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Sooner and later waiting time problems for Markovian Bernoulli trials

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Abstract: The waiting time problems introduced by Ebneshahrashoob and Sobel (1990) for independent trials are generalized to Markov correlated Bernoulli trials. A new waiting time problem arising due to mixed quotas is also discussed. A learning model is used as illustration.

Keywords: Frequency quota; later waiting time; Markovian Bernoulli trials; run quota; sooner waiting time.

1. Introduction

In many applications involving Bernoulli trials, the quantity of interest is the waiting time for a given number, a run or other specific pattern of events to occur. These variates arise naturally in the setting up of (sequential) acceptance sampling plans in quality control (Schilling, 1982). Other applications are found in the design of certain clinical trials in biomedical experimentation. Hahn and Gage (1983) presented an application to the modeling of the starting-up reliability of power-generation equipment. See Viveros and Balakrishnan (1993) for the corresponding statistical analysis. An application to the modeling and assessment of the occurrence of a number of defective fasteners adjacent to one another in an assembly process was given by Feder (1974). It should be mentioned here that waiting time problems in general have a long and rich history; see, for example, Feller (1968).

An interesting class of waiting time problems was proposed recently by Ebneshahrashoob and Sobel (1990) (to be referred as ES subsequently). The associated stopping criteria take into account both individual sequences of successes and failures observed in the trials as the experiment progresses. ES performed a probabilistic analysis of the associated waiting times under the assumption that the Bernoulli trials are independent and identically distributed.

Most of the literature on discrete waiting times is based on the assumption that the outcomes from different trials are independent events. A different point of view is taken in this article by permitting the

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Bernoulli trials to exhibit some degree of correlation. The type of correlation allowed is Markovian and is characterized by the parameters p_0, p_1 and p_2, defined as

$$p_0 = \Pr(O_1 = S), \qquad p_1 = \Pr(O_i = S \mid O_{i-1} = S) \quad \text{and} \quad p_2 = \Pr(O_i = S \mid O_{i-1} = F), \tag{1}$$

where O_i is the outcome of trial i ($i \geqslant 1$), and S and F denote success and failure, respectively. Let $q_j = 1 - p_j$, $j = 0, 1, 2$.

A simple waiting time problem under this correlation structure, namely the analogue of the negative binomial distribution, is discussed by Viveros, Balasubramanian and Balakrishnan (1993) (to be referred as VBB subsequently). Also, the waiting time problem discussed by Hahn and Gage (1983), which is based on a run quota of successes, has been extended to Markovian Bernoulli trials by Balakrishnan, Balasubramanian and Viveros (1993) (to be referred as BBV subsequently). Other relevant references that will be of interest to the readers in this regard are Rajarshi (1974), Wang (1981), Gerber and Li (1981), Schwager (1983) and Benvenuto (1984).

The main objective of this article is to extend to the present correlation structure the waiting time problems proposed by ES. Thus, the results obtained by ES can be deduced as special cases from the distributional results derived here. Apart from the extension itself, the methods of analysis put forward in this article differ from those of ES.

For the sake of completeness, it is convenient to associate with each of the stopping rules considered in the remaining sections the random vector $\mathbf{Z} = (S_0, F_0, S_1, F_1, S_2, F_2)$ where $S_0 = 1$ or 0 depending on whether the initial trial is successful or not, $F_0 = 1 - S_0$, S_1 (F_1) is the number of successful (unsuccessful) trials for which the previous trial is successful, and S_2 (F_2) is the number of successful (unsuccessful) trials for which the previous trial is unsuccessful. The variates of primary interest are

$$X = S_0 + S_1 + S_2, \qquad Y = F_0 + F_1 + F_2 \quad \text{and} \quad W = X + Y, \tag{2}$$

which are the total numbers of successes, failures, and trials, respectively. Usually, W is regarded as the waiting time variate.

The waiting times to be discussed will arise by setting quotas on both runs and frequencies of successes and failures. The 'sooner cases' refer to situations in which experimentation stops as soon as one of the quotas is reached (Sections 2–4), while the 'later cases' bring the experiment to a halt when both quotas are completed (Section 5).

2. Run quotas

The waiting time problem to be addressed in this section belongs to the 'sooner cases' and arises when imposing a quota on runs of successes and failures. More specifically, Markovian Bernoulli trials, as described in Section 1, are performed sequentially until either c consecutive successes or d consecutive failures are observed, whichever event occurs first.

