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# ON THE DISTRIBUTION OF VALUES OF ADDITIVE ARITHMETICAL FUNCTIONS

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*SUMMARY.* Theorems are proved which show that an additive arithmetical function which takes values in a regular set of real numbers in a regular fashion will have a distribution function.

## 1. Introduction

In 1947 Erdös raised the following question (see Erdös, 1947) :

*Question* A : Let $f$ be an additive arithmetical function. Suppose there is a finite interval $I$ such that $f^{-1}(I)$ has a positive density. Does it follow that $f$ has a distribution function?

This question still remains unanswered.

In 1972 a writer claimed that he had proved that the answer to the question was in the affirmative. However, subsequently he detected a flaw in his argument and withdrew (in a personal letter) the claim (see American Mathematical Society, 1973).

The main aim of the present paper is to prove some theorems suggested by Erdös's question.

We take this opportunity to raise a question in classical probability theory. An affirmative answer will at once imply an affirmative answer to Erdös's question; this will be evident after Theorem 4 below.

*Question* B : Let $\{X_n\}$, $n = 1, 2, \ldots$, be mutually independent random variables such that $\Pr\{X_n = 0\} \to 1$ as $n \to \infty$. Suppose there is a finite interval $I$ such that $\Pr\{(X_1+X_2+\ldots+X_n) \epsilon I\}$ converges to a *positive* limit $\lambda$ as $n \to \infty$. Does it follow that $\sum_1^{\infty} X_n$ converges with probability 1?

It may be noted that if the condition "$\Pr\{X_n = 0\} \to 1$" is replaced by the weaker condition "$X_n \to 0$ in law" , the answer to question B is in the negative. For example, let $Y_n$ take the values 0 and $\frac{1}{2^n}$ with probabilities $\frac{1}{2}$ and $\frac{1}{2}$ for $n = 1, 2, 3, \ldots$. Let $\{d_n\}$ be a sequence of real numbers such that the partial sums of $\sum_{1}^{\infty} d_n$ oscillate between the lower and upper limits 0 and $\frac{1}{10}$ (say) respectively and let $d_n \to 0$ as $n \to \infty$. Let $Z_n$ be a random variable taking the value $d_n$ with probability 1. Let the $Y$'s and $Z$'s be all mutually independent. Consider the infinite series $Y_1+Z_1+Y_2+Z_2+\ldots+Y_n+Z_n+\ldots$. Let $I$ be the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, for example.

Broadly, we shall use the notation used in Paul (1963). Thus $\bar{\lambda}(S)$ and $\bar{\pi}(S)$ will stand for the upper logarithmic and $\pi$-densities of $S$, respectively.

## 2. Some theorems guaranteeing the existence of distribution functions for additive arithmetical functions

Let $S$ be any set of positive integers. Let $J = \{j_1 < j_2 < \ldots\}$ be any increasing sequence of positive integers. By *the upper J-log density of S*, we mean $\limsup_{n} \frac{\mu\{S \cap [1, p_{j_n}]\}}{\log p_{j_n}}$. Similarly we define *lower J-log density of S*. We shall denote them by $\bar{\lambda}^J(S)$ and $\underline{\lambda}_J(S)$ respectively. $\mu$ here is as in Paul (1963, p. 273). If $\bar{\lambda}^J(S) = \underline{\lambda}_J(S)$, we shall denote the common value by $\lambda^J(S)$.

We shall now prove an improvement of Lemma 1 of Paul (1963). The proof given there will be simplified and made to yield the following improved equality.

Lemma 1 : *If $J$ is arbitrary and $S$ is a set of positive integers which is right-complete with respect to $J$, $P\{M_J(S)\} = \bar{\lambda}^J(S)$.*

*Proof:* Let $\lambda = \bar{\lambda}^J(S)$ and $\tau = P\{M_J(S)\}$. Evidently $\lambda \geqslant \tau$. Suppose $\lambda > \tau$; we shall construct a contradiction.

$$\text{Take an } \varepsilon > 0 \text{ such that } (\lambda-\tau-2\varepsilon) > 0. \qquad \ldots \quad (1)$$

$D_r$ has been defined in Paul 1963, p. 274). Let $r$ be a positive integer so large that

$$\text{(i)} \qquad \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\ldots\left(1-\frac{1}{q_{j_s}}\right) > \frac{1}{2\log q_{j_s}} \text{ for } s \geqslant r \qquad \ldots \quad (2)$$

and

$$\text{(ii)} \qquad P(D_{r+1} \cup D_{r+2} \cup \ldots) < \frac{\lambda-\tau-2\varepsilon}{3}. \qquad \ldots \quad (3)$$

