Sankhyā : The Indian Journal of Statistics
1977, Volume 39, Series A, Pt. 4, pp. 387–395.

A CHARACTERISATION OF MERELY POSITIVE SUBDEFINITE MATRICES AND RELATED RESULTS

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SUMMARY. A real symmetric matrix A is said to be positive subdefinite (PSubD) if for any vector x, $x'Ax < 0$ implies Ax is either nonpositive or nonnegative vector.

A PSubD matrix which is not positive semidefinite is called merely positive subdefinite (MPSubD) matrix.

In this paper two characterisations of MPSubD matrices are established and also MPSubD matrices possessing MPSubD g-inverses are characterised.

1. Introduction and preliminaries

The concept of quasiconvex and pseudoconvex quadratic forms which play an important role in mathematical programming problems lead to a new subclass of real symmetric matrices, namely positive subdefinite (PSubD) matrices. Martos (1969) made an interesting study of these matrices where he proves some nice properties of merely positive subdefinite (MPSubD) matrices which are PSubD but not positive semidefinite (PSemiD). He wondered whether some of the properties of these matrices, proved by him, would characterise the MPSubD matrices.

The object of this paper is to answer his question in affirmative, thereby obtaining an interesting characterisation of MPSubD matrices. We obtain another characterisation of MPSubD matrices similar to the one of PSemiD matrices. These characterisations provide an easy recognition of quasiconvex and pseudoconvex quadratic forms in view of Theorems 4 and 5 of Martos. We study these matrices with respect to the generalized inverse (g-inverse) also. It is well known that a PSemiD matrix has a PSemiD g-inverse. However as we show, barring trivial cases MPSubD matrices do not possess MPSubD g-inverses.

Matrices are denoted by capital letters and a_{ij}, A', det A, tr A, $R(A)$ and A_k denote respectively (i, j)-th element, transpose, determinant, trace, rank and k-th order leading principal minor matrix of the matrix A. Column vectors are denoted by lower case letters and x^i denote the i-th component of x. (x, y) denotes the usual Euclidean innerproduct of vectors x and y.

We call a vector x nonnegative (positive) denoted by $x \geqslant 0$ $(x > 0)$ if $x^i \geqslant 0$ $(x^i > 0)$ for all i and x is semi-unisigned (unisigned) denoted by

$x \gtrless 0$ ($x \gtrsim 0$) if $x^i \geqslant 0$ or $x^i \leqslant 0$ ($x^i > 0$ or $x^i < 0$) for all i and $|x|$ denotes the vector x, each element replaced by its modulus.

Definition 1 : Let A be an $m \times n$ matrix, then the matrix G of order $n \times m$ satisfying

$$\text{(i)} \quad AGA = A$$
$$\text{(ii)} \quad GAG = G$$
$$\text{(iii)} \quad (AG)' = AG$$
$$\text{(iv)} \quad (GA)' = GA$$

is defined as Moore-Penrose inverse of A, denoted by A^+. Any matrix G satisfying (i) and (ii) is defined as reflexive g-inverse of A and is denoted by A_r^- (Rao and Mitra, 1971).

Definition 2 : A real symmetric matrix A is said to be positive subdefinite (PSubD) if for any vector x, $x'Ax < 0$ implies Ax is either non-positive or non-negative vector.

Definition 3 : A PSubD matrix which is not positive semidefinite is called merely positive subdefinite (MPSubD) matrix.

2. Characterisations of MPSubD matrices

Theorem 1 : *A non-positive symmetric matrix, having exactly one (simple) negative eigen value, is MPSubD.*

Proof : Let A be a non-positive symmetric matrix having exactly one negative eigen value λ_1. Therefore the eigen vector of A corresponding to λ_1 is semi-unisigned. Consider the spectral decomposition of A.

$$A = \lambda_1 p_1 p_1' + \lambda_2 p_2 p_2' + \ldots + \lambda_r p_r p_r' + \ldots + \lambda_n p_n p_n'$$

where $\lambda_2, \ldots, \lambda_r$ are positive eigen values of A, $\lambda_{r+1}, \ldots, \lambda_n$ and zero eigen values of A and $p_1, p_2, \ldots, p_n$ is the orthonormal set of eigen vectors of A corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$. Without loss of generality let $p_1 \geqslant 0$ because p_1 is semi-unisigned. Therefore

$$A = \lambda_1 p_1 p_1' + \lambda_2 p_2 p_2' + \ldots + \lambda_r p_r p_r'. \qquad \ldots \quad (1)$$

