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Common Due-Date Assignment and Scheduling on Single Machine with Exponential Processing Times

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Abstract

This article deals with the problem of common due-date assignment and scheduling on single machine with exponential processing times. The objective is to minimize the expected total cost associated with the due-date and earliness/tardiness of jobs. For large due-date cost, we have derived a closed-form solution, which is applicable to the general processing times as well. In the other case, it is shown that an optimal sequence lies among the V-shaped sequences, and an algorithm is developed for the derivation of optimal solution.

Key words

Stochastic Scheduling, Due-date Assignment, Early/tardy.

1. Introduction

Consider a single machine with n independent jobs, all available for processing (non-preemptively) at time zero. All the jobs have common due-date which is unknown. The problem is to find an optimal value of the due-date and an optimal sequence which minimize the total cost based on due-date value and the earliness/tardiness of each job.

Traditionally, the due-date has been assumed to be externally determined (beyond the control of the decision-maker). Under such situation, the decision problem is essentially scheduling the jobs subject to' their prescribed due-dates. Conway [6] was the first to introduce the notion of attainable due-date which is internally determined. With this orientation, the decision problem is both assigning due-dates to jobs and scheduling them.

Recently, the issue of due-date assignemt has received considereable attention of the researchers in the field of scheduling. For instance, refer to De et al. [7, 8], Cheng and Gupta [5], Baker [4] etc., Analytical studies of the problem were initiated by Seidmann et al. [13] and Panwalkar et al. [12].

The present study is motivated by Panwalkar et al. [12]. They have given elegant solution to the problem wherein job processing time are known constants. However, in many real-life sitations, the processig times are likely to be random and this calls for scheduling analysis under uncertainty (refer to Frost [9], Al-Turki et al. [2, 3]). In such cases a common modelling assumption has been the use of exponentially distributed processing time. For example, see Glazebrook [10], Weiss and Pinedo [14], Agrawala et al. [1], Kampke [11] and so on.

In this paper, we study the common due-date (fixed) determination and sequencing problem on single machine, when the job processing times are exponentially distributed.

We first formulate the problem in Section 2 as the minimization of expected total cost. In Section 3, we present the preliminary results that are used later to derive the properties of optimal due-date and sequence. In this section, we derive the distribution of the sum of independent and distinct exponential variables. The main results are presented in Section 4. For large due-date cost, a closed form solution is provided. In the other case, it is shown that an optimal sequence can be found in the set of V-shaped sequences. An algorithm is also developed for this case.

2. Problem Formulation

The processing time X_i of job $i\,(1 \le i \le n)$ is assumed to be random having exponential distribution with parameter λ_i. We assume that X_i's are independent, and distinct (i.e., λ_i's are distinct). Let us denote the pdf (probability density function) and cdf (cumulative distribution function) of X_i by $f_i(.)$ and $F_i(.)$ respectively, so that

$$f_i(x) = \lambda_i e^{-\lambda_i x} \text{ and } F_i(x) = 1 - e^{-\lambda_i x} \text{ when } x \ge 0, \text{ for } i = 1,....,n.$$

For a sequence $\pi = (\pi_1,...,\pi_n)$ of n jobs, let $C_i(\pi)$ be the completion time of π_i (the *i-th* job in π)

It is also assumed that all the jobs have common due-date δ which is fixed but unknown. Next, let us denote the earliness and tradiness of the job π_i by

$E_i(\delta, \pi)$ and $T_i(\delta, \pi)$ respectively.

Therefore, for any fixed sequence π and given due-date δ,

$$C_i(\pi) = \sum_{j=1}^{i} X_{\pi_j},$$

$$E_i(\delta, \pi) = \max\{0, \delta - C_i(\pi)\}$$

and $$T_i(\delta, \pi) = \max\{0, C_i(\pi) - \delta\}$$

for $i = 1, \ldots, n$.

We denote the cost rate (per unit of time) for *(i)* due-date, *(ii)* earliness and *(iii)* tardiness by P_1, P_2 and P_3 respectively, and they are assumed to be fixed and known.

