/2-\alpha}) = o(n^{-c^2/2}(\log n)^{-1/2}). \quad \ldots \quad (3.4)$$

We shall prove (3.3) and the proof of (3.4) is similar. Towards this end, let us suppose that for every $x \in (Q-2l, Q+2l)$, $l > 0$,

$$\infty > M > F'(x) > m^{-1} > 0 \text{ and } |F''(x)| \leqslant b. \qquad \dots \quad (3.5)$$

Let I denote the interval $(Q-1, Q+1)$. Let us observe that

$$P((Q_n-Q)f+F_n(Q)-p > 7n^{-\frac{1}{2}-a})$$

$$\leqslant P((Q_n-Q)f+F_n(Q)-p > 7n^{-\frac{1}{2}-a}, \sup_{x \in I} |F_n(x)-F(x)| < 2M\, n^{-3\alpha})$$

$$+P(\sup_{x \in I} |F_n(x)-F(x)| \geqslant 2M\, n^{-3\alpha}). \qquad \dots \quad (3.6)$$

Partitioning the interval I into subintervals of length $n^{-3\alpha}$, it follows that

$$\sup_{x \in I} |F_n(x)-F(x)|$$

$$\leqslant \max_{|r| \leqslant 1 n^{3\alpha}+1} |F_n(Q+r\, n^{-3\alpha})-F(Q+r\, n^{-3\alpha})| + M\, n^{-3\alpha}. \qquad \dots \quad (3.7)$$

Using (3.7), Markov's inequality, Lemma 2 of Ghosh and Babu (1977), Lemma 2.2 of Babu and Singh (1978), Bonferroni inequality and the fact that $3\alpha < \frac{1}{2}$, one verifies that, under (i) as well as under (ii),

$$\text{the last term in the r.h.s. of (3.6)} = o(n^{-c^2/2}(\log n)^{\frac{1}{2}}. \qquad \dots \quad (3.8)$$

From now onwards the statements which follow are true after certain n (non-random) onwards. In the following statements, $\Longrightarrow$ stands for 'implies'.

We now prove that,

$$\sup_{x \in I} |F_n(x)-F(x)| < 2M\, n^{-3\alpha} \Longrightarrow |Q_n-Q| \leqslant 2Mm\, n^{-3\alpha}. \quad \dots \quad (3.9)$$

We have

$$\text{l.h.s. of (3.9)} \Longrightarrow F(x)-2Mn^{-3\alpha} < F_n(x)$$

for every $x \in I$ which, by substituting $F(x)-2M\, n^{-3\alpha} = y$, implies that

$$F_n^{-1}(y) \leqslant F^{-1}(y+2Mn^{-3\alpha}) \qquad \dots \quad (3.10)$$

for every $y \in (Q-l/2, Q+l/2)$. From (3.10) we conclude easily, with the help of mean-value theorem and (3.5) that

$$Q_n-Q \leqslant 2M\, m\, n^{-3\alpha}.$$

Similarly we obtain the otherway inequality to conclude (3.9).

The second order mean value theorem and (3.5) prove that if $|Q_n-Q| \leqslant 2M\, m\, n^{-3\alpha}$, then

$$|F(Q_n)-F(Q)-(Q_n-Q)f| \leqslant 2b(M\, m\, n^{-3\alpha})^2 \qquad \dots \quad (3.11)$$

Hence if $$|Q_n - Q| \leqslant 2Mmn^{-3\alpha}$$

$$|(Q_n-Q)f+F_n(Q)-p| \leqslant |F_n(Q)-F_n(Q_n)+(Q_n-Q)f|+|p-F_n(Q_n)|$$
$$\leqslant |F_n(Q_n)-F_n(Q)-F(Q_n)+F(Q)|+2b(Mmn^{-3\alpha})^2+|p-F_n(Q_n)|. \quad \ldots (3.12)$$

Also, $|Q_n-Q| \leqslant 2Mmn^{-3\alpha}$ implies, using the assumed continuity of F at Q, that

$$0 \leqslant F_n(Q_n)-p \leqslant F_n(Q_n)-F_n(Q_n-0)$$
$$\leqslant \sup_{|x|<4Mmn^{-3\alpha}} |F_n(Q_n+x)-F_n(Q_n)-F(Q_n+x)+F(Q_n)|$$
$$\leqslant 2 \sup_{|x|<4mn^{-3\alpha}} |F_n(Q+x)-F_n(Q)-F(Q+x)+F(Q)|. \quad \ldots (3.13)$$