Let $x = c_1 p_1 + c_2 p_2 + \ldots + c_n p_n$ be any vector, then

$$Ax = \lambda_1 c_1 p_1 + \lambda_2 c_2 p_2 + \ldots + \lambda_r c_r p_r$$

and

$$(Ax)^i = \lambda_1 c_1 p_1^i + \lambda_2 c_2 p_2^i + \ldots + \lambda_r c_r p_r^i \qquad \text{for } i = 1, 2, \ldots, n$$

$$x'Ax = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \ldots + \lambda_r c_r^2.$$

To show that $x'Ax < 0 \Longrightarrow Ax \gtreqless 0$. Let $c_1 \geqslant 0$.

$$x'Ax < 0 \Longrightarrow c_1 \neq 0,$$

therefore,

$$-\lambda_1 > \frac{1}{c_1^2}[\lambda_2 c_2^2 + \ldots + \lambda_r c_r^2] = (u, u) \qquad \ldots \quad (2)$$

where $u' = \frac{1}{c_1}(\sqrt{\lambda_2}c_2, \ldots, \sqrt{\lambda_r}c_r)$.

Since $A \leqslant 0$

$$a_{ii} = \lambda_1(p_1^i)^2 + \lambda_2(p_2^i)^2 + \ldots + \lambda_r(p_r^i)^2 \leqslant 0.$$

Now $\qquad p_1^i = 0 \Longrightarrow rp_2^i = p_3^i = \ldots = 0 \Longrightarrow (Ax)^i = 0$

Otherwise if $p_1^i \neq 0$

$$-\lambda_1 \geqslant \frac{1}{(p_1^i)^2}[\lambda_2(p_2^i)^2 + \ldots + \lambda_r(p_r^i)^2] = (v, v) \qquad \ldots \quad (3)$$

where $\qquad v' = \frac{1}{p_1^i}(\sqrt{\lambda_2}p_2^i, \ldots, \sqrt{\lambda_r}p_r^i)$

(2) and (3) $\Longrightarrow$

$$-2\lambda_1 > (u, u) + (v, v) \geqslant 2(u, v)$$

$$-\lambda_1 > (u, v)$$

$$-\lambda_1 > \frac{1}{c_1 p_1^i}(\lambda_2 c_2 p_2^i + \ldots + \lambda_r c_r p_r^i)$$

$$\Longrightarrow \lambda_1 c_1 p_1^i + \lambda_2 c_2 p_2^i + \ldots + \lambda_r c\ p_r^i < 0 \qquad \ldots \quad (4)$$

that is $(Ax)^i < 0$ for all i such that $p_1^i \neq 0$.

Therefore $\qquad (Ax)^i \leqslant 0$ for all i.

Hence $\qquad Ax \leqslant 0$.

Similarly if $c_1 < 0$ the inequality in (4) changes and $Ax \geqslant 0$. Hence the theorem.

In view of Theorem 1 of Martos (1969) we thus have

Theorem 2 : *A real symmetric matrix A is MPSubD if and only if*

(a) $A \leqslant 0$

and (b) *has exactly one (simple) negative eigen value.*

Remark 1 : It is interesting to note that if A is MPSubD and $x'Ax < 0$ then $(Ax)^i = 0$ if and only if $p_1^i = 0$, that is, if and only if i-th row and i-th column of A are null. Hence we have the following result.

An MPSubD matrix is strictly PSubD if and only if p_1 is unsigned that is, if and only if A is irreducible (Gantmacher, 1960).

Also it is easy to observe that if A is MPSubD then $a_{ij} = 0$ implies either a_{ii} or a_{jj} or both are zero, and in the case of strictly PSubD $a_{ij} = 0 \Longrightarrow a_{ii} = a_{jj} = 0$.

We need the following lemma in the proof of Theorem 3. This lemma is also of independent interest.

Lemma 1 : *If A is an $n \times n$ MPSubD matrix and B is a non-negative matrix of order $n \times p$ then $B'AB$ is also MPSubD, provided it is not null.*

Proof : $x'B'ABx < 0$

$$\begin{aligned} &\Longrightarrow y'Ay < 0 \qquad \text{where } y = Bx \\ &\Longrightarrow Ay = ABx \gtreqless 0 \\ &\Longrightarrow B'ABx \gtreqless 0 \qquad \text{since } B \geqslant 0. \end{aligned}$$

Thus $B'AB$ is MPSubD.

Remark 2 : The above lemma holds for $B \leqslant 0$ also.

