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The revelation approach to Nash implementation

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Recent work on implementation has established necessary and sufficient conditions for a social choice rule to be implementable in Nash equilibrium strategies. These conditions need to distinguish the case of two agents from that of three or more agents in the organization. We interpret and unify these conditions in terms of the ICSE (Incentive Compatibility and Selective Elimination) condition, which is necessary for implementation, irrespective of the number of agents or the underlying information structure.

Numerous papers in recent years have provided characterizations of social choice rules that can be implemented in Nash equilibrium strategies. For complete information environments, Moore and Repullo (1986) have identified necessary and sufficient conditions for a choice rule to be Nash-implementable, provided there are at least three agents. These conditions roughly amount to Maskin's (1977) monotonicity conditions plus a strengthened unanimity requirement. For the case of two agents, Moore and Repullo (1991) and Dutta and Sen (1991) have formulated an additional condition which combined with monotonicity and the unanimity requirement is again necessary and sufficient for Nash implementation. [1]

The purpose of this note is to relate the conditions of Moore–Repullo and Dutta–Sen to the ICSE (Incentive Compatibility and Selective Elimination) condition of Mookherjee and Reichelstein (1990). ICSE requires that there exist an incentive compatible revelation mechanism whose truth-telling equilibria 'agree' with the social choice rule. Furthermore, any undesired equilibrium of this revelation mechanism can be selectively eliminated in the following sense: some agent can be given an additional message option such that the outcomes associated with this message eliminate the undesired equilibrium without eliminating the truth-telling equilibrium. The necessity of ICSE follows directly from the 'Augmented Revelation Principle', regardless of the underlying information structure and the number of agents.

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[1] Nash-implementation with two agents has played an important part in some of the recent work on incomplete contracts, for example, Green and Laffont (1988) and Hart and Moore (1988).

Incentive compatibility has played a major role in the theory of implementation with incomplete information, but not when information is complete. The apparent reason is that with complete information and three or more agents incentive compatibility is trivially satisfied (by simply ignoring the report of the deviating agent). With two agents, however, incentive compatibility does impose restrictions. We show in this note that the ICSE condition is equivalent to monotonicity combined with the additional condition developed by Moore–Repullo and Dutta–Sen for the two-agent case. When there are three or more agents, ICSE reduces exactly to monotonicity. Accordingly, we find that ICSE in conjunction with the above-mentioned unanimity requirement is necessary and sufficient for Nash implementation. The unanimity requirement will be vacuously satisfied in many economic environments, for example, when private transfers are possible. Our results, therefore, interpret and unify the existing characterizations in terms of a single necessary (and 'frequently' sufficient) condition that applies to an arbitrary number of agents.

Suppose there are n agents $I=\{1,\ldots,n\}$. The set of feasible outcomes is X, and set of possible states of the environment is denoted Θ. The state defines preferences of every agent over outcomes in X. In any state $\theta\in\Theta$, agent i's preferences are represented by a reflexive, complete and transitive ordering $R_i(\theta)$ over X.

A *social choice correspondence* (SCC) is a correspondence $F:\Theta \twoheadrightarrow X$. A *mechanism* is a pair $\Lambda\equiv\langle M,\ g\rangle$, where $M=\times_{i\in I}M_i$ is the product of individual message sets M_i, and $g:M\rightarrow X$ is an outcome function. The message-tuple $(m_1,\ldots,m_n)\in M$ is a *Nash equilibrium in state* θ in the mechanism Λ if for every $i\in I$, $g(m_1,\ldots,m_n)R_i(\theta)g(\tilde{m}_i,\ m_{-i})$ for all $\tilde{m}_i\in M_i$. By $E_\Lambda(\theta)$ we denote the set of Nash equilibria of the mechanism Λ at $\theta\in\Theta$. The mechanism of $\Lambda=\langle M,g\rangle$ implements $F:\Theta\twoheadrightarrow X$ if for every $\theta\in\Theta$: $E_\Lambda(\theta)\neq\phi$, for all $m\in E_\Lambda(\theta)$: $g(m)\in F(\theta)$. For any set $A\subset X$, let $M_i(A,\ \theta)\equiv\{x\in A\mid xR_i(\theta)y$ for all $y\in A\}$ be the set of maximal elements for i in A, given $\theta\in\Theta$.

