OPSEARCH, Vol. 29, No. 4, 1992

SHORT COMMUNICATIONS

THE WEIGHTED MAJORITY GAME

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(*Received : August 1991, Revision Received : April 1992*)

Abstract

In this paper the author presents some results for weighted majority games. Also, a polynomial time algorithm is presented here, which accepts as inputs the minimal winning coalitions for a decisive simple game and produces as output either a quota weights vector which represent the game or a proof that the game is not a weighted majority game.

1. Introduction

Let N be a nonempty finite set. *A simple game* on N is a function $v : 2^N \to \{0,1\}$ satisfying

(*i*) $v(\phi) = 0$,

(*ii*) $v(N) = 1$,

(*iii*) $v(S) \leqslant v(T)$ whenever $S \subseteq T$.

Some authors refer to this as *monotone simple game* also. Elements of the set N are called *players*. The player set N is normally represented as $N = \{1,2,\ldots,n\}$. Elements of 2^N (subsets of N) are called *coalitions*. A coalition S is called *winning* if $v(S) = 1$; *losing* if $v(S) = 0$; and *blocking* if $v(N \setminus S) = 0$. A coalition S is called *minimal winning* if $v(S) = 1$ and $v(T) = 0 \ \forall\ T \subset S$; *maximal losing* if $v(S) = 0$ and $v(T) = 1 \ \forall\ T \supset S$; *minimal blocking* if $v(N \setminus S) = 0$ and $v(N \setminus T) = 1 \ \forall\ T \supset S$. A simple game is called *decisive* if $v(S) + v(N \setminus S) = 1$ for all $S \subseteq N$.

A *weighted majority* game is a simple game such that there exists $w \in \mathcal{R}^n$ and $q \in \mathcal{R}$ such that $\Sigma_{i \in S} w_i \geqslant q$ if $v(S) = 1$ and $\Sigma_{i \in S} w_i < q$ if $v(S) = 0$. A simple game is called a *pseudo weighted majority* game if (*i*) it is not a weighted majority game and (*ii*) there exists $w \in \mathcal{R}^n$ and $q \in \mathcal{R}$ such that

$\sum_{i\in S} w_i \geqslant q$ if $v(S) = 1$ and $\sum_{i\in S} w_i \leqslant q$ if $v(S) = 0$. Element w_i of the vector w is the weight of the player i and q is the quota.

Hilliard [3] in his thesis presented an algorithm which was designed to accept as input a list of the minimal winning coalitions for a (monotone) simple game and to produce as output either a quota and a set of weights which represent the game or a proof that the game is not a weighted majority game. Goel [2] presented a modified version of Hilliard's algorithm. The algorithms presented were non-polynomial time algorithms (For details refer to Goel [2]). In this paper, the author presents some simple results for weighted majority games and a polynomial time algorithm for finding a weighted majority representation of decisive simple games if it exists.

2. Preliminaries: Definitions and Notations

Let $N = \{1,2,\ldots,n\}$. For any subset S of N, $|S|$ denotes its cardinality; and $\{\ \}$ or ϕ denotes the empty set (or empty collection). Throughout the paper, we shall assume that all the sets are ordered subsets of N, i.e., $S \subseteq N$, $S = \{s_1, s_2, \ldots, s_k\}$ then $s_1 < s_2 < s_3 < \ldots < s_k$. Let Ω_k be the collection of all ordered subsets of N of cardinality k, i.e., $\Omega_k = \{S : S \subseteq N$ and $|S| = k\}$, where k is between 1 and n.

Definition 1. Consider S and T belonging to Ω_k:

$S < T$ if $s_i \leqslant t_i$ for $i = 1$ to k and strict inequality holds for at least one i,

$S \leqslant T$ if $s_i \leqslant t_i$ for all $i = 1$ to k,

$S = T$ if $s_i = t_i$ for all $i = 1$ to k,

S is comparable to T if $S \leqslant T$ or $T \leqslant S$, (Otherwise S is not comparable to T).

Definition 2. A non-empty collection B of subsets from Ω_k is called an antichain set if

(*i*) cardinality of collection B is one, or

(*ii*) $|B| > 1$; then any two distinct elements from B are not comparable to each other, i.e., $X, Y \in B$ and $X \neq Y \Rightarrow X \nleq Y$.

