Statistics & Decisions 12, 281 - 291 (1994)

SEQUENTIAL ESTIMATION BY ACCELERATED STOPPING TIMES IN A TWO-PARAMETER EXPONENTIAL FAMILY OF DISTRIBUTIONS

Arup Bose and Nitis Mukhopadhyay

Abstract. *We reconsider the minimum risk point estimation problem discussed in Bose and Boukai (1993) in the framework of a particular class of two-parameter exponential family of distributions. We make their purely sequential estimation rule more attractive by cutting sampling operations significantly via performing purely sequential sampling part of the way followed by appropriate batch sampling. In this note, we provide the asymptotic second-order approximation for the "regret" function associated with such accelerated sequential estimation methodology. The unified theory discussed in Mukhopadhyay and Solanky (1991) does not apply here and hence we opt for direct derivations.*

Keywords and Phrases: Minimum risk point estimation; weighted squared error loss; accelerated stopping rule; risk efficiency; regret; second-order asymptotics.

1. INTRODUCTION

We consider a *regular* two-parameter exponential family of distributions whose probability density function (p.d.f.), with respect to Lebesgue measure on $\mathbb{R}$, is given by

$$f(x;\underset{\sim}{\theta}) = a(x)\exp\left[\theta_1 U_1(x) + \theta_2 U_2(x) + c(\underset{\sim}{\theta})\right], \tag{1.1}$$

$\underset{\sim}{\theta} = (\theta_1, \theta_2)$. See Brown (1986). The natural parameter space is defined by

$$\Omega = \left\{\underset{\sim}{\theta} \in \mathbb{R}^2 : e^{-c(\underset{\sim}{\theta})} = \int a(x)\exp\left[\theta_1 U_1(x) + \theta_2 U_2(x)\right]dx < \infty\right\},$$

so that Ω^0 = interior of Ω, which is assumed to be nonempty. It is well known that for any $\underset{\sim}{\theta} \in \Omega^0$, the random vector $\underset{\sim}{U} = (U_1, U_2)$ has moments of all orders. In particular, one writes

$$E_{\underset{\sim}{\theta}}(\underset{\sim}{U}) = (\mu_1, \mu_2), \mu_i = -\partial c(\underset{\sim}{\theta})/\partial\theta_i, i = 1,2 \tag{1.2}$$

and

$$V_{\underset{\sim}{\theta}}(\underset{\sim}{U}) = (\sigma_{ij}), \sigma_{ij} = -\partial^2 c(\underset{\sim}{\theta})/\partial\theta_i\partial\theta_j, i,j = 1,2. \tag{1.3}$$

Here, $V_{\underset{\sim}{\theta}}(\underset{\sim}{U})$ is the associated positive definite variance - covariance matrix.

Let $X_1, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables having the p.d.f. given by (1.1). Let $T_{i:n} = \sum_{j=1}^{n} U_i(X_j)$ and denote by $\bar{T}_{i:n}, i = 1,2$, the usual averages. The joint distribution of $\underset{\sim}{T}_n = (T_{1:n}, T_{2:n})$ is a member of the same two-parameter exponential family (1.1) where

$$E_{\underset{\sim}{\theta}}(\underset{\sim}{T}_n) = (n\mu_1, n\mu_2), V_{\underset{\sim}{\theta}}(\underset{\sim}{T}_n) = (n\sigma_{ij}), i,j = 1,2. \tag{1.4}$$

Bar-Lev and Reiser (1982) considered a particular subfamily of (1.1) that is characterized by the following two conditions which we assume to hold throughout this discussion.

Assumption A1. *The parameter θ_2 can be represented as $\theta_2 = -\theta_1\psi'(\mu_2)$ where $\psi'(\mu_2) = \partial\psi(\mu_2)/\partial\mu_2$ for some function $\psi(\cdot)$.*

Assumption A2. $U_2(x) = h(x)$ *where $h(x)$ is a 1-1 function on the support of (1.1).*

The family of distributions (1.1) under the Assumptions A1 and A2 is known to include the normal, gamma and inverse Gaussian distributions. The problem here is to estimate μ_2 in the presence of the nuisance parameter μ_1. In the three examples mentioned above, this problem reduces to one of estimating the mean in the presence of appropriate nuisance parameters. In what follows, we collect some of the pertinent properties associated with the p.d.f. (1.1). See also Bose and Boukai (1993) for a few details. We mention that