Proposition 1. *The joint probability generating function (PGF) of* $\mathbf{Z}$ *is*

$$\phi_{\mathbf{Z}}(t_0, u_0, t_1, u_1, t_2, u_2) = E\left(t_0^{S_0} u_0^{F_0} t_1^{S_1} u_1^{F_1} t_2^{S_2} u_2^{F_2}\right) = (A + B)/Q, \tag{3}$$

where

$$A = (1 - p_1 t_1)(p_1 t_1)^{c-1}\left\{p_0 t_0 (1 - q_2 u_2) + q_0 u_0 p_2 t_2\left[1 - (q_2 u_2)^{d-1}\right]\right\},$$

$$B = (1 - q_2 u_2)(q_2 u_2)^{d-1}\left\{q_0 u_0 (1 - p_1 t_1) + p_0 t_0 q_1 u_1\left[1 - (p_1 t_1)^{c-1}\right]\right\},$$

$$Q = (1 - p_1 t_1)(1 - q_2 u_2) - p_2 t_2 q_1 u_1\left[1 - (p_1 t_1)^{c-1}\right]\left[1 - (q_2 u_2)^{d-1}\right].$$

Proof. Sequences of outcomes terminating in c consecutive successes, containing exactly k subsequences of failures and with the first trial being successful, can be described as

$$\underbrace{SSS\ldots S}_{i_1}\underbrace{FF\ldots F}_{j_1}\underbrace{SS\ldots S}_{i_2}\underbrace{FF\ldots F}_{j_2}\cdots\underbrace{SS\ldots S}_{i_k}\underbrace{FF\ldots F}_{j_k}\underbrace{SS\ldots S}_{c},$$

where $0 \leqslant i_1 \leqslant c-2$; $1 \leqslant i_a \leqslant c-1$, $a=2,3,\ldots,k$; $1 \leqslant j_a \leqslant d-1$, $a=1,2,\ldots,k$; and $k \geqslant 0$. The net contribution A_{1k} to the joint PGF of all such sequences is

$$\begin{aligned}
A_{1k} &= p_0t_0\left[1+p_1t_1+(p_1t_1)^2+\cdots+(p_1t_1)^{c-2}\right] \\
&\quad\times q_1u_1\left[1+q_2u_2+(q_2u_2)^2+\cdots+(q_2u_2)^{d-2}\right] \\
&\quad\times p_2t_2\left[1+p_1t_1+(p_1t_1)^2+\cdots+(p_1t_1)^{c-2}\right] \\
&\quad\times q_1u_1\left[1+q_2u_2+(q_2u_2)^2+\cdots+(q_2u_2)^{d-2}\right] \\
&\quad\vdots \\
&\quad\times p_2t_2\left[1+p_1t_1+(p_1t_1)^2+\cdots+(p_1t_1)^{c-2}\right] \\
&\quad\times q_1u_1\left[1+q_2u_2+(q_2u_2)^2+\cdots+(q_2u_2)^{d-2}\right] \\
&\quad\times p_2t_2(p_1t_1)^{c-1} \\
&= p_0t_0\left[p_2t_2\frac{1-(p_1t_1)^{c-1}}{1-p_1t_1}q_1u_1\frac{1-(q_2u_2)^{d-1}}{1-q_2u_2}\right]^k(p_1t_1)^{c-1}.
\end{aligned}$$

Adding all of these contributions gives

$$A_1=\sum_{k=0}^{\infty}A_{1k}=p_0t_0(p_1t_1)^{c-1}(1-p_1t_1)(1-q_2u_2)/Q,$$

where Q is as in (3).

Similarly, the total contribution to the joint PGF of terms beginning with a failure and ending with c consecutive successes can be shown to be

$$A_2=q_0u_0p_2t_2(p_1t_1)^{c-1}(1-p_1t_1)\left[1-(q_2u_2)^{d-1}\right]/Q.$$

By symmetry, the corresponding contributions B_1 and B_2 from sequences ending with d consecutive failures and beginning with a success or a failure, respectively, are easily derived. The joint PGF is then obtained as $A_1+A_2+B_1+B_2=(A+B)/Q$ where $A=(A_1+A_2)Q$ and $B=(B_1+B_2)Q$. □

Evaluating ϕ_Z of (3) at the appropriate values gives the marginal PGF's of X and W.