Therefore, using (3.9), (3.12) and (3.13) and the fact that $6\alpha > \frac{1}{2}+\alpha$, we conclude that

$$P((Q_n-Q)f+F_n(Q)-F(Q) > 7n^{-\frac{1}{2}-\alpha}, \sup_{x \in I} |F_n(x)-F(x)| < 2Mn^{-3\alpha})$$
$$\leqslant P\Big(\sup_{|x|<4Mmn^{-3\alpha}} |F_n(Q+x)-F_n(Q)-F(Q+x)+F(Q)| > 2n^{-\frac{1}{2}-\alpha}\Big). \quad \ldots (3.14)$$

Now dividing the interval $[-4Mmn^{-3\alpha}, 4Mmn^{-3\alpha}]$ into subintervals of length n^{-1}, it follows that

$$\sup_{|x|\leqslant 4Mmn^{-3\alpha}} |F_n(Q+x)-F_n(Q)-F(Q+x)+F(Q)|$$
$$\leqslant \max_{|r|\leqslant 4Mmn^{1-3\alpha}+1} |F_n(Q+rn^{-1})-F_n(Q)-F(Q+rn^{-1})+F(Q)|+n^{-\frac{1}{2}-\alpha}$$

and hence, one gets, using Bonferroni inequality, that

$$\text{r.h.s. of (3.14)} \leqslant n \sup_{|x|<K_2 n^{-3\alpha}} P(|F_n(Q+x)-F_n(Q)-F(Q+x)+F(Q)|>n^{-\frac{1}{2}-\alpha}) \quad \ldots (3.15)$$

for some absolute constant K_2.

We estimate the r.h.s. of (3.15) by the following lemma.

LEMMA 3.1 : *Let the process $\{X_n\}$ satisfy either of the conditions* (i) *and* (ii) *of Theorem* 3.1 *and let F has a bounded derivative in a neighbourhood of a point D. Define $x_i(D, D+d) = I(|X_i-D| \leqslant d)-P(|X_i-D| \leqslant d)$. For any $s > 0$, there exists a constant $c(s) > 0$ such that for all $d < K_3 n^{-3/8}$, $\infty > K_3 > 0$,*

$$P\left(\left|\sum_{i=1}^{n} x_i(D, D+d)\right| > n^{-\frac{1}{2}-\frac{1}{8}}\right) \leqslant c(s)n^{-s}$$

Remark 3.1 : $c(s)$ depends on the point D only through the upper bound of F' in a neighbourhood of D, a fact to be used in the proof of Remark 3.2.

Proof of Lemma 3.1 : For ϕ-mixing case the lemma follows from Lemma 2.1 of Babu and Singh (1978a) and for strong mixing case the lemma follows from Lemma 3.3 of Babu and Singh (1978a).

Clearly, (3.14), (3.15) and Lemma 3.1 complete the proof of (3.1). Similarly, we calculate PMD for $Q-Q_n$ and combine with (3.1) to get (3.2).

The proof of Theorem 3.1 is complete.

Expressions for PMD of quantiles for linear processes can be obtained if $a_i = O(i^{-2c^2-1})$ and F has bounded density. The proof runs parallel to the proof of the above theorem. One gets PMD for η_i's defined above by imitating the arguments of Ghosh and Babu (1977) and making use of the following modification of Lemma 3.2 of Philipp (1977).

Lemma 3.2 : *Let X and Y be random variables with*

$$E|X-Y| < \varepsilon.$$

Suppose that density of X is bounded by $A > 0$. Then for all $-\infty < t < \infty$,

$$E|I(X \leqslant t)-I(Y \leqslant t)| \leqslant 4(A+1)\varepsilon^{1/2}.$$

The details are too long to be presented here.

Nextly, we extend the technique of above theorem to get PMD of L-estimators.

Let Q_{nt} and Q_t denote t-th sample quantile and t-th population quantile respectively for $0 < t < 1$. Let K be a distribution function with support $J = [u, v]$, $0 < u \leqslant v < 1$.

We define the statistic

$$L_n = \int Q_{nt}\, dK(t)$$

as an estimator of the parameter

$$L = \int Q_t\, dK(t).$$

In the literature such estimators are referred to as L-estimators.

Let us suppose that for every $x \in (Q_u-l, Q_v+l)$, $l > 0$, $F''(x)$ exists and

$$\infty > H \geqslant F'(x) \geqslant h^{-1} > 0 \quad \text{and} \quad F''(x) \leqslant p < \infty. \quad \ldots \quad (3.16)$$

We adopt the notations $F'(Q_t) = f_t$, $\eta_{it} = (t-I(X_i \leqslant Q_t))f_t^{-1}$ and $\eta_i^* = \int \eta_{it} dK(t)$. Let I^* stand for the interval $(Q_u-l/2, Q_u+l/2)$.

