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Minimal Path Sets with Known Size in a Consecutive-2-out-of-*n:F* System

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Abstract

K.G. Ramamurthy [4] showed that the number of minimal path sets of a linear consecutive-2-out-of-*n:F* system is the rounded value of the expression $\rho^n (1+\rho)^2 / (2\rho+3)$ where ρ is the unique real root of the cubic equation $x^3 - x - 1 = 0$. This paper gives two others formulae for the same. The first formula is in terms of the binomial coefficients. While the second formula is in terms of the number of minimal path sets with known size of a linear consecutive-2-out-of-*n:F* system. It is shown that the number of minimal path sets of a circular consecutive-2-out-of-n:F system is the rounded value of ρ^n, for $n \geq 10$.

Key words

Consecutive-2-out-of-*n:F* system, Minimal path sets.

1. Introduction

A linear consecutive-k-out-of-*n:F* (con/*k*/*n:F*) system consists of *n* line ordered components and the system fails if and only if at least *k* consecu components fail. If components are arranged on a circle we have a circular con/*k*/ system. This system has been studied by various authors since 1980 [3] and fi applications in telecommunication and pipeline network [2], vacuum systems accelerators, computer networks, design of integrated circuits [1] etc. We assu binary state. All components and the system are in operating or fail state. Supp P denotes the subset of components, which are in operating state. We call I path set of a system when the system itself is in operating state. A path se of the system is said to be a minimal path set if $S \subset P$ implies that S is not a path set. In this paper, we confine our attention only to minimal path sets of con|2|n:F system. In Section 2, we give a closed formula for determining the number of minimal path sets of a linear con|2|n:F system. In Section 3, we give a closed formula for determining the number of minimal path sets with known size of a linear con|2|n:F system. Similar results for a circular con|2|n:F system are given in Section 4.

2. Minimal Path Sets of a Linear con|2|n:F System

Suppose p_n^L denotes the number of minimal path sets of a linear con|2|n:F system. The following lemma is required in the sequel.

Lemma 1. We have:

$$p_n^L = p_{n-2}^L + p_{n-3}^L \text{ for } n \geq 3$$

Proof. Let $p_n^L = |\alpha^L(n)|$ where $\alpha^L(n)$ is the collection of minimal path sets of a linear con|2|n:F system. We note that:

$$a^L(n) = \{P : P \in \alpha^L(n) \text{ and } n \in P\} \cup \{P : P \in \alpha^L(n) \text{ and } n \notin P\}$$

and the collections on the right hand side are disjoint. If $n \in P$ then $n-1 \notin P$ and $n-2 \in P$. And if $n \notin P$ then $n-1 \in P$. We have:

$$\alpha^L(0) = \alpha^L(1) = \{\phi\},\ \alpha^L(2) = \{\{1\}, \{2\}\},\ \alpha^L(3) = \{\{1,3\}, \{2\}\} \text{ and}$$

$$\alpha^L(4) = \{\{1,3\}, \{2,3\}, \{2,4\}\}$$

It is easy to verify that for $n \geq 3$ we have:

$$\{P : P \in \alpha^L(n) \text{ and } n \in P\} = \{P : P = T \cup \{n-2, n\} \text{ and } T \in \alpha^L(n-3)\}$$

and for $n \geq 2$ we have:

$$\{P : P \in \alpha^L(n) \text{ and } n \notin P\} = \{P : P = T \cup \{n-1\} \text{ and } T \in \alpha^L(n-2)\}$$

Therefore we have:

$$p_n^L = |\alpha^L(n)| = |\{P : P \in \alpha^L(n) \text{ and } n \in P\}| + |\{P : P \in \alpha^L(n) \text{ and } n \notin P\}|$$

$$= |\{P : P = T \cup \{n-2,n\} \text{ and } T \in \alpha^L(n-3)\}| + |P : P = T \cup \{n-1\} \text{ and } T \in \alpha^L(n-2)\}|$$

That is: $p_n^L = |\alpha^L(n-3)| + |\alpha^L(n-2)| = p_{n-2}^L + p_{n-3}^L$; $n \geq 3$. This completes the proof of the lemma.

Suppose $[x]$ denotes integer part of x and $\binom{m}{r}$ is usual binomial coefficient. We assume that $\binom{m}{r} = 0 \text{ for } m < r \text{ or } r < 0.$