Let $L = (q_{j_r}, q_{j_{r+m}}]$ where $m$ is so chosen that

(i) $$\frac{\mu(L \cap S)}{\mu(L)} > (\lambda - \varepsilon) \quad \ldots (4)$$

(ii) $$\frac{\mu\{L \cap (S_1 \cup \ldots \cup S_r)\}}{\mu(L)} < (T_r + \varepsilon) \quad \ldots (5)$$

where $T_r = P(D_1 \cup \ldots \cup D_r)$, and

(iii) $$\frac{\mu(L)}{\log q_{j_{(r+m)}}} > \frac{2}{3}. \quad \ldots (6)$$

From (4) and (5), we get

$$\frac{\mu\{[S-(S_1 \cup \ldots \cup S_r)] \cap L\}}{\mu(L)} > (\lambda - T_r - 2\varepsilon). \quad \ldots (7)$$

Again, we have

$$\{S-(S_1 \cup \ldots \cup S_r)\} \cap L \leqslant (S_{r+1} \cup \ldots \cup S_{r+m}) \quad \ldots (8)$$

as in step (12) in the proof in Paul (1963).

Repeating the argument in the lower half of p. 275 of Paul (1963), we get step (11) of that proof. ... (9)

From (2), (7) and (9), we get

$$P(D_{r+1} \cup \ldots \cup D_{r+m}) \geqslant (\lambda - T_r - 2\varepsilon). \frac{\mu(L)}{2 \log q_{j_{(r+m)}}}$$

$$\geqslant (\lambda - \tau - 2\varepsilon). \frac{\mu(L)}{2 \log q_{j_{(r+m)}}}$$

$$> \frac{(\lambda - \tau - 2\varepsilon)}{3} \quad \text{by (6).}$$

Now this contradicts (3).

Lemma 2 : *If $J$ is arbitrary and $S$ is any set of positive integers,* $P\{M_J^u(S)\} \geqslant \bar{\lambda}^J(S)$.

*Proof:* The proof of Lemma 2 of Paul (1963), goes through; we use Lemma 1 of the present paper.

Corollary : *If $S$ is any set of positive integers, $P\{M_L(S)\} \leqslant \underline{\lambda}(S) \leqslant \bar{\lambda}(S) \leqslant P\{M^u(S)\}$. This inequality was proved in Paul (1962). The proof given above, unlike the previous one, does not make use of Dirichlet series and Tauberian theorems.*

Lemma 3 : *Let $f$ be an additive arithmetic function such that for some sequence $J = \{j_1 < j_2 < j_3 < \dots\}$ of positive integers, $\lim_n f(2^{x_1} 3^{x_2} \dots p_{j_n}^{x_{j_n}}) = g(x)$ exists with probability* 1. *Let $A$ be any closed set of real numbers. Then*

$$\bar{\lambda}^J\{f^{-1}(A)\} \leqslant \Pr\{g(x) \in A\}.$$

*Proof:* Let $S = f^{-1}(A)$. Let $g$ be defined on the set $\Omega$ where $P(\Omega) = 1$. Now $M_J^u(S) \cap \Omega \subset E\{x : g(x) \in A\} \cap \Omega$, since $A$ is a closed set. So $\bar{\lambda}^J\{f^{-1}(A)\} \leqslant P\{M_J^u(S)\} \leqslant \Pr\{g(x) \in A\}$.

Lemma 4 : *$f$ is an additive arithmetical function such that a subsequence of the sequence of partial sums of $f(2^{x_1}) + \dots + f(p_n^{x_n}) + \dots$ converges with probability* 1 *to a random variable $g(x)$. If the distribution of $g(x)$ is discrete, $f$ has a distribution.*

*Proof:* $\sum_n f(p_n^{x_n})$ converges with probability 1 when centered; see Doob (1953, p. 121). Let $\sum_n \{f(p_n^{x_n}) + d_n\}$ converge with probability 1; by altering $d_1$, if necessary, we may suppose that the sum is $g(x)$ itself.