(a) $$V(U_2(X)) = -[\theta_1\psi''(\mu_2)]^{-1}, \tag{1.5}$$

where $\psi''(\mu_2) = \partial^2\psi(\mu_2)/\partial\mu_2^2$;

(b) $$\begin{aligned} c(\theta_1,\mu_2) &= \theta_1[\mu_2\psi'(\mu_2) - \psi(\mu_2)] - G(\theta_1), \\ \mu_1 &= \psi(\mu_2) + G'(\theta_1), \end{aligned} \tag{1.6}$$

where $G(\cdot)$ is an infinitely differentiable function for $\theta_1 \in \Omega_1$, an appropriate set;

(c) $\Omega_1 \subseteq \mathbb{R}^-$ or $\Omega_1 \subseteq \mathbb{R}^+$ and without loss of generality, we will assume that $\Omega_1 \subseteq \mathbb{R}^-$.

The minimum risk point estimation problem for μ_2 was introduced in the following way by Bose and Boukai (1993), and other relevant citations can also be obtained from the same paper. Having recorded $X_1,\ldots,X_n$, $n \geq 1$, we have already defined $T_{i:n}$ and $\bar{T}_{i:n}$, $i = 1,2$. Let θ_{1n} and μ_{2n} denote the respective maximum likelihood estimators of θ_1 and μ_2, and it can be shown that

$$\mu_{2n} = \bar{T}_{2:n} = n^{-1}\sum_{i=1}^{n} U_2(X_i) \text{ and} \tag{1.7}$$

$$nG'(\theta_{1n}) = T_{1:n} - n\psi\left(\bar{T}_{2:n}\right) = Z_n, \text{ say.} \tag{1.8}$$

We assume that the loss function in estimating μ_2 by $\bar{T}_{2:n}$ is given by

$$L_\rho(n) = \rho\,|\psi''(\mu_2)|\left(\bar{T}_{2:n} - \mu_2\right)^2 + n, \tag{1.9}$$

where $\rho\,(>0)$ is a known number. One may note that $\rho\,|\psi''(\mu_2)|$ represents, in some sense, the importance of the estimation error relative to the cost per unit observation. The risk associated with (1.9) is given by

$$R_\rho(n) = E(L_\rho(n)) = -\rho(n\theta_1)^{-1} + n, \tag{1.10}$$

which is minimized if $n = n_0$ where $n_0 \approx [\rho/(-\theta_1)]^{\frac{1}{2}}$. Here and throughout, we write $a(\rho) \approx b(\rho)$ if $a(\rho)/b(\rho) \to 1$ as $\rho \to \infty$ and asymptotic analysis is carried out as $\rho \to \infty$. The goal is to achieve approximately the minimum risk, namely,

$$R_\rho(n_0) = 2n_0. \tag{1.11}$$

Since n_0 is unknown, Bose and Boukai (1993) had proposed a purely sequential estimation procedure. In the next section, we briefly summarize some of the associated results.

1.1. Purely Sequential Methodology

One starts with $X_1, \ldots, X_m$ with $m \geq 1$ and proceeds sequentially by taking one sample at a time according to the stopping rule

$$N_0 = N_0(\rho) = \inf\left\{n \geq m : -\theta_{1n} > \rho^2 n^{-2}\right\}, \tag{1.12}$$

which can be equivalently written as

$$N_0 = \inf\left\{n \geq m : Z_n < nG'\left(-\rho/n^2\right)\right\}. \tag{1.13}$$

The parameter μ_2 is finally estimated by $\bar{T}_{2:N}$. Bose and Boukai (1993) had proved that $I(N_0 = n)$ is independent of $\bar{T}_{2:n}$ for all fixed $n \geq m$ and verified that the risk function associated with $\bar{T}_{2:N_0}$ is given by

$$R_\rho^* = E\left(L_{N_0}\right) = n_0^2 E\left(N_0^{-1}\right) + E\left(N_0\right). \tag{1.14}$$

Hence, the *risk efficiency* and *regret* functions, as introduced in Robbins (1959) and Starr (1966), are respectively given by

$$e(\rho) = R_\rho^*/R_\rho(n_0) = \tfrac{1}{2}\left\{E\left(N_0/n_0\right) + E\left(n_0/N_0\right)\right\}, \tag{1.15}$$

and

$$\omega(\rho) = R_\rho^* - R_\rho(n_0) = E\left\{\left(N_0 - n_0\right)^2/N_0\right\}. \tag{1.16}$$

Bose and Boukai (1993) imposed two further conditions on the function G and the initial sample size m in order to ensure sufficient smoothness on the stopping boundary.