Definition 1 (Condition μ). [2] There exists a subset $B\subseteq X$ such that for every $i\in I$, $\theta\in\Theta$ there exist sets $C_i(\theta)\subseteq B$ and outcome $y\in F(\theta)$ satisfying:

(1) $y\in\cap_{i\in I}M_i(C_i(\theta),\ \theta)$.
(2) for any $\theta^*\in\Theta$, if $y\in\cap_{i\in I}M_i(C_i(\theta),\ \theta^*)$ then $y\in F(\theta^*)$.
(3) for any $\theta^*\in\Theta$ and any $i\in I$, if $x\in M_i(C_i(\theta),\ \theta^*)\cap[\cap_{j\neq i}M_j(B,\ \theta^*)]$, then $x\in F(\theta^*)$.
(4) for any $\theta^*\in\Theta$, $x\in B$, if $x\in\cap_{i\in I}M_i(B,\ \theta^*)$, then $x\in F(\theta^*)$.

Moore and Repullo (1986) have shown that, with $n\geq 3$, Condition μ is necessary and sufficient for any SCC $F:\Theta\twoheadrightarrow X$ to be implementable. Parts (1) and (2) essentially amount to Maskin's (1977) monotonicity condition, [3] while parts (3) and (4) represent an unanimity condition: an outcome x lies in the choice set $F(\theta)$ if x is the top ranked alternative for all agents [either in $C_i(\theta)$ or B]. Consider now the following stengthened version of Condition μ.

Definition 2 (Condition μ^*). There are two agents, Condition μ holds, and:

(5) for any $\theta,\tilde{\theta}\in\Theta$, there exists an outcome $y(\theta,\tilde{\theta})\in C_1(\theta)\cap C_2(\tilde{\theta})$ such that for all $\theta^*\in\Theta$: $y(\theta,\ \tilde{\theta})\in M_1(C_1(\theta),\ \theta^*)\cap M_2(C_2(\tilde{\theta}),\ \theta^*)$ implies $y(\theta,\ \tilde{\theta})\in F(\theta^*)$.

[2] The condition stated here is slightly weaker than that of Moore and Repullo reflecting our weaker implementation requirement. In contrast to Definition 1, Moore and Repullo (1986, 1991) require *full implementation*, that is, all $y\in F(\theta)$ must be attainable by some Nash equilibrium.

[3] Monotonicity requires that: for all $\theta\in\Theta$, there exists $y\in F(\theta)$ such that if $L_i(y,\ \theta)\subset L_i(y,\ \theta^*)$ for all $i\in I$, then $y\in F(\theta^*)$. [Here $L_i(y,\ \theta)$ denotes the lower contour set, i.e., $L_i(y,\ \theta)=\{x\in X\mid yR_i(\theta)x\}$.] This condition is weaker than Maskin's original condition reflecting again that we are not concerned with full implementation.