It is known that a square matrix A is PSemiD if and only if all its principal minors are non-negative. A similar characterisation for MPSubD matrices is proved below using a separation theorem (Wilkinson, 1965, p. 103).

The eigen values $\lambda_1', \lambda_2', \ldots, \lambda_{n-1}'$ of the leading principal minor matrix A_{n-1} of the symmetric matrix A_n separate the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A_n.

Theorem 3 : *A nonpositive symmetric matrix $A\,(\neq 0)$ is MPSubD if and only if all its principal minors are nonpositive.*

Proof : 'If' part. The proof is by induction. Assuming $A_k = 0$ or MPSubD we will prove A_{k+1} is null or MPSubD. To show A_{k+1} is MPSubD it is enough to show that it has exactly one simple negative value as the result follows from Theorem 1.

To start with $A_1 = a_{11} \leqslant 0 \Longrightarrow A_1$ is 0 or MPSubD. Notice that

$$R(A_k) \leqslant R(A_{k+1}) \leqslant R(A_k)+2.$$

Case 1 : Let $A_k = 0$. If A_{k+1} is also null we are done. Otherwise if $R(A_{k+1}) = 1$ which implies there is only one nonzero eigen value of A_{k+1}, which has to be negative since $\mathrm{tr}(A_{k+1}) \leqslant 0$. On the other hand if $R(A_{k+1}) = 2$ then out of the nonzero eigen values of A_{k+1} one is positive and the other is negative because of separation theorem.

Case 2 : Let A_k be MPSubD. We will show that A_{k+1} is also MPSubD. Denoting by $\lambda_1', \lambda_2', \ldots, \lambda_k'$, the eigen values of A_k and $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ the eigen values of A_{k+1} in increasing order we have by separation theorem $\lambda_1 < 0$, $\lambda_2 \leqslant 0$ and $\lambda_3, \ldots, \lambda_{k+1}$ are non-negative. Now if $\lambda_2 = 0$ then λ_1 is the only negative eigen value of A_{k+1} and hence the result. Otherwise that is $\lambda_2 < 0$ we will show a contradiction.

Let $R(A_{k+1}) = m$. Then there exists an m-th order non-zero principal minor of A_{k+1}, which can be brought to m-th order leading principal minor of A_{k+1} by using same permutation on rows and columns of A_{k+1}. As MPSub definiteness is undisturbed by these operations (Lemma 1) without loss of generality we can assume that the m-th order leading principal minor of A_{k+1} is nonzero. Therefore by hypothesis m-th order leading principal minor of A_{k+1} is negative.

Considering the spectral decomposition of A_{k+1} we have

$$A_{k+1} = P \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \quad P' = \begin{bmatrix} P_1MP_1' & P_1MP_3' \\ P_3MP_1' & P_3MP_3' \end{bmatrix}$$

where $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ is an orthogonal matrix and M is diagonal matrix

of m-th order with diagonal elements as the nonzero eigen values of A_{k+1}. So det $M > 0$ as there are exactly two negative eigen values of A_{k+1}. Thus m-th order principal minor of A_{k+1}, that is det $P_1MP_1' \geqslant 0$ which is contradiction. Therefore $\lambda_2 = 0$. Hence λ_1 is the only negative eigen values of A_{k+1}. That proves the if part.

Only if part: Given A is MPSubD, to show that every principal minor of $A \leqslant 0$. We know that given any r-th order principal minor there exists a permutation matrix P such that the given minor is the leading principal minor of $B = PAP'$. Now consider

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad PAP' \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_r & 0 \\ 0 & 0 \end{bmatrix}$$

where I_r is the identity matrix of order r. From Lemma 1 and Corollary 2 of Martos (1969) it follows that B_r is MPSubD.

Hence the theorem.

3. g-INVERSES OF MPSUBD MATRICES

When does a PSubD matrix have a PSubD g-inverse ? Notice that a PSubD matrix is either a PSemiD matrix or MPSubD matrix. It is known that a PSemiD matrix always posesses a PSemiD g-inverse. So our main interest is towards the class of MPSubD matrices, that is, when does an MPSubD matrix have a PSubD g-inverse ? Noticing the fact that an MPSubD matrix cannot possess a PSemiD g-inverse, since a symmetric matrix A have a PSemiD g-inverse if and only if A is PSemiD. So the only possibility is MPSubD matrix to possess MPSubD g-inverse. A necessary and sufficient condition for an MPSubD matrix to have an MPSubD g-inverse is established in Theorem 5. Before that we prove a theorem on symmetric reflexive g-inverses of symmetric matrices.