Theorem 1. We have :

$$p_n^L = \sum_{i=[n/3]}^{[n/2]} \sum_{j=n-2i-2}^{n-2i} \binom{i}{j}$$

Proof. Let g(x) denotes the generating function of p_n^L the number of minimal path sets of a linear con|2|n:F system. Noting that $p_0^L = p_1^L = 1,\ p_2^L = 2$ we have:

$$g(x) = p_0^L + p_1^L x + p_2^L x^2 + p_3^L x^3 + \dots + p_n^L x^n + \dots$$

$$x^2 g(x) = p_0^L x^2 + p_1^L x^3 + \dots + p_{n-2}^L x^n + \dots$$

$$x^3 g(x) = p_0^L x^3 + \dots + p_{n-3}^L x^n + \dots$$

Using Lemma 1, we have: $(1-x^2-x^3)\ g(x) = 1+x+x^2$ This implies that $g(x) = \frac{1+x+x^2}{1-x^2-x^3} = \frac{1+x+x^2}{1-x^2(1+x)}$. For sufficiently small x such that $|x^2(1+x)|<1$, we then have:

$$g(x) = (1+x+x^2)\left[\sum_{i=0}^{\infty} x^{2i}(1+x)^i\right] = (1+x+x^2)\left[\sum_{i=0}^{\infty} x^{2i} \sum_{j=0}^{i} \binom{i}{j} x^j\right]$$

For a given i, the maximum value of power of x is 3i + 2. Hence for getting coefficient of x^n, the minimum value of i is the nearest integer greater than or equal to $\frac{n-2}{3}$. That is $[n/3] \le i$. Similarly, the minimum value of power of x is 2i. Hence $i \le [n/2]$. It is easy to see that coefficient of x^n in g(x) is:

$$p_n^L = \sum_{i=[n/3]}^{[n/2]} \sum_{j=n-2i-2}^{n-2i} \binom{i}{j},\ n \ge 0.$$

Remark 1. We note that in Ramamurthy's formula, $p_n^L = \left[\frac{(1+\rho)^2}{2\rho+3}\rho^n + 0.5\right]$ for higher values of n, calculation precision of ρ should be increased. For example, if ρ = 1.324717958 then Ramamurthy's formula gives exact values only for n $\leq$ 58. This is not case for our formula as stated in Theorem 1.

3. Minimal Path Sets with Known Size in a Linear con$|2|n$:F System

In this Section, we first give a recursive relation for the number of minimal path sets with known size of a linear con$|2|n$:F system and then we derive a closed expression for it. The next lemmae are also required in the sequel.

Lemma 2. Let $R = \{a_1, a_2, \ldots, a_r\}$ be a subset of components such that $a_1 < a_2 < \ldots a_r$. R is a minimal path set of a linear con$|k|n$:F system if and only if:

(i) $a_i - a_{i-1} \leq K$, for $i = 1, 2 \ldots, r+1$

(ii) $a_{i+1} - a_{i-1} \geq k+1$, for $i = 1, 2 \ldots, r$

where we take $a_0 = 0$ and $a_{r+1} = n+1$.

Proof. Recall that a con$|k|n$:F system fails if and only if at least k consecutive components fail. Hence a minimal cut set of a con$|k|n$:F system is of the form $\{i, i+1, \ldots, i+k-1\}$, $i = 1, 2 \ldots, n-k+1$. It is known that for any coherent system a subset $R \subseteq N = \{1, 2 \ldots, n\}$ is a path set if and only if it has non-empty intersection with every minimal cut set. (i) $\Leftrightarrow$ R is a path set of a con$|k|n$:F system. (ii) $\Leftrightarrow$ $R - \{a_i\}$ is not a path set. This means R is a minimal path set.

Remark 2. If R is a minimal path set of a linear con$|k|n$:F system then:

(i) $|R \cap \{1, 2 \ldots, k\}| = 1$

(ii) $|R \cap \{j, j+1, \ldots, j+k\}| \leq 2$ for $j = 1, 2, \ldots, n-k$

(iii) $|R \cap \{n-k+1, n-k+2, \ldots, n\}| = 1$

It can be seen that:

(i) If we put i = 1 in Lemma 2, we then have $a_1 \leq k$ (from Part (i)) and $a_2 \geq k+1$ (from Part (ii)). Therefore $|R \cap \{1, 2, \ldots, k\}| = |\{a_1\}| = 1$.

(ii) Suppose there exists $1 \leq j^* \leq n-k$ such that $|R \cap \{J^*, j^*+1, \ldots, j^*+k\}| \geq 3$ and let $\{j_1, j_2, j_3\} \subseteq R \cap \{j^*, j^*+1, \ldots, j^*+k\} (j^* \leq j_1 < j_2 < j_3 \leq j^*+k)$ We have $\{j_1, j_2, j_3\} \subseteq R$ and note that $j_3 - j_1 \leq k$. This contradicts the second part of Lemma 2. Therefore $|R \cap \{j, j+1, \ldots, j+K\}| \leq 2$ for all $1 \leq j \leq n-k$.

(iii) If we put i = r + 1 in Part (i) of Lemma 2, we have $n+1-a_r \leq k$. Therefore $a_r \geq n-k+1$. And if we put i = r in Part (ii) of Lemma 2 we then have $n+1-a_{r-1} \geq k+1$. Hence $a_{r-1} \leq n-k$. We get result $|R \cap \{n-k+1, n-k+2, \ldots, n\}| = |\{a_r\}| = 1$.