Definition 3. Define the following sets:

$W = \{T : v(T) = 1$, and T is an ordered set$\}$,

$L = \{T : v(T) = 0$, and T is an ordered set$\}$,

$B = \{T : v(N \setminus T) = 0$, and T is an ordered set$\}$,

$W_k = \{T : v(T) = 1, \mid T \mid = k$ and T is an ordered set$\}$,

$L_k = \{T : v(T) = 0, \mid T \mid = k$ and T is an ordered set$\}$,

$B_k = \{T : v(N \setminus T) = 0, \mid T \mid = k$ and T is an ordered set$\}$,

$W_k^s = \{S : S \in W_k$ and for all $T \in W_k, T \nless S\}$,

$L_k^l = \{S : S \in L_k$ and for all $T \in L_k, T \ngtr S\}$,

$W_k^m = \{S : S \in W_k$ and for all $T \in W, T \not\subset S\}$,

$L_k^m = \{S : S \in L_k$ and for all $T \in L, T \not\supset S\}$,

$B_k^m = \{S : S \in B_k$ and for all $T \in B, T \not\subset S\}$,

$W^s = \cup_{k=1}^n W_k^s, L^l = \cup_{k=1}^n L_k^l$,

$W^m = \cup_{k=1}^n W_k^m, \; L^m = \cup_{k=1}^n L_k^m, \; B^m = \cup_{k=1}^n B_k^m$

$W^{sm} = \{S : S \in W^s$ and for all $T \in W^s, T \not\subset S\}$,

$L^{lm} = \{S : S \in L^l$ and for all $T \in L^l, T \not\supset S\}$.

Definition 4. A simple game is called

Proper if $v(S) + v(N \setminus S) \leqslant 1 \;\forall\; S \subseteq N$,

Strictly Proper if it is proper and strict inequality holds for atleast one $S \subseteq N$,

Strong if $v(S) + v(N \setminus S) \geqslant 1 \;\forall\; S \subseteq N$,

Strictly Strong if it is strong and strict inequality holds for atleast one $S \subseteq N$,

Decisive if $v(S) + v(N \setminus S) = 1 \;\forall\; S \subseteq N$,

Mixed if it does not fall into any of the above categories, i.e., if it is neither proper nor strong.

3. Main Results

THEOREM 1. *For all simple games* $L^m = \{S : N \setminus S \in B^m\}$.

Proof. Follows from the fact that (maximal) losing coalitions are compliment of (minimal) blocking coalitions.

THEOREM 2.

(*i*) *A simple game which is proper but not strictly proper is decisive.*

(*ii*) *A simple game which is strong but not strictly strong is decisive.*

Proof. Follows easily from the definitions.

Remarks. Using the above result we can classify the simple games into the following four distinct categories:

(1) Strictly Proper.

(2) Strictly Strong.

(3) Decisive.

(4) Mixed.

THEOREM 3. *For any weighted majority game the following two cannot hold simultaneously* :

$$(i)\ v(S) + v(N \setminus S) < 1 \text{ for some } S \subseteq N,$$

$$(ii)\ v(T) + v(N \setminus T) > 1 \text{ for some } T \subseteq N.$$

Proof Since (*i*) implies that $q > \frac{1}{2} \sum_{i=1}^{n} w_i$ and (*ii*) $\Rightarrow q \leqslant \frac{1}{2} \sum_{i=1}^{n} w_i$ and obviously both cannot hold together for any choice of w_i's.

The above theorem states that the mixed simple games cannot be represented as weighted majority games. And that a weighted majority game is a simple game which is Strictly Proper or Strictly Strong or Decisive. Also, it does not imply that if a simple game is Strictly Proper or Strictly Strong or Decisive, then it has the weighted majority representation.

THEOREM 4.

(*i*) For a decisive simple game $W^m = B^m$,

(*ii*) For a strictly proper simple game $W \subset B$,

(*iii*) For a strictly strong simple game $W \supset B$.

Proof. (*i*) Follows from the fact that every blocking coalition is also winning and also every winning coalition is blocking, i.e., $W = B$. Others also follow from basic definitions.

Notation : For all $A \in \Omega_k$ let us donate

$$w(A) = w_{a_1} + w_{a_2} + \dots + w_{a_k}.$$

Consider the following LP problem:

$$\text{Maximise } q,$$

subject to
$$w(S) \geqslant q \ \forall\ S \in W^m,$$
$$w(N) = 1,$$
$$w_1, \ldots, w_n, q \geqslant 0.$$

Let us denote this LP problem as LP1. It can be easily seen that LP1 is always feasible irrespective of W^m.

THEOREM 5. *A decisive game v (i.e., $v(S) + v(N \setminus S) = 1$) specified by W^m is a weighted majority game iff the optimum value of the corresponding problem* LP1 *is greater than $\frac{1}{2}$.*

Proof. Suppose, there is a decisive game v (i.e., $v(S) + v(N \setminus S) = 1$) specified by W^m, which is a weighted majority game and the optimum value of the corresponding problem LP1 is less than $\frac{1}{2}$. This implies existance of a $S \in W^m$ such that $w(S) = q$ and $q \leqslant \frac{1}{2}$ then $w(N \setminus S) = 1 - q \geqslant \frac{1}{2} \geqslant q$ which contradicts the fact that the game is decisive.