Then, the total cost, denoted by $Z(\delta,\pi)$, associated with a specified value of due-date δ and a given sequence π, is

$$Z(\delta, \pi) = \sum_{i=1}^{n} [P_1\delta + P_2 E_i(\delta,\pi) + P_3 T_i(\delta,\pi)]. \tag{1}$$

It must be noted that $E_i(\delta,\pi)$ and $T_i(\delta,\pi)$ are random, and consequently $Z(\delta,\pi)$ is also random. Let us use the notation $\varepsilon(W)$ to represent the expectation of any random variable W.

The problem under consideration is to minimize the expected total cost, i.e.,

Find δ^* and π^* such that

$$\varepsilon[Z(\delta^*, \pi^*)] = \min_{\delta,\pi} \varepsilon[Z(\delta,\pi)]. \tag{2}$$

3. Preliminary Results

In this section, we present some preliminary results that are required to derive the optimal due-date and optimal sequence obtained in the next section.

Lemma 1 : For any r ($r \geq 2$) distinct numbers – a_i ($i = 1, 2, \ldots, r$),

(a) $$\sum_{i=1}^{r} \frac{1}{\prod_{j=1, j\neq i}^{r} (a_j - a_i)} = 0, \tag{3}$$

(b) $$\sum_{i=1}^{r} \frac{\prod_{\substack{j=1\\ \neq i}}^{r} a_j}{\prod_{\substack{j=1\\ \neq i}}^{r} (a_j - a_i)} = 1, \tag{4}$$

Proof : The proof is given in the Appendix.

The following lemma gives the density function of the sum of exponential random variables having distinct parameters.

Lemma 2 : Lex $X_i \cap$ exp (λ_i) for $i = 1, 2, ..., r$ $(r > 2)$ be distinct and mutually independent. Then the pdf of $\sum_{i=1}^{r} X_i$, denoted by g_r (t), is given by

$$g_r(t) = (\Pi_{j=1}^{r} \lambda_j) \sum_{i=1}^{r} \frac{e^{-\lambda_i t}}{\prod_{\substack{j=1\\ \neq i}}^{r} (\lambda_j - \lambda_i)}. \tag{5}$$

Proof : The proof is based on Lemma *1(a)* and is given in the Appendix.

Corollary 1 : The cdf of $\sum_{i=1}^{r} x_i$ (as defined in Lemma 2) is given by

$$G_r(t) = P\left[\sum_{i=1}^{r} X_i < t\right]$$

$$= 1 - \sum_{i=1}^{r} \frac{\prod_{\substack{j=1\\ \neq i}}^{r} \lambda_j}{\prod_{\substack{j=1\\ \neq i}}^{r} (\lambda_j - \lambda_i)} e^{-\lambda_i t}$$

Proof : It is obvious from Lemma 2.

Lemma 3 : For a specified value of due-date δ and a given sequence π, the expected total cost is

$$\varepsilon[Z(\delta,\pi)] = n(P_1 + P_2)\delta + P_3 \sum_{i=1}^{n} (n - i + 1)\mu_{\pi_i}$$

$$- (P_2 + P_3) \sum_{i=1}^{n} \int_{t=0}^{\delta} \overline{G_i}\,(t/\pi)dt \tag{6}$$

where $\overline{G}_i(t/\pi) = P[C_i(\pi) > t]$

and $\mu_{\pi_i} = \varepsilon[X_{\pi_i}]$.

Proof : We have, for any i $(1 \leq i \leq n)$,

$$\begin{aligned} E_i(\delta, \pi) &= \max\{0, \delta - C_i(\pi)\} \\ &= \delta - \min\{C_i(\pi), \delta\} \\ \text{and } T_i(\delta,\pi) &= \max\{0, C_i(\pi) - \delta\} \\ &= C_i(\pi) - \min\{C_i(\pi), \delta\}. \end{aligned}$$

This implies that

$$\varepsilon[E_i(\delta,\pi)] = \delta - \varepsilon[\min\{C_i(\pi),\delta\}] \tag{7}$$

$$\text{and } \varepsilon[T_i(\delta,\pi)] = \sum_{j=1}^{i} \mu_{\pi_j} - \varepsilon[\min\{C_i(\pi),\delta\}] \tag{8}$$