Lemma 3. Suppose $\bar{r}_k^{n,L}$ and $\underline{r}_k^{n,L}$ denote the maximum and minimum size of a minimal path set in a linear con/k/n:F system respectively. Then we have:

$$\bar{r}_k^{n,L} = \begin{cases} 2\left[\dfrac{n}{k+1}\right] & \text{if } \dfrac{n+1}{k+1} > \left[\dfrac{n+1}{k+1}\right] \\ 2\left[\dfrac{n}{k+1}\right]+1 & \text{if } \dfrac{n+1}{K+1} = \left[\dfrac{n+1}{k+1}\right] \end{cases} \text{ and } \underline{r}_k^{n,L} = [n/k].$$

Proof. From Lemma 2, we note that $R_1 = \{k, 2k, 3k, \ldots, k[n/k]\}$ is a minimal path set of size $[n/k]$. Now suppose $A \subseteq N$ and A is a path set of a con/k/n:F system. We note that $C_s = \{sk+1, sk+2, \ldots, (s+1)k\}$, for $s = 0, 1 \ldots, [n/k]-1$, are $[n/k]$ disjoint minimal cut sets. From Lemma 2, we have:

$|A \cap C_s| \geq 1 \; \forall s.$

Hence

$$|A| = \left|\bigcup_{s=0}^{[n/k]-1}(A \cap C_s) \cup \left\{\left(N - \bigcap_{s=0}^{[n/k]-1} C_s\right) \cap A\right\}\right| \geq \left|\bigcup_{s=0}^{[n/k]-1}(A \cap C_s)\right| = \sum_{s=0}^{[n/k]-1}|A \cap C_s| \geq [n/k] = |R_1|.$$

Hence R1 is a minimal path set of minimum size.

- if $\dfrac{n+1}{k+1} = \left[\dfrac{n+1}{k+1}\right] = t$ then we note that:

$R_2 = \{1, k+1, k+2, 2k+2, 2k+3, \ldots, pk+p, \; pk+p+1\}$ is a minimal path set (by conditions of Lemma 2) where $p = \left[\dfrac{n}{k+1}\right]$ and we have $|R_2| = 2p+1$.

Suppose

$C_1 = \{1, 2, \ldots, k\}$

$C_2 = \{k+1, k+2, \ldots, 2k+1\}$

$\bar{C}_3 = \{2k+2, 2k+3, \ldots, 3k+2\}$

$\vdots$

$\bar{C}_t = \{(t-1)(k+1), \ldots, (t-1)(k+1) + k = n\}$

and Let P be a minimal path set of a linear con$k$ln:F system. By Remark 2, we have: $|P \cap \bar{C}_1| = 1$ and $P \cap \bar{C}_j$ for $\leq 2 \leq j \leq t$. We note that $P = \bigcup_{j=1}^{t} (P \cap \bar{C}_j)$ hence:

$$|P| = \sum_{j=1}^{t} |P \cap \bar{C}_j| \leq 1 + 2(t-1) = 2t - 1 = 2\left\lceil \frac{n+1}{k+1} \right\rceil - 1$$

$$= 2\left(\frac{n+1}{k+1}\right) - 1 = 2\{[n/(k+1)] + 1\} - 1 = 2[n/(k+1)] + 1 = 2p + 1 = |R_2|.$$

Hence in this case R_2 is a minimal path set with maximum size.

- *if* $\frac{n+1}{k+1} > \left\lceil \frac{n+1}{k+1} \right\rceil = t$ then we note that:

$R_3 = \{1, k+1, k+2, 2k+2, 2k+3, \ldots, (p-1)k + p, pk + p\}$ is a minimal path set (by Lemma 2) and $|R_3| = 2p = 2\,[n/(k+1)]$ We have : $n+1 = (k+1) + s$ and $0 < s \leq k$. In this case we have :

$\bar{C}_t = \{(t-1)(k+1), \ldots, (t-1)(k+1) + k = n - s\}$

We define $\bar{C}_{t+1} = \{n-s+1, n-s+2, \ldots, n\}$.

Suppose $P \subseteq N$ is a minimal path set. By Remark 2, we have $|P \cap \bar{C}_1| = 1, |P \cap \bar{C}_j| \leq 2,$ for j = 2, 3, ..., t and $P \cap \bar{C}_{t+1} \leq 1\,(since\, \bar{C}_{t+1} \subseteq \{n-k+1, n-k+2, \ldots, n\}$ and

$|P \cap \{n-k+1, n-k+2, \ldots, n\}| = 1)$. We note that $P = \bigcup_{j=1}^{t+1} (P \cap \bar{C}_j)$. Hence

$|P| < 1 + (2(t-1) + 1 = 2t = 2\left[\dfrac{n+1}{k+1}\right] = 2[n/(k+1)] = 2p = |R_3|$. Therefore in this case R_3 is a minimal path set with maximum size. This completes the proof of the lemma.