Assumption A3. For some $\alpha > \frac{1}{2}$, $\sup_{x \geq 4|\theta_1|} x^\alpha G'(-x) \leq M < \infty$.

Assumption A4. The initial sample size m is such that for some $\beta > 2(2\alpha - 1)^{-1}$ and for all $\theta_1 \epsilon \Omega_1$, $E_{\theta_1}\left(Z_m^{-\beta}\right) < \infty$.

The following results were obtained in Bose and Boukai (1993). As $\rho \to \infty$,

(a) under Assumptions A1-A2, one has

$$N_0(\rho)/n_0 \to 1 \text{ a.s.}, \; E\left(N_0(\rho)\right)/n_0 \to 1; \tag{1.17}$$

(b) under Assumptions A1-A4, one has

$$e(\rho) \to 1; \tag{1.18}$$

(c) under Assumptions A1-A3 and A4 with $\beta > 3(2\alpha - 1)^{-1}$, one has

$$E(N_0(\rho)) = n_0 + \eta + o(1) \text{ where } \eta \text{ is a real number;} \tag{1.19}$$

(d) under Assumptions A1-A3 and A4 with $\beta > 5(2\alpha - 1)^{-1}$, one has

$$\omega(\rho) \to \left[4\theta_1^2 G''(\theta_1)\right]^{-1}. \tag{1.20}$$

Now, given all these, let us add that in specific cases, it is known that the distribution of N_0 given by (1.12) can be very skewed to the right. Hence, as a plausible alternative to (1.12), one may first proceed purely sequentially and go only partly by estimating a suitable fraction of n_0, followed by taking the remaining necessary samples, all in *one single batch*. Such a modification will then make the original sampling scheme (1.12) operationally more attractive. In other words, we opt for a suitable *accelerated version* of (1.12) in order to cut sampling operations and yet maintain second-order properties for the associated regret function along the lines of (1.20). Hall (1983) first proposed such a procedure for constructing a fixed-width confidence interval for the mean of a normal distribution when the variance is unknown. A general theory of *accelerated stopping times* has been recently developed in Mukhopadhyay and Solanky (1991) and several sequential estimation problems have been included in that paper. We should remark that the development in Mukhopadhyay and Solanky (1991) parallels that of Woodroofe (1977) in the sense that the boundary condition for stopping depended on the comparison between certain powers of "n" and the corresponding sample mean of n i.i.d. positive random variables. It is far from trivial to extend the existing unified theory of accelerated stopping times in the setup of Lai and Siegmund (1977, 1979). Bose and Boukai (1993) indeed combined the tools from Woodroofe (1977) and Lai and Siegmund (1977, 1979) in order to obtain (1.19) - (1.20) because the boundary condition in (1.13) involved a sample mean of i.i.d. positive random variables plus a sufficiently smooth random "fudge" factor.

In the next section, we propose a suitable accelerated version of the sequential stopping time given in (1.12). The machineries available in Mukhopadhyay and Solanky (1991) are not directly applicable in the present situation. However, in what follows, one will find a direct and straightforward approach that eventually leads to the asymptotic *second-order expansion* of the *regret function* associated with the corresponding estimator of μ_2. Some of the proofs are provided in Section 3. Throughout, $I(\cdot)$ stands for the indicator function of $(\cdot)$.

2. ACCELERATED SEQUENTIAL METHODOLOGY

Instead of (1.12), Bose and Boukai (1993) in fact proposed the following slightly general purely sequential scheme. Let

$$\tilde{N}_0 = \tilde{N}_0(\rho) = \inf\left\{n \ge m : Z_n a_n < nG'\left(-\rho/n^2\right)\right\} \tag{2.1}$$

where $a_n > 1$ and $a_n = 1 + a_0 n^{-1} + o(n^{-1})$. We write $\langle x \rangle$ for the largest integer $< x$ and provide the accelerated version of (2.1). Note that $\tilde{N}_0$ can be equivalently written as

$$\tilde{N}_0 = \tilde{N}_0(\rho) = \inf\left\{n \geq m : n\left[-g\left(\bar{Z}_n a_n\right)\right]^{\frac{1}{2}} > \rho^{\frac{1}{2}}\right\}, \tag{2.2}$$

where $g = G'^{-1}$. Indeed we utilize the representation (2.2) for developing the accelerated sequential methodology.