Now,

$$\begin{aligned} \varepsilon[\min\{C_i(\pi), \delta\}] &= \int_{t=0}^{\infty} P[\min\{C_i(\pi),\delta\} > t]\,dt \\ &= \int_{t=0}^{\delta} P[C_i(\pi) > t]\,dt \\ &= \int_{t=0}^{\delta} \overline{G}_i(t/\pi)dt. \end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned} \varepsilon[Z(\delta,\pi)] &= nP_1\delta + P_2 \sum_{i=1}^{n} \varepsilon[E_i(\delta,\pi)] + P_3 \sum_{i=1}^{n} \varepsilon[T_i(\delta,\pi)] \\ &= n(P_1 + P_2)\delta + P_3 \sum_{i=1}^{n}\sum_{j=1}^{i} \mu_{\pi_j} - (P_2 + P_3) \sum_{i=1}^{n} \int_{t=0}^{\delta} \overline{G}_i(t/\pi)dt \end{aligned}$$

(by using (7), (8) and (9).)

Hence the result follows.

Lemma 4 : Let δ ($\geq$ 0) be a constant. Let U and V be two non- negative continuous random variables which are independent. Denote the pdf and inverse cdf of $U(V)$ by $h_U(\cdot)$ $(h_V(\cdot))$ and $\bar{H}_U(\cdot)$ $(\bar{H}_V(\cdot))$ respectively. Then

$$E[min\ \{U + V,\delta\}] = \int_{t=0}^{\delta} \left[\int_{u=0}^{t} \bar{H}_V\ (t-u)h_U(u)du + \bar{H}_U(t)\right] dt.$$

Proof : We know that

$$\begin{aligned} & E[min\{U + V, \delta\}] \\ &= \int_{t=0}^{\infty} P[min\{U + V,\delta\} > t]dt \\ &= \int_{t=0}^{\delta} P[U + V > t]dt \\ &= \int_{t=0}^{\delta}\left[\int_{u=0}^{\infty} P[V > t - u]\ h_U(u)du\right] dt \\ &= \int_{t=0}^{\delta}\left[\int_{u=0}^{t} \bar{H}_V(t - u]\ h_U(u)du + \bar{H}_U(t)\right] dt. \end{aligned}$$

Hence the lemma holds.

4 Main Results

This section deals with the determination of optimal due-date and optimal sequence of the jobs in order to minimize the *expected total cost.*

The following lemma describes the behaviour of the *expected total cost* function.

Lemma 5 : For a given sequence π,

(a) $$\frac{d\varepsilon[Z(\delta,\pi)]}{d\delta} = n(P_1 - P_3) + (P_2 + P_3)\sum_{i=1}^{n} G_i(\delta/\pi),$$

(b) $\varepsilon[Z(\delta,\pi)]$ *is a convex function of* δ.

Proof : Using Lemma 3, it is easy to note that

$$\frac{d\varepsilon[Z(\delta,\pi)]}{d\delta} = n(P_1 - P_3) + (P_2 + P_3)\sum_{i=1}^{n} \bar{G}_i(\delta/\pi)$$

and $$\frac{d^2\varepsilon[Z(\delta,\pi)]}{d\delta^2} = (P_2 + P_3)\sum_{i=1}^{n} g_i(t/\pi),$$

where $g_i(\ /\pi)$ is the pdf of $C_i(\pi)$.

Hence the result follows.

We present below a theorem that gives solution to the problem if the cost rate for due-date is large enough.

Theorem 1 : If $P_1 \geq P_3$, then optimal due-date $\delta^* = 0$, and optimal sequence $\pi^* = (\pi_1,\ldots,\pi_n)$ is such that $\mu_{\pi_1} < \mu_{\pi_2} < \ldots < \mu_{\pi_n}$.

Proof : For a given sequence π, we get from Lemma 5 that

$$\frac{d\varepsilon[Z(\delta,\pi)]}{d\delta} = n(P_1 - P_3) + (P_2 + P_3)\sum_{i=1}^{n} G_i(\delta/\pi)$$

$$\geq 0 \ (\text{since } P_1 \geq P_3),$$

i.e., $\varepsilon[Z(\delta,\pi)]$ is an increasing function of δ. Therefore, $\delta^* = 0$ minimizes $\varepsilon[Z(\delta,\pi)]$ for any given π.