Let $M_n$ be the maximum probability carried by any single point in the distribution of $\{f(p_n^{x_n}) + d_n\}$; that is, in the distribution of $f(p_n^{x_n})$. Let $Q$ be the set of primes $p$ such that $f(p) \neq 0$. Let $p_m \in Q$. In the distribution of $f(p_m^{x_m})$, the maximum probability (carried by any single point) is $\leqslant \left(1 - \frac{1}{p_m} + \frac{1}{p_m^2}\right)$. Since $g(x)$ has a discrete distribution, we have, by Levy's theorem on convergent infinite convolutions of discrete distributions, $\prod_n M_n > 0$. Thus

$$\prod_{p_m \in Q} \left(1 - \frac{1}{p_m} + \frac{1}{p_m^2}\right) > 0.$$

Thus

$\sum_{p_m \in Q} \frac{1}{p_m} < \infty$. So $\sum_{n=1}^{\infty} f(p_n^{x_n})$ converges with probability 1, by the zero-one law and $f$ has a distribution.

Theorem 1 : *If $f$ is additive and if there is a countable compact set $D$ of real numbers such that $f^{-1}(D)$ has positive upper logarithmic density, then $f$ has a distribution.*

*Proof:* Let $S = f^{-1}(D)$. Let $J = \{j_1 < j_2 < \dots\}$ be such that $\bar{\lambda}^J(S) = \bar{\lambda}(S)$; by taking a suitable subsequence if necessary, we may suppose that $\lim_n f(2^{x_1} \dots p_{j_n}^{x_{j_n}}) = g(x)$ exists with probability 1. Now $0 < \bar{\lambda}(S) = \bar{\lambda}^J(S) \leqslant \Pr\{g(x) \in D\}$, by Lemma 3. So $g(x)$ has a discrete distribution. So by Lemma 4, $f$ has a distribution.

Corollary : *If $f$ is additive and if there is a real number $\alpha$ such that $f^{-1}(\alpha)$ has positive upper logarithmic density, then $f$ has a distribution. This generalizes a result given in Paul,* (1967).

Theorem 2 : *If $f$ is additive and if there is a compact set $A$ of real numbers such that the number 0 does not lie in the interior of the difference set of $A$, and if $\underline{\lambda} f^{-1}(A) > 0$, then $f$ has a distribution.*

*Remark :* $A$ will necessarily be of Lebesgue measure zero (see Halmos, 1950, p. 68).

*Proof :* Let $S = f^{-1}(A)$. Suppose $f$ has no distribution. Let $J$ be any sequence of positive integers such that $\lim_n \{f(2^{x_1})+\ldots+f(p_{j_n}^{x_{j_n}})\} = g(x)$ exists with probability 1.

$$0 < \underline{\lambda}(S) \leqslant P\{M_J^u(S)\} \leqslant \Pr\{g(x) \in A\}.$$

We now alter the sequence $J$ in such a way that the new subsequence of partial sums converges with probability 1 to the function $g(x)+\theta$ where $\theta$ is a constant.

So

$$\Pr\{g(x) \in A-\theta\} \geqslant \underline{\lambda}(S) > 0.$$

Choose $\theta_1, \theta_2, \ldots, \theta_\nu$ so that $A-\theta_1, A-\theta_2, \ldots, A-\theta_\nu$ are all disjoint and $\nu\underline{\lambda}(S) > 1$. Contradiction.

If $A$ is any set of real numbers, we shall mean by *the core difference set of* $A$ the set of all real numbers $d$ such that $d = (x-y)$ for *uncountably many* pairs of points $x, y$ occurring in $A$.

Theorem 3 : *Let $f$ be an additive arithmetical function and let $A$ be a compact set (of real numbers) such that the core difference set of $A$ does not have* 0 *in its interior. If $\underline{\lambda} f^{-1}(A) > 0$, $f$ has a distribution.*

*Proof :* Similar to the proof of the previous theorem. By Lemma 4, $g(x)$ has a continuous distribution and so every countable set gets measure zero. The intersection of $(A-\theta_m)$ and $(A-\theta_n)$ is at most countable.

The attention of the reader is invited to the problem posed at the end of this paper.

## 3. The inverse image of an interval with respect to an additive arithmetical function

The main aim of this section is to prove the following.

Theorem 4 : *Let $f$ be an additive arithmetical function such that $\bar{\lambda} f^{-1}(I) > 0$ for some finite interval $I$. Let $V$ be any set of real members such that the boundary of $V$ is finite or countably infinite. Then*

$$\bar{\lambda} f^{-1}(V) = \limsup_n \Pr\{[f(2^{x_1})+\ldots+f(p_n^{x_n})] \in V\}$$
$$= \bar{\pi} f^{-1}(V)$$

*and*

$$\underline{\lambda} f^{-1}(V) = \liminf_n \Pr\{[f(2^{x_1})+\ldots+f(p_n^{x_n})] \in V\}$$
$$= \underline{\pi} f^{-1}(V).$$

*Remark :* The most important case is when $V$ is an interval.