Moore and Repullo (1991) and Dutta and Sen (1991) have shown that, if $n = 2$, an SCC $F: \Theta \twoheadrightarrow X$ can be implemented if and only if F satisfies Condition μ^*. Unfortunately, part (5) of Condition μ^* does not have an apparent economic interpretation. We provide one such interpretation by relating Conditions μ and μ^* to the ICSE condition (Incentive Compatibility and Selective Elimination) of Mookherjee and Reichelstein (1990). There, it is shown that ICSE is necessary for implementation irrespective of the number of agents (and irrespective of whether information is complete or incomplete). ICSE is stated in terms of revelation mechanisms where $M_i = \Theta$ for all agents. By Λ_g we denote a revelation mechanism whose outcome function is $g: \Theta^n \to X$. A generic message of a revelation mechanism will be denoted by $\vec{\theta}$. If all agents announce the same $\theta \in \Theta$, we shall write $\vec{\theta} = \theta^n$. If Agent i only deviates to some $\tilde{\theta}_i \in \Theta$, we shall use the notation $\vec{\theta} = (\theta^{n-1}, \tilde{\theta}_i)$. Finally, let $U_i(y, \theta)$ denote an agent's upper contour set, i.e., $U_i(y, \theta) = \{x \in X \mid xP_i(\theta)y\}$.

Definition 3. The SCC $F: \Theta \twoheadrightarrow X$ satisfies ICSE if

(i) there exists a revelation mechanism Λ_g in which truthful reporting is an equilibrium, and $g(\theta) \in F(\theta)$ for all $\theta \in \Theta$;
(ii) if $\vec{\theta} \in E_{\Lambda g}(\theta^*)$ but $g(\vec{\theta}) \notin F(\theta^*)$, then there exists $j \in I$ for whom $U_j(g(\vec{\theta}), \theta^*) \neq \phi$. Furthermore, if $\vec{\theta} = (\theta^{n-1}, \tilde{\theta}_j)$ for some $\theta \in \Theta$, $\tilde{\theta}_j \in \Theta$, then $U_j(g(\vec{\theta}), \theta^*) \cap L_j(g(\theta^n), \theta) \neq \phi$.

ICSE requires existence of an incentive compatible revelation mechanism whose truthful equilibria agree with F. If this mechanism has an undesired equilibrium $\vec{\theta}$ in state θ^*, then some agent can be given an additional message in order to eliminate this equilibrium. The corresponding outcome is some $y \in U_j(g(\vec{\theta}), \theta^*)$. However, the elimination has to be selective in the sense that no truth-telling equilibria are upset. Hence, if $\vec{\theta} = (\theta^{n-1}, \tilde{\theta}_i)$, then the outcome y must belong to $L_j(g(\theta^n), \theta)$, ensuring that it is still a Nash equilibrium for Agent j to report truthfully in state θ.

Proposition 1. If $n \geq 3$, F satisfies ICSE if and only if it satisfies monotonicity, i.e., parts (1) and (2) of Condition μ.

Proof. To prove that monotonicity implies ICSE, let f be a single-valued selection from F to which the monotonicity hypothesis applies (compare footnote 2).

Consider the revelation mechanism Λ_g constructed as follows:

$$g(\vec{\theta}) = \begin{cases} f(\theta) & \text{if } \vec{\theta} = \theta^n \text{ for some } \theta \in \Theta, \\ f(\theta) & \text{if } \vec{\theta} = \left(\theta^{n-1}, \tilde{\theta}_i\right) \text{ for some } \theta, \tilde{\theta}_i \in \Theta, \\ f(\theta_n) & \text{otherwise, if } \vec{\theta} = \left(\vec{\theta}_{-n}, \theta_n\right) \text{ for } \theta_n \in \Theta. \end{cases}$$

Hence, the outcome $f(\theta)$ is chosen if all agents unanimously report θ, or at most one agent deviates. In all other cases, the outcome is determined by $f(\cdot)$ and the nth agent's report. Clearly, this mechanism is incentive compatible.