Next, with $\delta^* = 0$, we get from Lemma 3,

$$\varepsilon[Z(\delta^*,\pi)] = P_3\sum_{i=1}^{n}(n - i + 1)\mu_{\pi_i} \tag{10}$$

It is known that the sum on the right-hand-side of the equation (10) is minimized by $\pi^* = (\pi_1,\ldots,\pi_n)$ when $\mu_{\pi_1} < \mu_{\pi_2} < \ldots < \mu_{\pi_n}$.

This completes the proof.

Remark 1 : The π^* in Theorem 1 is called SEPT (Shortest Expected Processing Time) sequence, and the result is same as that of Panwalkar et al. [12] for the deterministic case.

When $P_1 \geq P_3$, the Theorem 1 provides the solution of the problem under consideration. We assume that $P_3 > P_1$ throughout the remaining part of this section.

Theorem 2 : For a given sequence π, $\varepsilon[Z(\delta, \pi)]$ has unique minimum at $\delta = \delta_0$ where δ_0 is solution of

$$\sum_{i=1}^{n} G_i(\delta/\pi) = \frac{n(P_3 - P_1)}{P_2 + P_3}. \tag{11}$$

Proof : The proof follows from Lemma 5.

Remark 2 : Using Corollary 1, we can solve the equation (11) for δ by numerical method.

Remark 3 : Notice that the Theorems 1 and 2 hold for arbitrary random job processing times.

The following lemma evaluates the effect on the *expected total cost* when two adjacent jobs are interchanged in a sequence.

Lemma 6 : Let $\pi(\pi_1,...,\pi_n)$ be any sequence of the jobs. The sequence π' is obtained from π by interchanging the jobs π_r and π_{r+1} only $(1 \leq r \leq n-1)$, i.e. $\pi' = (\pi_1,..., \pi_{r-1}, \pi_{r+1}, \pi_r, \pi_{r+2},..., \pi_n)$. For any fixed δ, the difference in expected total cost between π and π' is given by

$$\varepsilon[Z(\delta,\pi) - Z(\delta,\pi')] = (\mu_{\pi_r} - \mu_{\pi_{r+1}})[P_3 - (P_2 + P_3)G_{r+1}(\delta/\pi)].$$

Proof : Observe that $C_i(\pi)$ and $C_i(\pi')$ are identical except for $i = r$. Therefore, the result can easily be verified for $r = 1$.

Assume that $r \geq 2$. Then $C_r(\pi) = C_{r-1}(\pi) + X_{\pi_r}$ and $C_r(\pi') = C_{r-1}(\pi) + X_{\pi_{r+1}}$. Consequently, using Lemma 3, we have

$$\begin{aligned}
&\varepsilon[Z(\delta,\pi) - Z(\delta,\pi')] \\
&= P_3(\mu_{\pi_r} - \mu_{\pi_{r+1}}) - (P_2 + P_3)\,\varepsilon\,[\min\{C_{r-1}(\pi) + X_{\pi_r}, \delta\} \\
&\qquad - \min\{C_{r-1}(\pi) + X_{\pi_{r+1}}, \delta\}] \\
&= P_3(\mu_{\pi_r} - \mu_{\pi_{r+1}}) - (P_2 + P_3)\int_{t=0}^{\delta}\int_{y=0}^{t}\left[\bar{F}_r(t-y) - \bar{F}_{r+1}(t-y)\right] g_{r-1}(y/\pi)\,dy\,dt
\end{aligned}$$

(by Lemma 4)

$$= P_3(\mu_{\pi_r} - \mu_{\pi_{r+1}}) - (P_2 + P_3)\cdot K \quad (say).$$

For the sake of simplicity in notation, we denote λ_{π_i} by λ_i for i = 1,...r, in the following part of the proof only.

Now, with the help of Lemma 2, we can write

$$K = \left(\Pi_{j=1}^{r-1}\lambda_j\right) \sum_{i=1}^{r-1} \frac{1}{\Pi_{\substack{j=1\\ \neq i}}^{r-1}(\lambda_j - \lambda_i)} \int_{t=0}^{\delta}\left[\int_{y=0}^{t} \{e^{-\lambda_r(t-y)} - e^{-\lambda_{r+1}(t-y)}\}\, e^{-\lambda_i y}dy\right] dt$$