First we prove a lemma whose content is a known result. In fact, the lemma continues to be true even if "logarithmic density" occurring in the statement is replaced by "natural density"; (see for example, Billingsley (1964), remarks after Theorem 3.3). We prove the weaker lemma here so as to have a self-contained proof of Theorem 4.

Lemma 5 : *Let the additive arithmetical function $f$ have a discrete distribution, say $Q$. If $A$ is any set of real numbers, then $\lambda\{f^{-1}(A)\}$ exists and $= Q(A)$.*

Also, if $f$ takes the value $a$ once, it takes that value on a set having positive logarithmic density. Thus the points that carry positive probability in the distribution $Q$ are exactly the values taken by $f$.

*Proof :* Since, $f$ has a discrete distribution, $\sum_p \frac{1}{p}$, with $f(p) \neq 0$, is $<\infty$.

If $v$ is a value taken by $f$, then $Q(v) > 0$. Let $v = f(2^{x_1}\ldots p_m^{x_m})$. For $n = 1, 2, 3, \ldots$, let $B_n$ be the set of all points $x_r$ on the factor space $X_n$ such that $f(p_n^{x_r}) = 0$. Let $B = B_1 \times B_2 \times \ldots$ . We know that $P(B) > 0$. Now $\sum_n f(p_n^{x_n}) = g(x)$ takes the value $v$ with probability $\geqslant$ $P$-measure of the box $\{x_1\} \times \ldots \times \{x_m\} \times B_{m+1} \times B_{m+2} \times \ldots > 0$. So $Q(v) > 0$.

To prove the converse, let $V = \{v_1, v_2, \ldots\}$ be the set of values taken by $f$. Let $g(x) = \sum_1^\infty f(p_n^{x_n})$; $g_1(x) = \sum_2^\infty f(p_n^{x_n})$. Let $V_1$ be the set of values taken by the (finite) partial sums of the infinite series $\sum_2^\infty f(p_n^{x_n})$. It is then easy to see that $g_1^{-1}(V_1) \subseteq g^{-1}(V)$.

Again, let $V_2$ be the set of values taken by the finite partial sums of the infinite series $\sum_3^\infty f(p_n^{x_n})$. Let $g_2(x) = \sum_3^\infty f(p_n^{x_n})$. Then $g_2^{-1}(V_2) < g_1^{-1}(V_1) < g^{-1}(V)$ and so on. So $B \subset \bigcap_1^\infty g_n^{-1}(V_n) = T$. Since $T$ is a tail-event, $P(T) = 1$ by the zero-one law. Thus $P\{g^{-1}(V)\} = 1$ and $Q(V) = 1$.

Now let $a$ be a value taken by $f$. Let $Z$ be the subset of $X$ consisting of points $(z_1, z_2, \ldots)$ such that $z_n \epsilon B_n$ for all large $n$. Since $Z$ is a tail-event containing $B$, $P(Z) = 1$. It follows that $[g^{-1}\{a\} \cap Z] \subset M_L\{f^{-1}(a)\}$. Thus $\underline{\lambda}\{f^{-1}(a)\} \geqslant P[M_L\{f^{-1}(a)\}] \geqslant Q(a)$. The proof of Lemma 5 is now completed without difficulty.

Theorem 5 : *Let the additive arithmetical function $f$ have a discrete distribution. If $A$ is any set of real numbers,*

$$\lambda\{f^{-1}(A)\} = \lim_n Pr\{f(2^{x_1}) + \ldots + f(p_n^{x_n}) \epsilon A\} = \pi(A).$$

*Proof :* Let $C_k$ be the set of points $(x_1, x_2, \ldots)$ of $X$ such that $f(2^{x_1} \ldots p_k^{x_k}) \epsilon A$. Let $Z$ be the set described in the proof of Lemma 5. Since $P(Z) = 1$, we have

$$\begin{aligned} \bar{\lambda}\{f^{-1}(A)\} &\leqslant P[M^u\{f^{-1}(A)\} \cap Z] \\ &= P[M_L\{f^{-1}(A)\}] = P(\liminf_k C_k) \leqslant \liminf_k P(C_k). \end{aligned}$$

Using the corresponding inequality for $\underline{\lambda}\{f^{-1}(A)\}$, we get

$$\lambda\{f^{-1}(A)\} = \lim_n \Pr\{f(2^{x_1} \ldots p_n^{x_n}) \epsilon A\}.$$

Now we shall prove Theorem 4.