Lemma 4. Let $p_n^{r,L}$ denotes the number of minimal path sets of size r in a linear $con|2|n{:}F$ system. We have :

$$p_n^{r,L} = p_{n-2}^{r-1,L} + p_{n-3}^{r-2,L},\ n \geq 3, \bar{r}_2^{n,L} \leq r \leq \bar{\bar{r}}_2^{n,L}.$$

We assume that :

$$p_n^{r,L} = \begin{cases} 0 & \text{if } n < 0 \\ 0 & \text{if } n = 0, 1 \text{ and } r \neq 0 \\ 1 & \text{if } n = 0, 1 \text{ and } r = 0 \\ 0 & \text{if } n = 2 \text{ and } r \neq 1 \\ 2 & \text{if } n = 2 \text{ and } r = 1 \end{cases}$$

Proof. If n = 3, we have :

$$p_n^{r,L} = \begin{cases} 1 & \text{if } r = 1 \text{ or } r = 2 \\ 0 & \text{otherwise} \end{cases}$$

and if n = 4 we have :

$$p_n^{r,L} = \begin{cases} 3 & \text{if } r = 2 \\ 0 & \text{otherwise} \end{cases}$$

Hence the lemma is trivially true for n = 3 and 4. Now consider the case where $n \geq 5$. Let P_n^r denote the collection of all minimal path sets of size r in a $con|2|n{:}F$ system. We note that :

$$P_n^r = \{S : S \in P_n^r \text{ and } n \in S\} \cup \{S : S \in P_n^r \text{ and } n \notin S\}$$

and the collections on the right hand side are disjoint. We have :

$|\{S : S \in P_n^r \text{ and } n \in S\}| = |P_{n-3}^{r-2}| = p_{n-3}^{r-2,L}$ (since $n-1 \notin S, n-2 \in S$)

$|\{S : S \in P_n^r \text{ and } n \in S\}| = |P_{n-3}^{r-2}| = p_{n-3}^{r-L}$ (since $n-1 \in S$)

It follows that : $p_n^{r,L} = |P_n^r| = |P_{n-3}^{n-2}| + |P_{n-2}^{r-1}| = p_{n-2}^{r-1,L} + p_{n-3}^{r-2,L}$

This completes the proof of the lemma.

Theorem 2. We have :

$$p_n^{r,L} = \binom{n-r+1}{2n-3r} \text{ for } \bar{r}_2^{n,L} \le r \le \bar{\bar{r}}_2^{n,L} \text{ and } n \ge 2$$

Proof. Let g(x,y) denote the generating function of $p_n^{r,L}$, the number of minimal path sets of size r in a linear $con|2|n:F$ system. We have :

$$g(x,y) = \sum_{n=0}^{\infty} \sum_{r=\bar{r}_2^{n,L}}^{\bar{\bar{r}}_2^{n,L}} x^n y^r p_n^{r,L}$$

$$= 1 + x + 2x^2 y + x^3 (y + y^2) + 3x^4 y^2 + x^5 (y^2 + 3y^3) + x^6 (4y^3 + y^4) + \ldots$$

$$= 1 + x + 2x^2 y + \sum_{n=3}^{\infty} x^n \sum_{r=\bar{r}_2^{n,L}}^{\bar{\bar{r}}_2^{n,L}} y^r p_n^{r,L}$$

Using Lemma 4, we have :

$$g(x,y) = = 1 + x + 2x^2 y + \sum_{n=3}^{\infty} x^n \sum_{r=\bar{r}_2^{n,L}}^{\bar{\bar{r}}_2^{n,L}} y^r \left(p_{n-2}^{r-1,L} + p_{n-3}^{r-2,L}\right)$$

$$= 1 + x + 2x^2 y + x^2 y \sum_{n=3}^{\infty} x^{n-2} \sum_{r=\bar{r}_2^{n,L}}^{\bar{\bar{r}}_2^{n,L}} y^{r-1} p_{n-2}^{r-1,L} + x^3 y^2 \sum_{n=3}^{\infty} x^{n-3} \sum_{r=\bar{r}_2^{n,L}}^{\bar{\bar{r}}_2^{n,L}} y^{r-2} p_{n-3}^{r-2,L}$$

$$= 1 + x + 2x^2 y + x^2 y \{g(x,y) - 1\} + x^3 y^2 \{g(x,y)\}.$$

It implies that :

$$g(x,y) = \frac{1 + x + x^2 y}{1 - x^2 y - x^3 y^2}.$$

For sufficiently small x and y such that $|x^2 y(1+xy)| < 1$, we then have :

$$g(x,y) = (1+x+x^2y)\left\{\sum_{i=0}^{\infty} x^{2i} y^i (1+xy)^i\right\}$$

$$= (1+x+x^2y)\left\{\sum_{i=0}^{\infty} x^{2i}\ y^i \sum_{j=0}^{i} (xy)^j \binom{i}{j}\right\}.$$