$$= \left(\Pi_{j=1}^{r-1}\lambda_j\right) \sum_{i=1}^{r-1} \frac{1}{\Pi_{\substack{j=1\\ \neq i}}^{r-1}(\lambda_j - \lambda_i)} \cdot \frac{(\lambda_{r+1} - \lambda_r)}{\lambda_r\lambda_{r+1}}$$

$$\bullet \left[-\frac{\lambda_r\lambda_{r+1}}{\lambda_i(\lambda_r - \lambda_i)(\lambda_{r+1} - \lambda_r)} \left\{e^{-\lambda_i\delta} - 1\right\} + \frac{\lambda_{r+1}}{(\lambda_r - \lambda_i)(\lambda_{r+1} - \lambda_r)} \left\{e^{-\lambda_r\delta} - 1\right\} - \frac{\lambda_r}{(\lambda_{r+1} - \lambda_i)(\lambda_{r+1} - \lambda_r)} \left\{e^{-\lambda_{r+1}\delta} - 1\right\}\right] \text{(on simplification)}$$

$$= \frac{(\lambda_{r+1} - \lambda_r)}{\lambda_r\lambda_{r+1}} \left[-\sum_{i=1}^{r-1} \frac{\Pi_{\substack{j=1\\ \neq i}}^{r+1}\lambda_j}{\Pi_{\substack{j=1\\ \neq i}}^{r+1}(\lambda_j - \lambda_i)} \left\{e^{-\lambda_i\delta} - 1\right\} \right.$$

$$+ \frac{\lambda_{r+1}\cdot\Pi_{j=1}^{r-1}\lambda_j}{(\lambda_{r+1} - \lambda_r)} \sum_{i=1}^{r-1} \frac{1}{(\lambda_r - \lambda_i)\,\Pi_{\substack{j=1\\ \neq i}}^{r-1}(\lambda_j - \lambda_i)} \left\{e^{-\lambda_r\delta} - 1\right\}$$

$$\left. - \frac{\lambda_r\cdot\Pi_{j=1}^{r-1}\lambda_j}{(\lambda_{r+1} - \lambda_r)} \sum_{i=1}^{r-1} \frac{1}{(\lambda_{r+1} - \lambda_i)\,\Pi_{\substack{j=1\\ \neq i}}^{r-1}(\lambda_j - \lambda_i)} \left\{e^{-\lambda_{r+1}\delta} - 1\right\} \right]$$

$$= \frac{(\lambda_{r+1} - \lambda_r)}{\lambda_r\lambda_{r+1}} \left[-\sum_{i=1}^{r-1} \frac{\Pi_{\substack{j=1\\ \neq i}}^{r+1}\lambda_j}{\Pi_{\substack{j=1\\ \neq i}}^{r+1}(\lambda_j - \lambda_i)} \left\{e^{-\lambda_i\delta} - 1\right\} \right.$$

$$- \frac{\prod_{\substack{j=1 \\ \neq r}}^{r+1} \lambda_j}{\prod_{\substack{j=1 \\ \neq r}}^{r+1} (\lambda_j - \lambda_r)} \left\{ e^{-\lambda_r \delta} - 1 \right\}$$

$$\left. - \frac{\prod_{\substack{j=1 \\ \neq r+1}}^{r+1} \lambda_j}{\prod_{\substack{j=1 \\ \neq r+1}}^{r+1} (\lambda_j - \lambda_{r+1})} \left\{ e^{-\lambda_{r+1} \delta} - 1 \right\} \right] \text{ (by applying Lemma 1(a))}$$

$$= \frac{(\lambda_{r+1} - \lambda_r)}{\lambda_r \lambda_{r+1}} \left[\sum_{i=1}^{r+1} \frac{\prod_{\substack{j=1 \\ \neq i}}^{r+1} \lambda_j}{\prod_{\substack{j=1 \\ \neq i}}^{r+1} (\lambda_j - \lambda_i)} \left\{ 1 - e^{-\lambda_i \delta} \right\} \right]$$

$$= (\mu_{\pi_r} - \mu_{\pi_{r+1}}) G_{r+1}(\delta / \pi) \text{ (using Lemma 1(b) and Corollary 1).}$$

Therefore,

$$\varepsilon[Z(\delta,\pi) - Z(\delta,\pi')] = P_3(\mu_{\pi_r} - \mu_{\pi_{r+1}}) - (P_2 + P_3)(\mu_{\pi_r} - \mu_{\pi_{r+1}}) G_{r+1}(\delta/\pi)$$

$$= (\mu_{\pi_r} - \mu_{\pi_{r+1}}) [P_3 - (P_2 + P_3) G_{r+1}(\delta/\pi)].$$

Hence the lemma holds.