Corollary 1. *The PGF's of X and W, defined in* (2), *are*

$$\phi_X(t)=E(t^X)=\frac{p_1^{c-1}(1-q_0q_2^{d-1})(1-p_1t)t^c+q_2^{d-1}\left\{q_0(1-p_1t)+p_0q_1t\left[1-(p_1t)^{c-1}\right]\right\}}{1-p_1t-q_1(1-q_2^{d-1})t\left[1-(p_1t)^{c-1}\right]},\tag{4}$$

$$\phi_W(t)=E(t^W)=(C+D)/R,\tag{5}$$

where

$$C=(1-p_1t)p_1^{c-1}t^c\left\{p_0(1-q_2t)+q_0p_2t\left[1-(q_2t)^{d-1}\right]\right\},$$
$$D=(1-q_2t)q_2^{d-1}t^d\left\{q_0(1-p_1t)+p_0q_1t\left[1-(p_1t)^{c-1}\right]\right\},$$
$$R=(1-p_1t)(1-q_2t)-q_1p_2t^2\left[1-(p_1t)^{c-1}\right]\left[1-(q_2t)^{d-1}\right]. \quad \square$$

In view of the symmetry in the present waiting time problem, the marginal PGF of Y is easily obtained from (4). Two particular cases of (5) are of interest. First, replacing $p_0=p_1=p_2=p$ in (5) gives the result of ES (formula (2)) for the independent case. Second, letting $d\to\infty$ in (5) yields the PGF of the total number of trials until a run of c consecutive successes is observed, a result derived by BBV (formula (4)).

Most of the important features of W can be deduced from (5). For instance, evaluating $\phi_W'(t)$ at $t=1$ gives the expected waiting time which, after some simplification, reduces to

$$\mu_W=E(W)$$
$$=\frac{(1-p_2p_1^{c-1})(1-q_1q_2^{d-1})+p_2-p_1+(p_2-p_0)q_1p_1^{c-1}(1-q_2^{d-1})-(p_1-p_0)p_2q_2^{d-1}(1-p_1^{c-1})}{q_1p_2[1-(1-p_1^{c-1})(1-q_2^{d-1})]}. \tag{6}$$

Again, note that replacing $p_0=p_1=p_2=p$ in (6) gives the expected waiting time obtained by ES (formula (3)) for the independent case.

Alternatively, (5) can also be used for the calculation of probabilities about W. Recall that the values of the probability mass function (PMF) of W, $f_W(w)=\Pr(W=w)$, $w\geqslant 0$, are precisely the coefficients of (5) when expanded in powers of t. Writing (5) as $R\phi_W(t)=C+D$ yields a simple way of calculating $f_W(w)$ recursively by equating the coefficients of identical powers of t on both sides. This method yields the recurrence relation

$$f_W(w)=\alpha_w+(p_1+q_2)f_W(w-1)+(p_2-p_1)f_W(w-2)-q_1p_2p_1^{c-1}f_W(w-c-1)$$
$$-q_1p_2q_2^{d-1}f_W(w-d-1)+q_1p_2p_1^{c-1}q_2^{d-1}f_W(w-c-d), \tag{7}$$

for $w\geqslant 1$ where $C+D=\Sigma_w\alpha_wt^w$. These relationships are in fact complete and may be used systematically to compute all probabilities of W in a simple recursive manner for any specified c and d. Note that $f_W(w)=0$ for $w<\min\{c, d\}$.

3. Run and frequency quotas

The 'sooner case' of interest in this section arises when a run quota is imposed on the successes and only a frequency quota on the failures. Specifically, Markovian Bernoulli trials from structure (1) are performed sequentially until either c consecutive successes or d failures in total are observed, whichever event occurs first. It may be noted that this case has not been dealt with by ES in the independent case.

Arguments similar to those employed in the proof of Proposition 1 can be used to derive the joint PGF of Z for the present waiting time problem. The result will be stated without proof.