Suppose, there is decisive game v (i.e., $v(S) + v(N \setminus S)$ specified by W^m and the optimum value of the corresponding problem LP1 is greater than $\frac{1}{2}$ then the solution of LP1 shall be the weights and optimal objective value shall be the quota of related weighted majority game. Hence the result.

Consider the following LP problem:

$$\text{minimise } q,$$

subject to
$$w(S) \leqslant q \ \forall\ S \in L^m.$$
$$w(N) = 1,$$
$$w_1, \ldots, w_n, q \geqslant 0.$$

Let us denote this LP problem as LP2. It can be easily seen that LP2 is always feasible irrespective of L^m.

THEOREM 6. *A given simple game has a weighted representation only if the optimal value of corresponding* LP1 *(say, z_1) and* LP2 *(say, z_2) satisfies the following;*

(*i*) $z_1 > z_2 \geqslant \frac{1}{2}$ *for strictly proper game (also, $z_1 + z_2 > 1$),*

(*ii*) $\frac{1}{2} \geqslant z_1 > z_2$ *for strictly strong game (also, $z_1 + z_2 < 1$),*

(*iii*) $z_1 > \frac{1}{2} > z_2$ *for decisive game (also, $z_1 + z_2 = 1$).*

THEOREM 7. *A given simple game has a pseudo weighted representation only if the optimal value of corresponding* LP1 (*say* z_1) *and* LP2 (*say* z_2) *satisfies the following*:

(*i*) $z_1 = z_2 > \frac{1}{2}$ *for strictly proper game,*

(*ii*) $z_1 = z_2 < \frac{1}{2}$ *for strictly strong game.*

(*iii*) $z_1 = z_2 = \frac{1}{2}$ *for decisive game.*

Remark. Normally L^m is not known. The above two theorems can be used as a necessary condition by checking the inequalities for the LP1 as W^m is known.

THEOREM 8. *If* $a_{ki} = |\{S \mid i \in S, |S| = k$ *and* $S \in W^m\}|$ *where* W^m *is the set of minimal winning coalitions for a weighted majority game, then there is a quota-weight vector with the following properties*:

(1) *if* $a_{ki} = a_{kj}$ *for all* $k \leqslant n$, *then* $w_i = w_j$,

(2) *if* $a_{ki} = a_{kj}$ *for all* $k < l \leqslant n$ *and* $a_{li} > a_{lj}$, *then* $w_i > w_j$.

Proof. See Hilliard [3]

Given W^m, we can use Theorem 8 to reorder the players (if necessary), according to lexicographical order of the vectors $(a_{1i}, a_{2i}, \ldots, a_{ni})$ to ensure that if there is a solution (that is, if there is a weighted representation) then it satisfies $w_1 \leqslant w_2 \leqslant \ldots \leqslant w_n$. Consider the following LP problem:

$$\text{Minimise} \quad w(N) - q,$$

subject to

$$w(S) \geqslant q \;\forall\; S \in W^{sm},$$

$$w_1 \geqslant 1,$$

$$w_{i+1} - w_i \geqslant 0 \;\forall\; i = 1 \text{ to } n-1,$$

$$w_1, \ldots, w_n, q \geqslant 0.$$

Let us denote this LP problem as LP3. It can be easily seen that LP3 is always feasible irrespective of W^m or W^{sm}.

THEOREM 9. *A decisive game* v (*i.e.*, $v(S) + v(N \setminus S) = 1$) *specified by* W^m *is a weighted majority game iff the optimum solution to the corresponding problem* LP3 *should be such that* $q > \frac{1}{2} \sum_{i=1}^{n} w_i$.

Proof. Follows from Theorem 5 and equivalenace of the two problems LP1 and LP3.

Algorithm

To check whether a given decisive simple game (W^m is given) has a weighted representation, we solve either LP1 or LP3 with an additional constraint given as follows:

$q > \frac{1}{2}$ for LP1.

$q > \frac{1}{2}\, w(N)$ for LP3.

and if the problem with the additional constraint is feasible than the given decisive game has a weighted majority representation with (q,w) the quota-weight vector as any feasible solution to above problem.

ACKNOWLEDGEMENTS

I am indebted to Dr. K.G. Ramamurthy and Dr. T. Parthasarathy for introducing me to this subject and constantly encouraging and guiding me to carry out the research work. I wish to sincerely thank Dr. S.R. Mohan for many helpful discussions. I am thankful to the referee for his valuable remarks which led to significant improvement in this paper.

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