For finding coefficient of x^n as in the proof of Theorem 1, we note that $[n/3] \le i \le [n/2]$. We have :

$$\sum_{r=\bar{r}_2^{n,L}}^{\bar{\bar{r}}_2^{n,L}} y^r\, p_n^{r,L} = \sum_{i=[n/3]}^{[n/2]} y^i \left\{ \sum_{j=n-2i-1}^{n-2i} y^j \binom{i}{j} + y^{n-2i-1}\binom{i}{n-2i-2}\right\}$$

Let $\ell = [n/2]-i$ then

$$\sum_{r=\bar{r}_2^{n,L}}^{\bar{\bar{r}}_2^{n,L}} y^r\, p_n^{r,L} = \sum_{\ell=0}^{[n/2]-[n/3]} y^{[n/2]-\ell} \left\{ \sum_{j=n-2[n/2]+2\ell-1}^{n-2[n/2]+2\ell} y^j \binom{[n/2]-\ell}{j} + y^{n-2[n/2]+2\ell-1}\binom{[n/2]-\ell}{n-2[n/2]+2\ell-2}\right\}$$

For a given r, suppose $r=[n/2]+k'$ for some $k'=0,1,\ldots,\ \bar{\bar{r}}_2^{n,L}-[n/2]$. Then the coefficient of $y^{[n/2]+k'}$ is :

$$\binom{n-[n/2]-k'-1}{2[n/2]-n+2k'+1} + \binom{n-[n/2]-k'-1}{2[n/2]-n+2k'} + \binom{n-[n/2]-k'}{2[n/2]-n+2k'}$$

$$= \binom{n-r-1}{2r-n+1} + \binom{n-r-1}{2r-n} + \binom{n-r}{2r-n}$$

First and second binomial coefficients are corresponding to $\ell = 2[n/2]-n+k'+1$ (if $k' \le n-[n/2]-[n/3]+1$). We note that $\ell \ge 0$, the third binomial coefficient is corresponding to $\ell = 2[n/2]-n+k'$ (if $k' \ge n-2[n/2]$). We also note that $\ell \le 2[n/2]-n+\bar{\bar{r}}_2^{n,L}-[n/2] = [n/2]-n+\bar{\bar{r}}_2^{n,L} \le [n/2]-[n/3]$

Therefore we have :

$$p_n^{r,L} = \binom{n-r}{2r-n} + \binom{n-r-1}{2r-n+1} + \binom{n-r-1}{2r-n} = \binom{n-r+1}{2n-3r}. \text{ (after simplification)}$$

This completes the proof of the theorem.

Remark 3.

(i) If n is odd then the minimal path set with minimum size $(r = \bar{r}_2^{n,L} = [n/2])$ *1)* is unique. Since in this case we have $p_n^{r,L} = 1$.

(ii) If $n/3 = [n/3]$ then the minimal path set with maximum size $(r = \bar{\bar{r}}_2^{n,L})$ *)* is unique. Because we have $(n+1)/3 > [(n+1)/3] = n/3$. From Lemma 3 we have $\bar{\bar{r}}_2^{n,L} = 2[n/3] = 2n/3$. It implies that

$$p_n^{\bar{\bar{r}}_2^{n,L}} = \binom{n-\bar{\bar{r}}_2^{n,L}+1}{2n-3\bar{\bar{r}}_2^{n,L}} = \binom{n-2n/3+1}{2n-2n} = 1.$$

(iii) We can introduce second formula for p_n^L, the number of minimal path sets of a linear $con|2|n:F$ system as follows : $p_n^L = \sum_{r=\bar{r}_2^{n,L}}^{\bar{\bar{r}}_2^{n,L}} p_n^{r,L} = \sum_{r=\bar{r}_2^{n,L}}^{\bar{\bar{r}}_2^{n,L}} \binom{n-r+1}{2n-3r}$.

The number of minimal path sets of a linear $con|k|n:F$ system $(k \geq 3)$ is difficult to compute directly. We have the following special cases.

Lemma 5. Let p_n^k denote the number of minimal path sets of a linear $con|k|n:F$ system, where $k \geq 2$ and $k \leq n \leq 2k$, we then have

$$p_n^k = k + \frac{(n-k)(n-k-1)}{2} = k + \binom{n-k}{2}.$$

Proof. Suppose $n = k+t, 0 \leq t \leq k$ then minimal path sets of size 1 are : $\{t+1\}, t+2\}, \ldots, \{k\} (t < k)$. And the minimal path sets of size 2 are : (if $0 < t$)

$\{1, k+1\}$

$\{2, k+1\}, \{2, k+2\}$

$\vdots$

$\{t, k+1\}, \{t, k+2\}, \dots, \{t, k+t\}$.

Remark 4. For $k=3$ we have next recurrence relation :

$p_n^3 = p_{n-2}^3 + p_{n-3}^3 + p_{n-4}^3 - p_{n-6}^3, n \geq 6$ with $p_0^3 = p_1^3 = p_2^3 = 1, p_3^3 = p_4^3 = 3$ and $p_5^3 = 4$.