Now suppose that $\vec{\theta} \in E_{\Lambda g}(\theta^*)$ but $g(\vec{\theta}) \notin F(\theta^*)$. By construction, $g(\vec{\theta}) = f(\theta)$ for some $\theta \in \Theta$. Monotonicity says that there exists an agent $j \in I$ for whom $L_j(f(\theta), \theta) \not\subset L_j(f(\theta), \theta^*)$. Hence, $L_j(f(\theta), \theta) \cap U_j(f(\theta), \theta^*) \neq \phi$, proving ICSE. Mookherjee and Reichelstein (1990) prove that ICSE is always at least as strong a condition as monotonicity. □

So with $n \geq 3$, ICSE amounts to Condition μ except for the unanimity requirements (3) and (4). We note that with three agents, incentive compatibility is trivially satisfied since the report of any deviating agent can simply be ignored. However, if $n = 2$ the incentive compatibility condition has 'bite'. Define Condition $\hat{\mu}$ to be Condition μ^* without requirements (3) and (4), i.e., only items (1), (2) and (5) of Condition μ^* are required.

Proposition 2. If $n = 2$, F satisfies ICSE if and only if it satisfies Condition $\hat{\mu}$.

Proof. Suppose F satisfies ICSE, and A_g is the associated revelation mechanism satisfying (i) and (ii) of Definition 4. To demonstrate Condition $\hat{\mu}$ holds, set $B = X$ and choose for any given θ: $y = g(\theta, \theta)$ and $C_i(\theta) = L_i(y, \theta)$. Then condition (1) is met by definition. To show (2), suppose there exists θ^* with $y \in \cap_{i=1}^{2} M_i(L_i(y, \theta), \theta^*)$. Now IC implies that $g(\theta', \theta) \in L_1(y, \theta)$ and $g(\theta, \theta') \in L_2(y, \theta)$ for any θ'. Hence $g(\theta, \theta) R_1(\theta^*) g(\theta', \theta)$ and $g(\theta, \theta) R_2(\theta^*) g(\theta, \theta')$, i.e., (θ, θ) is an equilibrium in state θ^*. If $g(\theta, \theta) \notin F(\theta^*)$, then SE implies that there exists Agent j and outcome x such that $x \in L_j(y, \theta)$ but $x P_j(\theta^*) y$, which contradicts $y \in M_j(L_j(y, \theta), \theta^*)$.

To show condition (5), let $y(\theta, \tilde{\theta}) = g(\tilde{\theta}, \theta)$ for any two states $\theta, \tilde{\theta}$. By IC it follows that $y(\theta, \tilde{\theta}) = g(\tilde{\theta}, \theta) \in L_1(g(\theta, \theta), \theta) \cap L_2(g(\tilde{\theta}, \tilde{\theta}), \tilde{\theta}) = C_1(\theta) \cap C_2(\tilde{\theta})$. Suppose for some state θ^*: $g(\tilde{\theta}, \theta) \in M_1(L_1(g(\theta, \theta), \theta), \theta^*) \cap M_2(L_2(g(\tilde{\theta}, \tilde{\theta}), \tilde{\theta}), \theta^*)$. For any $\hat{\theta} \in \Theta$, IC implies that $g(\hat{\theta}, \theta) \in L_1(g(\theta, \theta), \theta)$ and $g(\tilde{\theta}, \hat{\theta}) \in L_2(g(\tilde{\theta}, \tilde{\theta}), \tilde{\theta})$. Hence, $g(\tilde{\theta}, \theta) R_1(\theta^*) g(\hat{\theta}, \theta)$ and $g(\tilde{\theta}, \theta) R_2(\theta_2^*) g(\tilde{\theta}, \hat{\theta})$. Hence $(\tilde{\theta}, \theta)$ is an equilibrium in the revelation mechanism A_g. If $g(\tilde{\theta}, \theta) \notin F(\theta^*)$, then SE implies that there exists Agent j and outcome x such that (if $j = 1$): $x \in L_1(g(\theta, \theta), \theta) \cap U_1(g(\tilde{\theta}, \theta), \theta^*)$, which contradicts $g(\tilde{\theta}, \theta) \in M_1(L_1(g(\theta, \theta), \theta), \theta^*)$. If $j = 2$, a similar argument applies. We have shown that ICSE implies Condition $\hat{\mu}$.