Remark 3.2 : If $\{X_n\}$ satisfies either of the conditions (i) and (ii) of Theorem 3.1 and F satisfies (3.16), one has, for a fixed $c > 0$,

$$P(L_n - L > c\sigma^*(\log n/n)^{\frac{1}{2}}) \sim (2\pi c^2 \log n)^{\frac{1}{2}} n^{-c^2/2} \quad \ldots \quad (3.17)$$

$$P(|L_n - L| > c\sigma^*(\log n/n)^{\frac{1}{2}}) \sim 2(2\pi c^2 \log n)^{-\frac{1}{2}} n^{-c^2/2} \quad \ldots \quad (3.18)$$

where $\sigma^{*2} = V(\eta_1^*) + 2\sum_{i=1}^{\infty} \text{cov}(\eta_1^*, \eta_{i+1}^*)$

alternatively, $\sigma^{*2} = \int k(s, t)dK(s)dK(t)$, $k(s, t)$

$$= \text{cov}(\eta_{1s}, \eta_{1t})$$

$$+ 2\sum_{i=1}^{\infty} \text{cov}(\eta_{1s}, \eta_{(1+i)t}).$$

Proof : The proof runs parallel to that of previous theorem. We present here only the essential points. Clearly

$$|L_n - L - \frac{1}{n}\sum_1^n \eta_i^*| \leqslant h \sup_{t \in J} |(Q_{nt} - Q_t)f_t + F_n(Q_t) - t| = D(\text{say}).$$

Hence we have, for $\alpha = 1/8$,

$$\left| P(L_n - L > c\sigma^*(\log n/n)^{\frac{1}{2}}) - P\left(\frac{1}{n}\sum_1^n \eta_i^* > c\sigma^*(\log n/n)^{\frac{1}{2}} \pm 7n^{-\frac{1}{2}-\alpha}\right)\right|$$

$$\leqslant P(D > 7n^{-\frac{1}{2}-\alpha}) \quad \ldots \quad (3.19)$$

To neglect the r.h.s. of (3.19) we note that for

$$T = \sup_{x \in I^*} |F_n(x) - F(x)|,$$

$$P(D > 7n^{-\frac{1}{2}-\alpha}) \leqslant P\{(D > n^{-\frac{1}{2}-\alpha}) \cap (T < 2Hn^{-3\alpha})\} + P(T \geqslant 2Hn^{-3\alpha}) \ldots \quad (3.20)$$

The second term in the r.h.s. of (3.20) is neglected using arguments similar to (3.7). To neglect the first term in the r.h.s. of (3.20) one proves, imitating the proof of (3.9), that for n sufficiently large

$$\sup_{x \in I^*} |F_n(x) - F(x)| < 2Hn^{-3\alpha} \Longrightarrow \sup_{t \in J} |Q_{nt} - Q_t| \leqslant 2Hh\, n^{-3\alpha}.$$

Other changes are similar. At the last stage, we shall show, by dividing the interval J into subintervals of length n^{-1}, that

$$\sup_{x \in [Q_u, Q_v)} \sup_{b \leqslant 4Hhn^{-3\alpha}} |F_n(x+b) - F_n(x) - F(x+b) + F(x)|$$

$$\leqslant \max_{0 \leqslant q \leqslant d_n} \max_{|r| \leqslant r_n} |F_n(Q_u + (q+r)n^{-1}) - F_n(Q_u + qn^{-1}) - F(Q_u + (q+r)n^{-1}$$

$$+ F(Q_u + qn^{-1})| + n^{-\frac{1}{2}-\alpha},$$

where q and r denote integers, $d_n = [(Q_v - Q_u)n]+1$ and $r_n = [4Hhn^{1-3\alpha}]+1$. Finally Bonferroni inequality and Remark 3.1 is made use of.

Choosing K suitably, one gets PMD of trimmed means and Winsorised means using Remark 3.2.

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References

Babu, G. J., Ghosh, M. and Singh, K. (1978): On rates of convergence to normality for ϕ-mixing processes. Submitted for publication.

Babu, G. J. and Singh, K. (1978): Probabilities of moderate deviations for some stationary strong-mixing processes. *Sankhyā*, A, 40, 38-43.

——— (1978a): On deviations between quantile and empirical processes for mixing random variables. Submitted for publication.

Chanda, K. C. (1976): Some comments on sample quantiles for dependent processes. *Commun. Statist. Theo. Math.*, A_5, (14), 1385-1392.

Ghosh, M. and Babu, G. J. (1977): Probabilities of moderate deviations for some stationary ϕ-mixing processes. *Ann. Prob.*, 5, 222-234.

Goredtski, V. V. (1977): On the strong mixing properties for linear sequences. *Teoria Verojatnostei i ee prim.* 22, 421-423.

Michel, R. (1974): Results on probabilities of moderate deviations. *Ann. Prob.*, 2, 349-353.

Philipp, W. (1977): A functional law of iterated logarithms for empirical distribution functions of weakly dependent random variables. *Ann. Prob.*, 5, 319-350.

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