4. Minimal Path Sets in a Circular con |2| n:F System

In this Section, we establish a relationship between number of minimal path sets of a linear $con|2|n:F$ system with that of in a circular $con|2|n:F$ system. Using this, we derive a closed formula for p_n^C (number of minimal path sets in a circular $con|2|n:F$ system) and $p_n^{r,C}$ (number of minimal path sets in a circular $con|2|n:F$ system of size r). We further show that the same recursive relation holds in a linear $con|2|n:F$ system as well as a circular $con|2|n:F$ system but with different initial values.

In a circular $con|k|n:F$ system we have n minimal cut sets as given below:

$C_1 = \{1, 2, \dots, k\}$

$C_2 = \{2, 3, \dots, k+1\}$

$\vdots$

$C_{n-k+1} = \{n-k+1, n-k+2, \dots, n\}$

$C_{n-k+2} = \{n-k+2, \dots, n, 1\}$

$\vdots$

$C_n = \{n, 1, 2, \dots, k-1\}$

Lemma 6. Let $R = \{a_1, a_2, \dots, a_r\}$ be a minimal path set of a circular $con|k|n:F$ system such that $a_1 < a_2 < \dots a_r$. We then have : $a_1 + n - a_r \leq k$ and $a_1 + n - a_{r-1} \geq k+1$.

Proof. We note that we should have : $\{a_r + 1, a_r + 2, \dots, n, 1, 2, \dots, a_1\} \leq k$ otherwise R is not a path set of a circular $con|k|n:F$ system. Therefore $n - a_r + a_1 \leq k$.

On the other hand we also note that :
$|\{a_{r-1}+1, a_{r-1}+2, \ldots, a_r, a_r+1, \ldots, n, 1, 2, \ldots, a_1\}| \geq k+1$ otherwise we can delete a_r from R and still be path set, that is R is not a minimal path set and this results in a contradiction. Therefore we have $n - a_{r-1} + a_1 \geq k+1$.

Lemma 7. Let $R = \{a_1, a_2, \ldots, a_r\}$ be a subset of $N = \{1, 2, \ldots, n\}$ such that $a_1 < a_2 < \ldots < a_r$. Then R is a minimal path set of a circular $con|k|n:F$ system if and only if we have :

(i) $a_i - a_{i-1} \leq k$ for $i = 1, 2, \ldots, r$

(ii) $a_{i+1} - a_{i-1} \geq k+1$ for $i = 1, 2, \ldots, r$

where $a_0 = a_r - n$ and $a_{r+1} = a_1 + n$.

Proof. The proof of part (i) for $i = 2, 3, \ldots, r$ and the proof of part (ii) for $i = 1, 2, \ldots, r-1$ are same as that of Lemma 2 and in view of Lemma 6 the proof of part (i) for $i = 1$ and the proof of part (ii) for $i = r$ are trivial.

Remark 5. If R is a minimal path set of a circular $con|k|n:F$ system then:

(i) $|R \cap \{1, 2, \ldots, k\}| \leq 2$

(ii) $|R \cap \{j, j+1, \ldots, J+k\}| \leq 2$ for $j = 1, 2, \ldots, n-k$

(iii) $|R \cap \{n-k+1, n-k+2, \ldots, n, 1\}| \leq 2$

$|R \cap \{n-k+2, n-k+3, \ldots, n, 2\}| \leq 2$

$\vdots$

$|R \cap \{n, 1, 2, \ldots, k-1\}| \leq 2$

Let P_C and P_L denote minimal path sets in a circular $con|k|n:F$ system and in a linear $con|k|n:F$system, respectively. We note that the only difference between P_C and P_L is :

(i) $|C_i \cap P_C| \leq 2$ for $i = 1, 2, \ldots, n$

(ii) $|C_i \cap P_L| = 1$ for $i = 1, n-k+1$

(iii) $|C_i \cap P_L| \leq 2$ for $i = 2, 3, \ldots, n-k$

Lemma 8. We have : $p_n^C = 2p_{n-3}^L + p_{n-6}^L ; n \geq 6$ where p_n^L is the number of minimal path sets in a linear $con|2|n:F$ system.

Proof. Suppose P_C is a minimal path set of a circular $con|2|n:F$ system. We have the following mutually exhaustive cases.

(i) Let $1 \in P_C$ and $2 \in P_C$.

We then have $n \notin P_C$ and $3 \notin P_C$. Hence $n-1 \in P_C$ and $4 \in P_C$. In this case P_C is a minimal path set for a circular $con|2|n:F$ system if and only if $P_C \cap \{5, 6, ..., n-2\}$ is a minimal path set for a linear $con|2|n-6:F$ subsystem with $\{5, 6, ..., n-2\}$ component set. Hence we have : $p_n^C = p_{n-6}^L$. If $n = 6$ we know that the only minimal path set in this case is : $P_C = \{1, 2, 4, 5\}$ and also we note that $p_n^C = p_{n-6}^L = 1$.