We choose and fix $\gamma, 0 < \gamma < 1$ and define

$$t = t(\rho) = \inf\left\{n \geq m : n\left[-g\left(\bar{Z}_n a_n\right)\right]^{\frac{1}{2}} > \gamma\rho^{\frac{1}{2}}\right\}. \tag{2.3}$$

With $q \geq 0$, let

$$N_1 = N_1(\rho) = \left\langle\left[-\rho/g\left(\bar{Z}_t a_t\right)\right]^{\frac{1}{2}} + q\right\rangle + 1, \tag{2.4}$$

$$N = N(\rho) = \max\{t(\rho), N_1(\rho)\}. \tag{2.5}$$

One starts sampling with $X_1, \ldots, X_m$, $m \geq 1$ and proceeds purely sequentially by taking one sample at a time according to the stopping rule (2.3) and obtains $X_1, \ldots, X_m, \ldots, X_t$. Observe that the stopping variable t estimates γn_0, a fraction of n_0. Now, based on $X_1, \ldots, X_t$ one estimates n_0 by means of N_1. If $t > N_1$, then we do not take any more samples. However if $N_1 > t$, then we sample the difference $(N_1 - t)$, all *in one single batch*. The sampling operations needed in the accelerated sequential sampling scheme (2.3) - (2.5) amounts to about 100 γ% of that needed in the purely sequential methodology (2.2). In other words, the operational convenience of (2.3) - (2.5) over (2.2) is quite obvious. Once one determines N and obtains $X_1, \ldots, X_N$ by means of (2.3) - (2.5), μ_2 is estimated by $\bar{T}_{2:N}$. The expressions of the *risk efficiency* and *regret* functions associated with the corresponding $\bar{T}_{2:N}$ are again given by (1.15) and (1.16) respectively once one replaces N_0 by N.

From the analysis given in Bose and Boukai (1993), one can conclude the following properties for the stopping variable $t = t(\rho)$ defined in (2.3). As $\rho \to \infty$, one has

(a) $$t/\gamma n_0 \to 1 \text{ } a.s., \; E(t)/\gamma n_0 \to 1; \tag{2.6}$$

(b) for every $\epsilon > 1$ and all $n > \epsilon\gamma n_0$, there exists $c^*(> 0)$ depending on ϵ and $G(\cdot)$ such that

$$P\{t > n\} \leq \exp\{-(n - \gamma n_0)c^*\}; \tag{2.7}$$

(c) Under Assumptions A1-A2,
$$(t - \gamma n_0)/(\gamma n_0)^{\frac{1}{2}} \xrightarrow{\mathcal{L}} N\left(0, [4\theta_1^2 G''(\theta_1)]^{-1}\right); \tag{2.8}$$

Under Assumptions A1-A4, one has as $\rho \to \infty$, for any $\varepsilon \in (0,1)$,

$$E\left\{n_0 t^{-1} I(t \leq \varepsilon\gamma n_0)\right\} \to 0. \tag{2.9}$$

Under Assumptions A1-A3 and A4 with $\beta > 3(2\alpha - 1)^{-1}$, one has as $\rho \to \infty$ and for any $\varepsilon \in (0,1)$,

$$\text{(d)} \qquad n_0 P\{t \le \varepsilon\gamma n_0\} \to 0; \tag{2.10}$$

$$\text{(e)} \qquad E\left\{(n_0/t)^2 I(t \le \varepsilon\gamma n_0)\right\} \to 0. \tag{2.11}$$

Under Assumptions A1-A3 and A4 with $\beta > (1+2k)/(2\alpha - 1)$ for $k > 1$, one has as $\rho \to \infty$, and for any $\varepsilon \in (0,1)$,

$$\text{(f)} \qquad n_0^k P\{t \le \varepsilon\gamma n_0\} \to 0. \tag{2.12}$$

We recall that $(Z_1, \ldots, Z_n)$ and $T_{2:n}$ are independent for every $n \ge m$ and also one has the representation $Z_n = \sum_{i=1}^n Y_i - \xi_n$ where $\{Y_i : i \ge 1\}$ is a sequence of i.i.d. random variables such that

i) $E(Y_1) = G'(\theta_1), V(Y_1) = G''(\theta_1)$; and (2.13)

ii) $\{\xi_n : n \ge 1\}$ is a sequence of *slowly changing* random variables; (2.14)

iii) $\bar{Z}_n \to G'(\theta_1)$ a.s. and $n^{\frac{1}{2}}\left[\bar{Z}_n - G'(\theta_1)\right] \xrightarrow{\mathcal{L}} N(0, G''(\theta_1))$ as $n \to \infty$. (2.15)

The main results are now summarized in the following Theorems.