We know that for any due-date δ and sequence p, $G_1 (\delta | \pi) \geq \ldots \geq G_n (\delta | \pi)$. Let $Q = P_3/(P_2 + P_3)$. Nete that $0 < Q \leq 1$.

Lemma 7 : Assume that due-date δ is given. For a sequence $\pi = (\pi_1, \ldots, \pi_n)$, suppose that

$$G_k (\delta | \pi) \geq Q > G_{k+1} (\delta / \pi) \text{ for some } 0 \leq k \leq n, \tag{12}$$

where $G_o (\delta / \pi) = 1$ and $G_{n+1} (\delta / \pi) = 0$. We have

(a) if $\mu_{\pi_r} < \mu_{\pi_{r+1}}$ for some $1 \leq r \leq k - 1$, then $\pi' = (\pi_1, \ldots, \pi_{r-1}, \pi_{r+1}, \pi_r, \pi_{r+2}, \ldots, \pi_n)$ is at least as good as π,

(b) if $\mu_{\pi_s} > \mu_{\pi_{s+1}}$ for some $k \leq s \leq n - 1$, then $\pi'' = (\pi_1, \ldots, \pi_{s-1}, \pi_{s+1}, \pi_s, \pi_{s+2}, \ldots, \pi_n)$ is at least as good as π,

Proof : Using Lemma 6, the proof is simple.

The following result helps us to reduce the effort in the search for optimal sequence.

Theorem 3 : Let the due-date δ be given. In order to minimize the expected total cost $\varepsilon[Z(\delta, \pi)]$, it is enough to consider the sequences - π's which satisfy the condition :

$$\mu_{\pi_1} > \dots > \mu_{\pi_k} < \dots < \mu_{\pi_n} \text{ for some } 1 \leq k \leq n, \quad (13)$$

where $\pi = (\pi_1, \dots, \pi_n)$.

Proof : The proof follows from repeated applications of Lemma 7.

Remark 4 : A sequence having property (13) is called V-shaped. Let us represent the set of all such sequences by V.

Finally, we present an algorithm for minimization of $\varepsilon[Z(\delta, \pi)]$ (when $P_3 > P_1$) based on the results obtained.

Algorithm :

Step 0 : Set $Z_o = \infty$.

Step 1 : If $V = \phi$, then goto Step 4. Else, take $\pi \in V$ and update $V \leftarrow V \setminus \{\pi\}$.

Step 2 : Let δ be the solution of the equation (11). Evaluate $\varepsilon[Z(\delta, \pi)]$ using the relation (6).

Step 3 : If $\varepsilon[Z(\delta, \pi)] < Z_o$, $\leftarrow \varepsilon\ [Z(\delta, \pi)]$, $\pi^* \leftarrow \pi$ and $\delta^* \leftarrow \delta$. Goto Step 1.

Step 4 : Return as optimal due-date δ^* and optimal sequence π^* with Z_o as the corresponding minimum expected total cost.

5. Discussion

In this aritcle, we have dealt with a stochastic version of the common due-date determination and scheduling problem on single machine, where the object is to minimize the *expected total cost* associated with the due-date and earliness/tardiness of the jobs.

With $P_1 \geq P_3$, we have derived optimal solution (Theorem 1) for general processing times. This solution is same as that of Panwalkar et al. [12] for the deterministic processing times. The case of $P_3 \geq P_1$ is analysed for distinct exponential processing times. It is shown that the search for optimal sequence may be confined to V-shaped sequences only (Theorem 3). The procedure (Theorem 2) for the determination of optimal due-date is also obtained. (In fact, it holds for genral processing times.) Based on these results, we have developed an algorithm for the dervation of optimal solution.

A closer view of the Theorems 2 and 3 suggest that similar characterization of the problem with general (or even with arbitrary exponential) processing times is quite difficult.