Proposition 2. *The joint PGF of* Z *is*

$$\phi_Z(t_0, u_0, t_1, u_1, t_2, u_2)=p_0t_0(p_1t_1)^{c-1}+AB^{d-1}+p_2t_2(p_1t_1)^{c-1}A(1-B^{d-1})/(1-B), \tag{8}$$

where

$$A = q_0u_0 + p_0t_0q_1u_1\left[1-(p_1t_1)^{c-1}\right]/(1-p_1t_1),$$
$$B = q_2u_2 + p_2t_2q_1u_1\left[1-(p_1t_1)^{c-1}\right]/(1-p_1t_1). \qquad \square$$

Corollary 2. *The marginal PGF's of X, Y and W are*

$$\phi_X(t) = CD^{d-1} + p_1^{c-1}t^c\left[p_0 + p_2C(1-D^{d-1})/(1-D)\right], \tag{9}$$
$$\phi_Y(t) = p_0p_1^{c-1} + p_2p_1^{c-1}at\left[1-(bt)^{d-1}\right]/(1-bt) + ab^{d-1}t^d, \tag{10}$$
$$\phi_W(t) = t^dCD^{d-1} + p_1^{c-1}t^c\left\{p_0 + p_2tC\left[1-(tD)^{d-1}\right]/(1-tD)\right\}, \tag{11}$$

where

$$C = q_0 + p_0q_1t\left[1-(p_1t)^{c-1}\right]/(1-p_1t), \qquad D = q_2 + q_1p_2t\left[1-(p_1t)^{c-1}\right]/(1-p_1t),$$
$$a = 1-p_0p_1^{c-1} \quad \textit{and} \quad b = 1-p_2p_1^{c-1}. \qquad \square$$

Note that letting $c \to \infty$ in (11) gives $\phi_W(t) = GH^{d-1}$, where $G = q_0t + p_0q_1t^2/(1-p_1t)$ and $H = q_2t + q_1p_2t^2/(1-p_1t)$, which is the PGF obtained by VBB (formula (2)) for the negative binomial analogue under Markovian Bernoulli trials. Also, the corresponding results for the independent case can be obtained from (9)–(11) by replacing $p_0 = p_1 = p_2 = p$; this case has been considered in a general multinomial set-up by Sobel and Ebneshahrashoob (1992) recently.

Again, many features of X, Y and W can be deduced from the PGF's (9)–(11). In particular, it can be seen from (10) that the PMF of the total number of failures Y assumes the explicit form

$$f_Y(0) = p_1^{c-1}(p_0 + p_2a), \qquad f_Y(y) = p_2p_1^{c-1}ab^{y-1}, \quad 1 \leqslant y \leqslant d-1, \qquad f_Y(d) = ab^{d-1}, \tag{12}$$

where a and b are as in (10).

Results for the case when a run quota on failures and a frequency quota on successes are imposed can be obtained by symmetry.

4. Frequency quotas

The last of the 'sooner cases' to be addressed relates to the situation in which frequency quotas are in place on both successes and failures. In this 'sooner' case, Markovian Bernoulli trials are performed sequentially until either a total of c successes or a total of d failures is completed, whichever event comes up first.

As far as the analysis is concerned, this case is closely related to the negative binomial analogue for Markovian Bernoulli trials discussed in Section 2 of VBB. Although conceptually simpler than the previous cases, the joint PGF of **Z** here does not assume a simple form.

Proposition 3. *The joint PGF of* **Z** *can be described as*

$$\phi_{\mathbf{Z}}(t_0, u_0, t_1, u_1, t_2, u_2) = \sum\left(\textit{all terms of degree at most } c+d-1 \textit{ in } \phi_{\mathbf{Z}}^{(1)} + \phi_{\mathbf{Z}}^{(2)}\right), \tag{13}$$

where

$$\phi_Z^{(1)} = \phi_Z^{(1)}(t_0, u_0, t_1, u_1, t_2, u_2) = \left[p_0 t_0 + \frac{q_0 u_0 p_2 t_2}{1 - q_2 u_2}\right]\left[p_1 t_1 + \frac{q_1 u_1 p_2 t_2}{1 - q_2 u_2}\right]^{c-1},$$

$$\phi_Z^{(2)} = \phi_Z^{(2)}(t_0, u_0, t_1, u_1, t_2, u_2) = \left[q_0 u_0 + \frac{p_0 t_0 q_1 u_1}{1 - p_1 t_1}\right]\left[q_2 u_2 + \frac{p_2 t_2 q_1 u_1}{1 - p_1 t_1}\right]^{d-1}. \qquad \square$$

Although formal expressions for the marginal PGF's of X, Y and W can be derived from (13), the authors found it useful to explore other avenues. The analysis will focus on the waiting time variable W.