Theorem 4 : *If A is an n-th order symmetric matrix of rank r, then every symmetric reflexive g-inverse G of A can be written as*

$$G = \frac{1}{\lambda_1} q_1 q_1' + \ldots + \frac{1}{\lambda_r} q_r q_r'$$

where $\lambda_1, \ldots, \lambda_r$ are nonzero eigen values of A and $q_1, \ldots, q_r$ are independent vectors.

Proof: Consider spectral decomposition of A

$$A = P \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} P' = \lambda_1 p_1 p_1' + \lambda_2 p_2 p_2' + \ldots + \lambda_r p_r p_r'$$

where $P = [p_1 : p_2 : \ldots : p_r : \ldots p_n]$ is orthogonal matrix of eigen vectors of A and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$.

It is well known that G is a symmetric reflexive g-inverse of A if and only if G is of the form

$$G = P \begin{bmatrix} \Lambda^{-1} & U \\ U' & U'\Lambda U \end{bmatrix} P'$$

where U is arbitrary. Partition P as $[P_1 : P_2]$ and let $U' = (u_1 u_2 \ldots u_r)$. Consider

$$q_i = p_i + \lambda_i P_2 u_i \qquad \text{for } i = 1, 2, \ldots, r$$

(Notice that q_i's are independent).

Therefore

$$Q = [q_1 : q_2 : \ldots : q_r] = P_1 + P_2 U'\Lambda.$$

Now

$$\begin{aligned} G &= P \begin{bmatrix} \Lambda^{-1} & U \\ U' & U'\Lambda U \end{bmatrix} P' \\ &= (P_1 : P_2) \begin{bmatrix} \Lambda^{-1} & U \\ U' & U'\Lambda U \end{bmatrix} \begin{bmatrix} P_1' \\ \cdots \\ P_2' \end{bmatrix} \\ &= P_1\Lambda^{-1}P_1' + P_1UP_2' + P_2U'P_1' + P_2U'\Lambda UP_2' \\ &= (P_1 + P_2U'\Lambda)\Lambda^{-1}(P_1' + \Lambda UP_2') \\ &= Q\Lambda^{-1}Q' \\ &= \frac{1}{\lambda_1} q_1q_1' + \frac{1}{\lambda_2} q_2q_2' + \dots + \frac{1}{\lambda_r} q_rq_r' \end{aligned}$$

which proves the result.

Remark 2 : Observe that $p_i'q_j = 0$ for $i \neq j$ and $p_i'q_i = 1$.

Hence Q is a right inverse of P_1'.

Again if Q is any right inverse of P_1' then $Q\Lambda^{-1}Q'$ is a symmetric reflexive g-inverse of A. Thus we have the following result.

If A is a symmetric matrix then G is a symmetric reflexive g-inverse of A if and only if it is of the form

$$G = Q\Lambda^{-1}Q'$$

where Q is a right inverse of P_1'.

Now we prove

Theorem 5 : *Let A be any MPSubD matrix, then the following statements are equivalent:*

(a) *there exists an MPSubD g-inverse of A*

(b) *$R(A) = 1$ or $R(A) = 2$ and the two nonzero eigen values of A are of same magnitude.*

(c) *A^+ is MPSubD.*

Proof: (a) $\Longrightarrow$ (b). Let G_1 be an MPSubD g-inverse of an MPSubD matrix A. Then $G = G_1AG_1$ is a reflexive g-inverse of A. From Lemma 1 it follows that G is also MPSubD. Let $R(A) = r$

$$A = \lambda_1 p_1p_1' + \lambda_2 p_2p_2' + \dots + \lambda_r p_rp_r'$$

as in Theorem 1. From Theorem 4 it follows that

$$G = \frac{1}{\lambda_1} q_1 q_1' + \frac{1}{\lambda_2} q_2 q_2' + \dots + \frac{1}{\lambda_r} q_r q_r'$$

where $q_i = p_i + \lambda_i P_2 u_i$ for some u_i.

$$G p_1 = \frac{1}{\lambda_1} q_1 \Longrightarrow q_1 \geqslant 0 \qquad \text{since } G \leqslant 0 \text{ and } p_1 \geqslant 0.$$

Since G is MPSubD we have

$$g_{ii} = \frac{1}{\lambda_1} (q_1^i)^2 + \frac{1}{\lambda_2} (q_2^i)^2 + \dots + \frac{1}{\lambda_r} (q_r^i)^2 \leqslant 0 \qquad \dots \quad (5)$$