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Appendix

Lemma 1 : For any r (r $\geq$ 2) distinct numbers – a_i (i = 1, 2,...,r),

$$(a)\ \sum_{i=1}^{r} \frac{1}{\prod_{\substack{j=1\\ \neq i}}^{r}(a_j - a_i)} = 0, \tag{14}$$

$$(b)\ \sum_{i=1}^{r} \frac{\prod_{\substack{j=1\\ \neq i}}^{r} a_j}{\prod_{\substack{j=1\\ \neq i}}^{r}(a_j - a_i)} = 1. \tag{15}$$

Proof of Part (a) : We prove it by induction.

For m = 2, the result is trivial. Suppose, the result is true for m = r – 1. We wil show that the same holds for m = r.

Let $a_1, a_2, \ldots, a_r$ be the distinct numbers. Denote the left-hand-side of (14) by L. Then

$$\prod_{k=1}^{r-1}(a_k - a_r).\ L = -\sum_{i=1}^{r-1} \frac{\prod_{\substack{k=1\\ \neq i}}^{r-1}(a_k - a_r)}{\prod_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)} + 1$$

$$= -\frac{\prod_{k=2}^{r-1}(a_k - a_r)}{\prod_{j=2}^{r-1}(a_j - a_1)} - \sum_{i=2}^{r-1} \frac{\prod_{\substack{k=1\\ \neq i}}^{r-1}(a_k - a_r)}{\prod_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)} + 1$$

$$= \sum_{i=2}^{r-1} \frac{\prod_{k=2}^{r-1}(a_k - a_r)}{\prod_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)} - \sum_{i=2}^{r-1} \frac{\prod_{\substack{k=1\\ \neq i}}^{r-1}(a_k - a_r)}{\prod_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)} + 1. \tag{16}$$

This is because of the assumption that the result is true for m = r – 1, i.e.,

$$\sum_{i=2}^{r-1} \frac{1}{\prod_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)} = 0$$

which implies that

$$\frac{1}{\Pi_{j=2}^{r-1}(a_j - a_1)} = - \sum_{i=2}^{r-1} \frac{1}{\Pi_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)}.$$

Now, on simplification, we get from (16),

$$\Pi_{k=1}^{r-1}(a_k - a_r).\ L = \sum_{i=2}^{r-1} \frac{(a_i - a_1)\,\Pi_{\substack{k=2\\ \neq i}}^{r-1}(a_k - a_r)}{\Pi_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)} + 1$$

$$= - \sum_{i=2}^{r-1} \frac{\Pi_{\substack{k=2\\ \neq i}}^{r-1}(a_k - a_r)}{\Pi_{\substack{j=2\\ \neq i}}^{r-1}(a_j - a_i)} + 1$$

$$= \Pi_{k=2}^{r-1}(a_k - a_r) \cdot \left[\sum_{i=2}^{r-1} \frac{1}{\Pi_{\substack{j=2\\ \neq i}}^{r}(a_j - a_i)} + \frac{1}{\Pi_{k=2}^{r-1}(a_k - a_r)} \right]$$

$$= \Pi_{k=2}^{r-1}(a_k - a_r) \cdot \left[\sum_{i=2}^{r} \frac{1}{\Pi_{\substack{j=2\\ \neq i}}^{r}(a_j - a_i)} \right].$$

Therefore,

$$L = \frac{1}{(a_1 - a_r)} \cdot \left[\sum_{i=2}^{r} \frac{1}{\Pi_{\substack{j=2\\ \neq i}}^{r}(a_j - a_i)} \right] = 0$$

since the result holds for $a_2, a_3, \ldots, a_r$.

This completes the proof of Part (a).

Proof of Part (b) : Denote the left-hand-side of (15) by U(r). We can write

$$U(r) = \sum_{i=1}^{r-1}\left[\frac{\prod_{\substack{j=1\\ \neq i}}^{r-1} a_j}{\prod_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)} + \frac{\prod_{j=1}^{r-1} a_j}{\prod_{\substack{j=1\\ \neq i}}^{r}(a_j - a_i)}\right] + \frac{\prod_{j=1}^{r-1} a_j}{\prod_{j=1}^{r-1}(a_j - a_r)}$$

$$= \sum_{i=1}^{r-1}\frac{\prod_{\substack{j=1\\ \neq i}}^{r-1} a_j}{\prod_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)} + \left(\prod_{j=1}^{r-1} a_j\right)\sum_{i=1}^{r}\frac{1}{\prod_{\substack{j=1\\ \neq i}}^{r}(a_j - a_i)}$$

$$= \sum_{i=1}^{r-1}\frac{\prod_{\substack{j=1\\ \neq i}}^{r-1} a_j}{\prod_{\substack{j=1\\ \neq i}}^{r-1}(a_j - a_i)} \quad \text{(by applying Part (a) of this lemma)}$$

$$= U(r-1).$$

Using the above recursive relation, one can easily prove the result.