THEOREM 1. *For the accelerated sequential estimation procedure (2.3) - (2.5), we have as* $\rho \to \infty$:

i. $N/n_0 \to 1$ a.s.;
ii. $E(N/n_0) \to 1$;
iii. $n_0^{-\frac{1}{2}}(N - n_0) \xrightarrow{\mathcal{L}} N\left(0, [4\gamma\theta_1^2 G''(\theta_1)]^{-1}\right)$;

under *Assumptions A1 - A2.*

THEOREM 2. *Suppose that Assumptions A1 - A3 hold. Then, for the accelerated sequential estimation procedure (2.3) - (2.5), we have as* $\rho \to \infty$:

i. $e(\rho) \to 1$, if G and m also satisfy Assumption A4;
ii. $\omega(\rho) \to [4\gamma\theta_1^2 G''(\theta_1)]^{-1}$, if G and m also satisfy Assumption A4 with $\beta > 5(2\alpha - 1)^{-1}$;

where $e(\rho)$ and $\omega(\rho)$ have respectively been defined in (1.15) and (1.16), replacing N_0 by N.

The part (i) of Theorem 2 shows that the accelerated sequential estimation methodology is in fact asymptotically first-order risk efficient in the sense of Ghosh and Mukhopadhyay (1981). The same result is referred to as the asymptotic risk efficiency property in the sense of Robbins (1959). Part (ii) of Theorem 2 provides the asymptotic second-order expansion of the associated regret function $\omega(\rho)$. One may contrast Theorem 2 (ii) with (1.20) and observe that the asymptotic regret for the accelerated sequential methodology exceeds that for the purely sequential methodology of Bose and Boukai (1993). This is quite apparent since $\gamma \in (0,1)$. On the other hand, this "loss in efficiency," while implementing the accelerated sequential analog, can be thought of as a direct reflection on the "gain" obtained via substantial operational convenience. In specific instances, the experimenter will perhaps attempt to achieve some type of practical balance between the "loss" in one component and "gain" in the other.

3. PROOFS OF THEOREMS

In this Section, we provide the proofs for Theorems 1 and 2.

3.1. Proof of Theorem 1

Part (i) follows immediately from (2.5) and the first parts of (2.6) and (2.15). To prove part (ii), note that $tn_0^{-1}I(t > N_1) \le tn_0^{-1}$ which is uniformly integrable in view of (2.6). Thus, observing that $P(t > N_1) \to 0$, it follows that $E\left\{tn_0^{-1}I(t > N_1)\right\} = o(1)$. Hence it will suffice to show that

$$\limsup E\left\{N_1 n_0^{-1} I(N_1 \ge t)\right\} \le 1. \tag{3.1}$$

Now, one combines (2.3) - (2.4) to write

$$\begin{aligned} E\left\{N_1 n_0^{-1} I(N_1 \ge t)\right\} &\le E\left[\left\{\rho/-g\left(\bar{Z}_t a_t\right)\right\}^{\frac{1}{2}}\right] n_0^{-1} + (q+1)\,n_0^{-1} \\ &\le E\left[t/(\gamma n_0)\right] + (q+1)\,n_0^{-1}, \end{aligned} \tag{3.2}$$

and thus (3.1) follows from (2.6).

For part (iii), it will suffice to prove the same result having replaced "N" by means of $\tilde{N}_1 = \rho^{\frac{1}{2}}/f^*\left(\bar{Z}_t\right)$ where $f^*(x) = \{-g(x)\}^{\frac{1}{2}}$. Now, from (2.6), (2.15) and Anscombe's (1952) result, we immediately note that $(\gamma n_0)^{\frac{1}{2}}\left(\bar{Z}_t - G'(\theta_1)\right) \xrightarrow{\mathcal{L}} N\left(0, G''(\theta_1)\right)$ as $\rho \to \infty$, and hence the claim, since one can write

$$\begin{aligned} \left(\tilde{N}_1 - n_0\right)/n_0^{\frac{1}{2}} &= -n_0^{\frac{1}{2}}\left[f^*\left(\bar{Z}_t\right) - \rho^{\frac{1}{2}} n_0^{-1}\right]/f^*\left(\bar{Z}_t\right) \\ &= -n_0^{\frac{1}{2}}\left[f^*\left(\bar{Z}_t\right) - f^*\left(G'(\theta_1)\right)\right]/f^*\left(\bar{Z}_t\right). \end{aligned} \tag{3.3}$$