A simple way of computing the PMF of W is by means of the relationship

$$f_W(w) = f_{N_c}(w) + f_{M_d}(w), \quad w = w_0, w_0 + 1, \ldots, c + d - 1, \tag{14}$$

where $w_0 = \min\{c, d\}$, and N_c and M_d are the numbers of trials needed to complete a total of c successes and a total of d failures, respectively. N_c and M_d are analogues of negative binomial variates for Markovian Bernoulli trials, and several ways of computing their PMF's have been discussed by VBB (Section 2).

For the independent case ($p_0 = p_1 = p_2 = p$), N_c and M_d have standard negative binomial distributions, thus (14) yields the apparently unnoticed identity

$$\sum_{i=c}^{c+d-1} \binom{i-1}{c-1} p^c q^{i-c} + \sum_{i=d}^{c+d-1} \binom{i-1}{d-1} q^d p^{i-d} = 1.$$

Regarding the calculation of moments of W, no closed-form expressions appear feasible in this approach even for the independent case. However, since the range of W is finite, any desired moment can be obtained numerically once $f_W(w)$ has been calculated using (14). See Section 6 for an illustration. It should be mentioned here that Sobel, Uppuluri and Frankowski (1985) have given explicit formulas for the moments of W (in the independent case) in terms of Dirichlet integrals of Type 2.

5. Later cases

Associated with each of the 'sooner cases' in Sections 2-4 there is a 'later case' obtained by stopping the experiment when both quotas are fulfilled. For instance, for the case of run quotas discussed in Section 2, the corresponding 'later case' will arise when stopping the experiment upon completion of both a run of c successes and a run of d failures.

In relation to the probabilistic analysis, all the relevant aspects can be deduced from the results already derived for the 'sooner cases'. This claim finds justification in a duality, already noticed by ES for independent Bernoulli trials, between each 'sooner case' and the corresponding 'later case'. This duality comes about by recognizing that a 'sooner case' arises as the union of two events while the associated 'later case' relates to the corresponding intersection. As far as the calculation of the joint PGF of $\mathbf{Z}$ is concerned, this duality implies that

$$\phi_{\mathrm{LZ}}^{(c,d)} = \phi_{\mathrm{SZ}}^{(c,\infty)} + \phi_{\mathrm{SZ}}^{(\infty,d)} - \phi_{\mathrm{SZ}}^{(c,d)},$$

where $\phi_{\mathrm{SZ}}^{(c,d)}$ and $\phi_{\mathrm{LZ}}^{(c,d)}$ are the PGF's of $\mathbf{Z}$ for the 'sooner' and 'later' cases, respectively. Note that $\phi_{\mathrm{SZ}}^{(c,\infty)}(\phi_{\mathrm{SZ}}^{(\infty,d)})$ is the PGF of $\mathbf{Z}$ when the quota is imposed only on the c successes (d failures). A similar relationship holds for the marginal PGF's of each variate X, Y and W. This also yields a corresponding relationship for the factorial moments of the variates.

6. Illustrative example: A learning model

As noted by VBB, the correlation structure (1) is nothing but a 2-state Markov chain with state space $\{S, F\}$, transition matrix

$$P = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix},$$

and with initial distribution $\pi_1 = (p_0, q_0)$. Therefore, any practical situation giving rise to such Markov chains will provide an application for the results discussed in this article. Further, arbitrary discrete Markov chains could be examined under this theory by proper grouping of states.

Bush and Mosteller (1951) proposed the following model for representing individual learning to make the correct response to a certain stimulus. The description and interpretation given here have been taken from Bailey (1964, p. 51). Let S be a correct response and F an incorrect one. If it can be assumed that, in a series of successive trials, the response at any trial depends only on the response at the previous trial, then the series can be represented by a 2-state Markov chain. Here p_1 and p_2 are the chances of a correct response following, respectively, a correct one and an incorrect one. When $p_1 = p_2$, each response is independent of the previous one; when $p_1 < 0.5$ and $p_2 > 0.5$, there is a tendency to oscillate from one response to the other; and when $p_1 > q_2$, there is some kind of preference for correct responses. See Bailey (1964, pp. 51–53) for a Markov chain analysis.