Lemma 2 : Let $X_i \cap \exp(\lambda_i)$, $i = 1, 2, \ldots, r$ $(r \geq 2)$ be distinct and mutually independent. Then the pdf of $\sum_{i=1}^{r} X_i$, denoted by $G_r(t)$, is given by

$$g_r(t) = \left(\prod_{j=1}^{r}\lambda_j\right)\sum_{i=1}^{r}\frac{e^{-\lambda_i t}}{\prod_{\substack{j=1\\ \neq i}}^{r}(\lambda_j - \lambda_i)}. \tag{17}$$

Proof : The proof is by induction.

With $m = 2$, it can be seen that the result holds. Assume that the result is true for $m = r$, i.e.,

$$g_r(t) = \left(\prod_{j=1}^{r}\lambda_j\right)\sum_{i=1}^{r}\frac{e^{-\lambda_i t}}{\prod_{\substack{j=1\\ \neq i}}^{r}(\lambda_j - \lambda_i)}.$$

We will show that (17) holds for m = $r + 1$. *Let* $f_{r+1}(t)$ be the pdf of X_{r+1} and $\lambda_{r+1} \neq \lambda_i$ *for* $i = 1, 2, \ldots, r$. Then, the pdf of $\sum_{i=1}^{r+1} X_i$, denoted by $g_{r+1}(t)$, can be written as

$$g_{r+1}(t) = \int_{y=0}^{t} g_r(t-y) f_{r+1}(y)\,dy$$

$$= \left(\Pi_{j=1}^{r+1}\lambda_j\right) \sum_{i=1}^{r} \frac{1}{\Pi_{\substack{j=1\\ \neq i}}^{r} (\lambda_j - \lambda_i)} \int_{y=0}^{t} e^{-\lambda_i(t-y)} . e^{-\lambda_{r+1} y} dy$$

$$= \left(\Pi_{j=1}^{r+1}\lambda_j\right) \sum_{i=1}^{r} \frac{1}{\Pi_{\substack{j=1\\ \neq i}}^{r} (\lambda_j - \lambda_i)} \cdot \frac{e^{-\lambda_i t}}{(\lambda_i - \lambda_{r+1})} \left[e^{(\lambda_i - \lambda_{r+1})t} - 1\right]$$

$$= \left(\Pi_{j=1}^{r+1}\lambda_j\right) \sum_{i=1}^{r} \frac{e^{-\lambda_i t}}{\Pi_{\substack{j=1\\ \neq i}}^{r+1} (\lambda_j - \lambda_i)} - \left(\Pi_{j=1}^{r+1}\lambda_j\right) \left[\sum_{i=1}^{r} \frac{1}{\Pi_{\substack{j=1\\ \neq i}}^{r+1} (\lambda_j - \lambda_i)}\right] e^{-\lambda_{r+1} t}$$

$$= \left(\Pi_{j=1}^{r+1}\lambda_j\right) \sum_{i=1}^{r} \frac{e^{-\lambda_i t}}{\Pi_{\substack{j=1\\ \neq i}}^{r+1} (\lambda_j - \lambda_i)} + \left(\Pi_{j=1}^{r+1}\lambda_j\right) \cdot \frac{1}{\Pi_{j=1}^{r} (\lambda_j - \lambda_{r+1})} \cdot e^{-\lambda_{r+1} t}$$

(by using Lemma 1(a))

$$= \left(\Pi_{j=1}^{r+1}\lambda_j\right) \sum_{i=1}^{r+1} \frac{e^{-\lambda_i t}}{\Pi_{\substack{j=1\\ \neq i}}^{r+1} (\lambda_j - \lambda_i)}.$$

Hence the result holds.