3.2. Proof of Theorem 2

It is easy to prove part (i) and hence we omit this. To prove part (ii), define $B_1 = \left\{t < \frac{1}{2}\gamma n_o\right\}$, $N^* = n_0^{-\frac{1}{2}}(N - n_0)$. Note that the regret function

$$\omega(\rho) = E\left[N^{*2} n_0 / N\right] = A_1 + A_2, \text{ say} \tag{3.4}$$

where $A_1 = E\left[N^{*2} n_0 N^{-1} I(B_1)\right]$, $A_2 = E\left[N^{*2} n_0 N^{-1} I(B_1^c)\right]$. Observe that $A_1 \le c^* n_0^2 P(B_1)$ where c^* is a positive generic constant independent of ρ. Thus, in view of (2.12), $A_1 \to 0$ as $\rho \to \infty$. On the other hand, $N^{*2} n_0 N^{-1} I(B_1^c) \le 2\gamma^{-1} N^{*2}$, and hence $A_2 = E(N^{*2}) + o(1)$, which completes the proof in view of Theorem 1, part (iii) and the following result.

Lemma 1. *For the stopping variable N defined by (2.5), N^{*2} is uniformly integrable.*

Proof: Fix $1 < \beta < \gamma^{-1}$ and define $C_1 = \{t \le \beta^{-1}\gamma n_0\}$, $C_2 = \{t \ge \beta\gamma n_0\}$, $B = \{\beta^{-1}\gamma n_0 < t < \beta\gamma n_0\}$, $A = \{t \ge N_1 + 1\}$, $J = \{k : \beta^{-1}\gamma n_0 < k < \beta\gamma n_0\}$, $D = \left\{t \ge \gamma\left(n_0 - n_0^{\frac{1}{2}} x\right)\right\}$. Let $t^* = n_0^{-\frac{1}{2}}(t - \gamma n_0)$. Observe that

$$t \le N_1 \le \gamma^{-1} t + q + 1 \tag{3.5}$$

and

$$N^{*2} = N^{*2} I(C_1) + N^{*2} I(B) + N^{*2} I(C_2). \tag{3.6}$$

It is easy to see that

$$\begin{aligned} E\left[N^{*2} I(C_2)\right] &\le E\left[(\gamma^{-1} t + q + 1)^2 I(C_2)\right] + n_0 P\{t > \beta\gamma n_0\} \\ &\to 0 \text{ as } \rho \to \infty, \end{aligned} \tag{3.7}$$

in view of (2.7). Also, one has

$$E\left[N^{*2} I(C_1)\right] \le c^* n_0 P(C_1) \to 0 \text{ as } \rho \to \infty, \tag{3.8}$$

in view of (2.10). Now, we have

$$\begin{aligned} E\left[N^{*2} I(B \cap A)\right] &\le c^* n_0 P(A \cap B) \\ &\le c^* n_0 P\{Z_k \le k G'(-\rho/k^2) \text{ for some } k \in J\}. \end{aligned} \tag{3.9}$$

One can use the martingale argument of page 500 of Bose and Boukai (1993) to verify that the bound given in (3.9) does not exceed $c^* n_0^{-1}$, that is

$$E\left[N^{*2} I(B \cap A)\right] \to 0 \text{ as } \rho \to \infty. \tag{3.10}$$