Consider the case $p_1 = 0.7$, $p_2 = 0.4$ and the initial distribution $p_0 = q_0 = 0.5$. Some inclination to respond as in the previous trial is noted in this subject, with a random guess at the initial trial. Consider the total number of trials W performed until either a run of 5 correct responses or a run of 3 incorrect responses is completed, whichever occurs first. Thus, this is a 'sooner case' of the type discussed in Section 2 with $c = 5$ and $d = 3$. Some values of the PMF of W, as calculated from recursion (7), are reported in Table 1. Recall that $f_W(w) = 0$ for $w < d$. Note that the chance that the experiment stops by the sixth trial is $\Pr(W \leqslant 6) = 0.522$ or 52.2%, while the chance that the experiment has to go beyond the 20th trial is $\Pr(W > 20) = 0.027$ or about 3%. The mean waiting time, as calculated from (6), is $\mu_W = 7.8$ trials.

As a numerical illustration of the 'sooner case' based on frequency quotas discussed in Section 4, consider the waiting time W until either a total of $c = 15$ correct responses or a total of $d = 7$ incorrect responses is completed, whichever comes up first. The PMF of W has been calculated using (14) and the results are reported in Table 2. One should realize in this case that the minimum value for W is $\min\{c, d\} = 7$ and the maximum value of W is $c + d - 1 = 21$.

Note that since $\Pr(W \geqslant 16) = 0.5$, the subject will be relieved from the experiment, on an average, by trial 15 half the time. The mean and the standard deviation of the waiting time are readily computed

Table 1
PMF of the waiting time W for the learning model under run quotas in the 'Sooner case' with $c = 5$ and $d = 3$, when $p_0 = 0.5$, $p_1 = 0.7$ and $p_2 = 0.4$

w	$f_W(w)$	w	$f_W(w)$	w	$f_W(w)$	w	$f_W(w)$	w	$f_W(w)$	w	$f_W(w)$
3	0.180	7	0.097	11	0.040	15	0.018	19	0.008	23	0.003
4	0.054	8	0.061	12	0.032	16	0.014	20	0.006	24	0.003
5	0.179	9	0.057	13	0.026	17	0.012	21	0.005	25	0.002
6	0.109	10	0.049	14	0.021	18	0.009	22	0.004	$\geqslant 26$	0.010

Table 2
PMF of the waiting time W for the learning model under frequency quotas in the 'sooner case' with $c = 15$ and $d = 7$, when $p_0 = 0.5$, $p_1 = 0.7$ and $p_2 = 0.4$

w	$f_W(w)$	w	$f_W(w)$	w	$f_W(w)$	w	$f_W(w)$	w	$f_W(w)$
7	0.023	10	0.056	13	0.069	16	0.072	19	0.086
8	0.035	11	0.063	14	0.068	17	0.076	20	0.091
9	0.047	12	0.067	15	0.070	18	0.081	21	0.094

from Table 2 to be $\mu_W = \Sigma_{w=7}^{21} w f_W(w) = 15.2$ trials and $\sigma_W = [\Sigma_{w=7}^{21}(w - \mu_W)^2 f_W(w)]^{1/2} = 4.1$ trials, respectively.

7. Conclusions

The waiting time problems discussed by ES have been generalized here to the case when the Bernoulli trials are correlated in a Markovian fashion. It should be mentioned here that further generalizations are also possible. Extension of the work of ES to the case when the population is finite will be of great interest in acceptance sampling framework as lots are of finite size. The problem where the probabilities of success in different trials do not remain constant will be quite relevant in start-up demonstration testing in an industrial setup; here, the probabilities of success change since the experimenter takes corrective action on the equipment being tested when the first failure (or after a certain number of failures) occurs. See, for example, Hahn and Gage (1983) and BBV. Problems involving multi-stage acceptance/rejection criteria based on run and frequency quotas may also be studied and some work in this direction has already been done by Sobel, Ebneshahrashoob and Lin (1989). Results presented in this paper can also be extended to higher-order dependence models. Work on some of these problems is currently in progress.

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