(ii) Let $1 \in P_C$ and $2 \notin P_C$.

We then have $3 \in P_C$. Similarly P_C is a minimal path set for a circular $con|2|n:F$ system if and only if $P_C \cap \{4, 5, ..., n\}$ is a minimal path set for a linear $con|2|n-3:F$ subsystem with $\{4, 5, ..., n\}$ component set. Hence we have : $P_n^C = p_{n-3}^L$.

(iii) Let $1 \notin P_C$ and $2 \in P_C$.

We then have $n \in P_C$ and P_C is a minimal path set for a circular $con|2|n:F$ system if and only if $P_C \cap \{3, 4, ..., n-1\}$ is a minimal path set for a linear $con|2|n-3:F$ subsystem with $\{3, 4, ..., n-1\}$ component set. Hence we have : $P_n^C = p_{n-3}^L$.

From these cases we get the result : $P_n^C = p_{n-6}^L + 2p_{n-3}^L$ for $n \geq 6$.

Remark 6. We know that : $p_n^L = p_{n-2}^L + p_{n-3}^L$; $n \geq 3$ with $p_0^L = p_1^L = 1$ and $p_2^L = 2$. We define : $p_{-1}^L = 1, p_{-2}^L = 0, p_{-3}^L = 1, p_{-4}^L = 0$, $p_{-5}^L = 0$ and $p_6^L = 1$. Therefore the relation $p_n^L = p_{n-2}^L + p_{n-3}^L$ holds true for all $n \geq -3$. Hence using Lemma 8, we get : $p_0^C = 2p_{-3}^L + p_{-6}^L = 3, p_1^C = 0, p_2^C = 2, p_3^C = 3, p_4^C = 2$, and $p_5^C = 5$. Therefore Lemma 8 holds true for all $n \geq 0$.

Lemma 9. We have :

$p_n^C = p_{n-2}^C + p_{n-3}^C \; ; n \geq 3$

Proof. From Lemma 8 and Remark 6, we have : $p_n^C = 2p_{n-3}^L + p_{n-6}^L$ and $n \geq 0$ and $p_n^L = p_{n-2}^L + p_{n-3}^L$ for $n \geq -3$.

Hence for $n \geq -3$ we can write :

$$p_n^C = 2(p_{n-5}^L + p_{n-6}^L) + (p_{n-8}^L + p_{n-9}^L) = (2p_{n-5}^L + p_{n-8}^L) + (2p_{n-6}^L + p_{n-9}^L).$$

If we again use Lemma 8, we get result : $p_n^C = p_{n-2}^C + p_{n-3}^C$.

In other words same recursive relation holds in a circular $con|2|n:F$ system as well as a linear $con\,2\,n:F$ system but with different initial conditions $(p_0^L = p_1^L = 1, p_2^L = 2$ and $p_0^C = 3, p_1^C = 0, p_2^C = 2)$.

Up to now we have given closed formulae for the number of minimal path sets of a linear $con\,2\,n:F$ system (three formulae are proposed) and also for $p_n^{r,L}$ as stated in Theorem 2. Hence applying Lemma 8 and Remark 3 (part (iii)) we can derive a closed formula for p_n^C. Now we use Lemma 9, to derive a closed formula for p_n^C directly.

Let $g_c(x)$ denote the generating function of p_n^C, the number of minimal path sets of a circular $con\,2|n:F$ system. By using Lemma 9 we have :

$$g_C(x) = \frac{p_0^C + p_1^C x + (p_2^C - p_0^C)\,x^2}{1 - x^2 - x^3} = \frac{3 - x^2}{1 - x^2 - x^3}$$

For partial fraction expansion of $g_C(x)$ let $\frac{3 - x^2}{1 - x^2 - x^3} = \frac{a}{1 - \rho x} + \frac{b}{1 - \sigma x} + \frac{c}{1 - \bar{\sigma} x}$ where ρ is the real root and σ and $\bar{\sigma}$ (conjugate of σ) are the complex roots of the cubic $x^3 - x - 1 = 0$. (We note that $1/\rho, 1/\dot{\sigma}$ and $1/\sigma$ are the roots of the equation $1 - x^2 - x^3 = 0$). It is easy to see that :

$$\rho\sigma\bar{\sigma} = 1, \rho + \sigma + \bar{\sigma} = 0, R(\sigma) = -\rho/2, I^2(\sigma) = \frac{3-\rho}{4\rho}, |\rho - \sigma|^2 = \frac{2\rho + 3}{\rho}.$$