Now suppose Condition $\hat{\mu}$ holds. For any state θ, define $g(\theta, \theta) = y$ as stipulated by Condition $\hat{\mu}$. For any pair $(\tilde{\theta}, \theta)$ of states, define $g(\tilde{\theta}, \theta) = y(\theta, \tilde{\theta})$ as given by (5) of $\hat{\mu}$. By requirement (1), $g(\theta, \theta) \in M_i(C_i(\theta), \theta)$, while by (5), $g(\tilde{\theta}, \theta) \in C_1(\theta) \cap C_2(\tilde{\theta})$. Hence truthful reporting is an equilibrium in this revelation mechanism, and $g(\theta, \theta) \in F(\theta)$. To establish part (ii) of ICSE, suppose initially that (θ, θ) is an equilibrium in state θ^*, where $g(\theta, \theta) \notin F(\theta^*)$. Then by requirement (2), $g(\theta, \theta) \notin \cap_{i=1}^{2} M_i(C_i(\theta), \theta^*)$. Hence there exists Agent j and outcome $x \in C_j(\theta)$ such that $x P_j(\theta^*) g(\theta, \theta)$. Since $g(\theta, \theta) \in M_j(C_j(\theta), \theta)$, it follows that $x \in U_j(g(\theta, \theta), \theta^*) \cap L_j(g(\theta, \theta), \theta)$. Finally, if $(\tilde{\theta}, \theta)$ is an equilibrium in state θ^*, with $g(\tilde{\theta}, \theta) \notin F(\theta^*)$, we can use an analogous argument [where condition (5) rather than (2) is invoked]. □

As demonstrated in Mookherjee and Reichelstein (1990), there exist choice rules which satisfy monotonicity as well as incentive compatibility, but not the 'selective elimination' property. However, ICSE is always at least as strong as monotonicity. [4] Requirement (5) of Condition μ^* represents exactly the additional strengthening of monotonicity needed to obtain ICSE in the case of two agents. This provides an interpretation of condition (5) and its special role in the case of two agents.

Given the preceding results, one would expect that ICSE combined with the unanimity requirements in (3) and (4) of Condition μ will be necessary and sufficient for implementation. This turns out to be true; a formal proof can be found in an earlier version of this paper [Mookherjee and Reichelstein (1991)]. As a consequence, one obtains a single condition which unifies the

[4] With incomplete information ICSE is generally stronger than Bayesian monotonicity [for slightly different definitions of Bayesian monotonicity, see Palfrey and Srivastava (1989) and Jackson (1991)]. However, ICSE and monotonicity can be shown to be equivalent when there are two agents, information is complete, and there exists a 'bad outcome', i.e., an outcome that is always ranked worse than any other outcome by all agents.

..o-agent case with that of three or more agents. In many economic environments the unanimity requirements in (3) and (4) of Condition μ are trivally satisfied. In such environments implementation then hinges entirely on ICSE.

Proposition 3. Suppose for all $\theta, \theta^* \in \Theta$:

(*i*) *For any* $y \in F(\theta)$, $i \in I$: $M_i(L_i(y, \theta), \theta^*) \cap [\cap_{j \neq i} M_j(X, \theta^*)] = \phi$,
(*ii*) $\cap_{j=1}^{n} M_j(X, \theta) = \phi$.

Then ICSE is necessary and sufficient for the implementability of F.

For a formal proof of this result, the reader is again referred to Mookherjee and Reichelstein (1991). The presence of a divisible private good is one particular case where conditions (i) and (ii) will be met. It suffices that starting at any $y \in F(\theta)$ the mechanism designer can impose (small) transfers in terms of a private good that all agents value. Consider, specifically, a bilateral trading problem involving a given endowment of $l \geq 2$ goods. If agents' consumption sets are the non-negative orthant in $\mathbb{R}^2$ and their preferences are monotone, then ICSE is necessary and sufficient for Nash implementation of a SCC F, provided F never assigns an agent zero consumption of all goods.

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