If $f$ has a discrete distribution, our assertion follows from Theorem 5. If $f$ has a continuous distribution, $\lambda\{f^{-1}(V)\}$ exists and $= \Pi\{f^{-1}(V)\}$, since the result holds for every open interval. So now suppose $f$ has no distribution. Then we have a sequence $\{d_n\}$ of real numbers such that $\sum_n \{f(p_n^{x_n}) + d_n\}$ converges with probability 1. Also $d_n \to 0$ as $n \to \infty$.

Let $J = \{j_1 < j_2 < \ldots\}$ be a sequence of positive integers such that $\lambda^J(S) = \bar{\lambda}(S)$. By choosing a subsequence of $J$ if necessary, we may suppose that $\lim_n f(2^{x_1} \ldots p_{j_n}^{x_{j_n}}) = g(x)$ exists with probability 1. By Lemma 4, $g(x)$ has a continuous distribution.

$$\Pr\{g(x) \epsilon V\} = P\{M_J^u(S)\} = P\{M_L^J(S)\} = \lambda^J(S) = \bar{\lambda}(S).$$

So $$\Pr\{f(2^{x_1})+\ldots+f(p_{j_n}^{x_{j_n}})\,\epsilon\, V\}\to\bar{\lambda}(S)$$

and so $$\overline{\Pi}(S)\geqslant\bar{\lambda}(S).$$

Now let $\beta=\limsup_n \Pr\{f(2^{x_1}\ldots p_n^{x_n})\,\epsilon\, V\}=\overline{\Pi}(S)$.

Let $K=\{k_1, k_2, \ldots\}$ be a sequence of positive integers such that $f(2^{x_1}\ldots p_{k_n}^{x_{k_n}})$ converges with probability 1 to some random variable $h(x)$; by Lemma 4, $h(x)$ has a continuous distribution; let $K$ be so chosen that $\Pr\{f(2^{x_1}\ldots p_{k_n}^{x_{k_n}})\,\epsilon\, V\}\to\overline{\Pi}(S)$. As before, $\Pr\{h(x)\,\epsilon\, V\}=P\{M_K^g(S)\}=P\{M_L^K(S)\}=\lambda^K(S)\leqslant\bar{\lambda}(S)$. So $\overline{\Pi}(S)\leqslant\bar{\lambda}(S)$. This proves Theorem 4.

Theorem 4 shows at once that an affirmative answer to question B implies an affirmative answer to Erdös's question A.

Now let $f$ be an additive arithmetical function and suppose there is a finite interval $(a, b)$ such that $\lambda\{f^{-1}(a, b)\}=\beta>0$ exists. Suppose $f$ has no distribution function. Let $\sum_{n=1}^{\infty}\{f(p_n^{x_n})+d_n\}=g(x)$ with probability 1. Let $\liminf_n (d_1+\ldots+d_n)=v$ and $\limsup_n (d_1+\ldots+d_n)=w$. So $v<w$. We may modify $d_1$ in such a way that $v<0<w$; we assume that such a modification has been made. Let $J=\{j_1<j_2<\ldots\}$ be such that $(d_1+\ldots+d_{j_n})\to 0$ as $n\to\infty$. So $\lim_n f(2^{x_1}\ldots p_{j_n}^{x_{j_n}})=g(x)$ p.p. By Theorem 4,

$$\Pr\{g(x)\,\epsilon\,(a, b)\}=\beta.$$

If $v\leqslant\theta\leqslant w$, $\Pr\{g(x)\,\epsilon\,(a-\theta,\ b-\theta)\}=\beta$; this we show by using, instead of $J$, a suitable new sequence $K$. Thus the distribution of the random variable $g(x)$ is peculiar in that the pattern of distribution on $[b-w,\ b-v]$ is the same as that on $[a-w,\ a-v]$.

It is also seen easily that the distribution function of the random variable $g(x)$ is strictly increasing throughout $(-\infty, \infty)$.

These facts give us the following theorem.

Theorem 6 : *Let $f$ be an additive arithmetical function. Let $a_1<a_2<\ldots\to\infty$. If for each $n$, $\lambda\{f^{-1}(a_n, a_{n+1})\}$ exists and is $>0$, $f$ has a distribution function.*

We conclude by posing the following *problem* : Construct a specific set $A$ (of real numbers) having positive Lebesgue measure, such that every additive arithmetical function $f$ for which $\lambda\{f^{-1}(A)\}$ exists and is positive will have a distribution function.

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