Now $q_1^i = 0 \Longrightarrow q_2^i = q_3^i = \dots = q_r^i = 0 \Longrightarrow g_{ii} = 0$. Otherwise $q_1^i \neq 0 \Longrightarrow$

$$1 \geqslant (x, x) \quad \text{where } x' = \frac{1}{q_1^i} \left(\sqrt{\frac{-\lambda_1}{\lambda_2}}\, q_2^i, \dots, \sqrt{\frac{-\lambda_1}{\lambda_r}}\, q_r^i \right).$$

Since A is MPSubD

$$a_{ii} = \lambda_1 (p_1^i)^2 + \lambda_2 (p_2^i)^2 + \dots + \lambda_r (p_r^i)^2 \leqslant 0 \qquad \dots \quad (6)$$

$$\Longrightarrow 1 \geqslant (y, y) \text{ where } y' = \frac{1}{p_1^i} \left(\sqrt{\frac{\lambda_2}{-\lambda_1}}\, p_2^i, \dots, \sqrt{\frac{\lambda_r}{-\lambda_r}}\, p_r^i \right)$$

provided $p_1^i \neq 0$. In case $p_1^i = 0$ then $a_{ii} = 0$.

Since $2(x, y) \leqslant (x, x) + (y, y)$

$$\Longrightarrow \frac{2}{p_1^i q_1} (p_2^i q_2^i + \dots + p_r^i q_r^i) \leqslant (x, x) + (y, y) \leqslant 2$$

for all i such that $p_1^i \neq 0$ and $q_1^i \neq 0$

$$\Longrightarrow p_1^i q_1^i \geqslant p_2^i q_2^i + p_3^i q_3^i + \dots + p_r^i q_r^i \text{ for all } i.$$

Summing over i, we have

$$\begin{aligned} \Sigma\, p_1^i q_1^i &\geqslant \Sigma\, p_2^i q_2^i + \dots + \Sigma\, p_r^i q_r^i \\ &\Longrightarrow 1 \geqslant r-1 \\ &\Longrightarrow r \leqslant 2. \end{aligned} \qquad \dots \quad (7)$$

Since $A \neq 0$ therefore $r = 1$ or 2.

If $r = 2$ then $A = \lambda_1 p_1 p_1' + \lambda_2 p_2 p_2'$

$$\Longrightarrow \text{Equality sign occurs in (7)}$$
$$\Longrightarrow \text{Equality sign occurs in (6)}$$
$$\Longrightarrow \text{tr } A = 0$$
$$\Longrightarrow \lambda_1 = -\lambda_2.$$

that is, both the nonzero eigen values are of same magnitude.

(b) $\Longrightarrow$ (c) If $r = 1$

$$A = \lambda_1 p_1 p_1'$$
$$\Longrightarrow A^+ = \frac{1}{\lambda_1} p_1 p_1' = \left(\frac{1}{\lambda_1}\right)^2 A.$$

Therefore A^+ is MPSubD.

If $r = 2$ and $\lambda_1 = -\lambda_2$

$$A = \lambda_1 (p_1 p_1' - p_2 p_2')$$
$$A^+ = \frac{1}{\lambda_1} (p_1 p_1' - p_2 p_2') = \left(\frac{1}{\lambda_1}\right)^2 A$$
$$\Longrightarrow A^+ \text{ is MPSubD.}$$

(c) $\Longrightarrow$ (a) is obvious.

Remark 3: When $r = 2$ and $\lambda_1 = -\lambda_2$ in the above theorem then $p_1 = |p_2|$. Because

$$0 = a_{ii} = \lambda_1[(p_1^i)^2 - (p_2^i)^2]$$
$$\Longrightarrow (p_1^i)^2 = (p_2^i)^2$$
$$\Longrightarrow p_1 = |p_2|.$$

Thus we observe that barring trivialities an MPSubD matrix does not possess an MPSubD g-inverse.

References

Gantmacher, F. R. (1960): *The Theory of Matrices*, 2, Chelsea Publishing Company, New York.

Martos, B. (1969): Subdefinite matrices and quadratic forms, *SIAM J. Appl. Math.*, 17, No. 6, 1215-1223.

Rao, C. R. and Mitra, S. K. (1971): *Generalized Inverses of Matrices and Its Applications*, Wiley, New York.

Wilkinson, J. H. (1965): *The Algebraic Eigen Value Problem*, Clarendon Press, Oxford.

Paper received : November, 1973.