Next, observe that $P\{N^* \ge x, t \in A^c \cap B\} \le P\left\{t^* \ge x\gamma - n_0^{-\frac{1}{2}}(q+1), t \in B\right\}$, and this does not exceed $c^* x^{-4}$, by the arguments of Bose and Boukai (1993). Also,

$$P\{N^* \le -x, t \in A^c \cap B \cap D^c\} \le P\{t^* \le -\gamma x, t \in B\},$$

and this does not exceed $c^* x^{-4}$, as in Bose and Boukai (1993). But, on the set $A^c \cap B \cap D \cap \{N^* \leq -x\}$, one obtains $Z_t < tG'\left(-\rho/(n_0 - n_0^{\frac{1}{2}}x)^2\right)$, and hence

$$
\begin{aligned}
&P\{N^* \leq -x, t \in A^c \cap B \cap D\} \\
&\leq P\Big\{T_{1:k} - k\mu_1 - k\left[\psi\left(\bar{T}_{2:k}\right) - \psi\left(\mu_2\right)\right] \\
&\quad < k\left[G'\left(-\rho(n_0 - n_0^{\frac{1}{2}}x)^{-2}\right) - G'\left(\theta_1\right)\right], t \in B \cap D\Big\}.
\end{aligned}
\tag{3.11}
$$

By means of Taylor expansion and using the fact that $t \in B \cap D$, one can show that the set on the right hand side of (3.11) is exactly of the form of the set considered on the top of page 500 in Bose and Boukai (1993). Hence, the bound in (3.11) does not exceed $c^* x^{-4}$. That is,

$$P\{|N^*| \geq x, t \in A^c \cap B\} \leq J^*(x), \tag{3.12}$$

where $xJ^*(x)$ is integrable. Now, one combines (3.6)-(3.10) and (3.12) to complete the proof.

Remark 3.1. Note that we have not been able to provide the expression for $\lim_{\rho \to \infty} E(N - n_0)$ where N is the accelerated stopping time defined in (2.5). From Theorem 3 of Bose and Boukai (1993), we can claim that

$$E(t) = \gamma n_0 + b_0 + o(1) \tag{3.13}$$

for an appropriate real number $b_0 = b_0(\theta_1, \gamma)$ under Assumptions A1 - A3 and A4 with $\beta > 3(2\delta - 1)^{-1}$. Now, from (3.5) and (3.13), one obtains

$$E(N) \leq n_0 + \left(b_0\gamma^{-1} + q\right) + o(1) \tag{3.14}$$

under similar assumptions.

ACKNOWLEDGEMENT

The referee pointed out a flaw in an earlier version of the proof of Lemma 1. We express our sincerest gratitude to the referee for this and other helpful comments.

Bibliography

[1] Anscombe, F.J. (1952). Large sample theory of sequential estimation. *Proc. Camb. Phil. Soc., 48,* 600-607.

[2] Bar-Lev, S.K. and Reiser, B. (1982). An exponential subfamily which admits UMPU test based on a single test statistic. *Ann. Statist., 10,* 979-989.

[3] Bose, A. and Boukai, B. (1993). Sequential estimation results for a two-parameter exponential family of distributions. *Ann. Statist., 21,* 484-502.

[4] Brown, L.D. (1986). *Fundamentals of Statistical Exponential Families.* IMS Lecture Notes, Vol. 9.

[5] Ghosh, M. and Mukhopadhyay, N. (1981). Consistency and asymptotic efficiency of two-stage and sequential procedures. *Sankhya, Ser. A, 43,* 220-227.

[6] Hall, P. (1983). Sequential estimation saving sampling operations. *J. Roy. Statist. Soc., Ser. B, 45,* 219-223.

[7] Lai, T.L. and Siegmund, D. (1977). A nonlinear renewal theory with applications to sequential analysis I. *Ann. Statist., 5,* 946-954.

[8] Lai, T.L. and Siegmund, D. (1979). A nonlinear renewal theory with applications to sequential analysis II. *Ann. Statist., 7,* 60-76.

[9] Mukhopadhyay, N. and Solanky, T.K.S. (1991). Second order properties of accelerated stopping times with applications in sequential estimation. *Seq. Anal., 10,* 99-123.

[10] Robbins, H. (1959). Sequential estimation of the mean of a normal population. *Prob. and Statist., H. Cramer Vol.* (U. Grenander ed.). Almquist and Wiksell, Stockholm.

[11] Starr, N. (1966). On the asymptotic efficiency of a sequential procedure for estimating the mean. *Ann. Math. Statist., 37,* 1173-1185.

[12] Woodroofe, M. (1977). Second order approximations for sequential point and interval estimation. *Ann. Statist., 5,* 984-995.

Arup Bose
Stat-Math Unit
Indian Statistical Institute
203 B.T. Road
Calcutta 700035 India

Nitis Mukhopadhyay
Department of Statistics, UBox 120
University of Connecticut
196 Auditorium Road
Storrs, CT 06269 U.S.A.