We then have :

$$a=\frac{3-1/\rho^2}{(1-\sigma/\rho)(1-\bar{\sigma}/\rho)}=\frac{3\rho^2-1}{(\rho-\sigma)(\rho-\bar{\sigma})}=\frac{3\rho^2-1}{|\rho-\sigma|^2}=\frac{\rho(3\rho^2-1)}{2\rho+3}=\frac{3\rho^3-\rho}{2\rho+3}=\frac{3(1+\rho)-\rho}{2\rho+3}=1$$

$$b=\frac{3-1/\sigma^2}{(1-\rho/\sigma)(1-\bar{\sigma}/\sigma)}=\frac{3\sigma^2-1}{(\sigma-\rho)(\sigma-\bar{\sigma})}=-\frac{(\rho-\bar{\sigma})(3\sigma^2-1)}{2(\sqrt{-1})I(\sigma)|\rho-\sigma|^2}$$

$$c=\frac{3-1/\bar{\sigma}^2}{(1-\rho/\bar{\sigma})(1-\sigma/\bar{\sigma})}=\frac{3\bar{\sigma}^2-1}{(\bar{\sigma}-\rho)(\bar{\sigma}-\sigma)}=\frac{(\rho-\bar{\sigma})(3\bar{\sigma}^2-1)}{2(\sqrt{-1})I(\sigma)|\rho-\sigma|^2}$$

It follows that :

$$p_n^C=\rho^n-\frac{(\rho-\bar{\sigma})(3\sigma^2-1)}{2(\sqrt{-1})\,I(\sigma)|\rho-\sigma|^2}\sigma^n+\frac{(\rho-\bar{\sigma})(3\bar{\sigma}^2-1)}{2(\sqrt{-1})I(\sigma)|\rho-\sigma|^2}\bar{\sigma}^n$$

Suppose $\sigma=r\,(\cos\theta+\sqrt{-1}\,\sin\theta),c_1=\dfrac{2+\rho}{2\rho+3}$ and $c_2=\dfrac{-\sqrt{\rho}\,\rho^2}{(2\rho+3)\sqrt{3-\rho}}$. It is easy to verify that (see for example Spickerman [5])

$p_n^C=\rho^n+3h_n-h_{n-2}$, where $h_n=r^n\,(c_1\cos n\theta+c_2\sin n\theta)$.

Theorem 3. The number of minimal path sets of a circular con|2|n:F system for all $n\geq 10$ is given by $p_n^C=\lfloor\rho^n+0.5\rfloor$ where ρ is the unique real root of the cubic equation $x^3-x-1=0$.

Proof. From Lemma 9, we have $p_n^C=p_{n-2}^C+p_{n-3}^C, n\geq 3$ with $p_0^C=3, p_1^C=0,\ p_2^C=2$. Using this up to n = 12 we get, $p_{10}^C=17,\ p_{11}^C=22$ and $p_{12}^C=29$. Applying Cardan's formula to the cubic equation $x^3-x-1=0$, we have approximately ρ = 1.324717958. It can be verified that the theorem is trivially true for $n=10,\ 11,\ 12$. Now suppose $n\geq 13$. We have already shown that $p_n^C=\rho^n+3h_n-h_{n-2}$ Therefore it is enough to show that $|3h_n-h_{n-2}|<0.5$ for $n\geq 13$ We have:

$$|3h_n-h_{n-2}|\ \leq\ 3|h_n|\ +\ |h_{n-2}|\ \leq\ 3r^n\sqrt{c_1^2+c_2^2}+r^{n-2}\ \sqrt{c_1^2+c_2^2}=(3r^n+r^{n-2})\sqrt{c_1^2+c_2^2}$$

Let $H_1(\rho)=\sqrt{c_1^2+c_2^2}=\sqrt{\left(\frac{2+\rho}{2\rho+3}\right)^2+\frac{\rho^5}{(2\rho+3)(2-\rho)}}=\frac{2}{\sqrt{-2\rho^2+3\rho+9}}$ (after simplification).

We note that $H_1(\rho)$ is an increasing function for $\rho>3/4$

Hence we have $H_1(\rho)<H_1(1.325)=0.650127$ since we know that $\rho<1.325$. We also note that $\rho\sigma\bar{\sigma}=1$, hence $\rho r^2=1$ or $r=1/\sqrt{\rho}$ If $n\geq 13$, we then have:

$$3r^n+r^{n-2}\leq 3r^{13}+r^{11}=\frac{3}{\sqrt{\rho\rho^6}}+\frac{1}{\sqrt{\rho\rho^5}}=\frac{3}{\sqrt{\rho(1+\rho)^2}}+\frac{1}{\sqrt{\rho}\,(1+\rho)\rho^2}$$

$$=\frac{3}{\sqrt{\rho(1+\rho)^2}}+\frac{1}{\sqrt{\rho}\,(\rho^2+\rho+1)}\ \text{(since } \rho^3-\rho-1=0).$$

Let $H_2(\rho)=\frac{3}{\sqrt{\rho(1+\rho)^2}}+\frac{1}{\sqrt{\rho}\,(\rho^2+\rho+1)}$. We note that $H_2(\rho)$ is a decreasing function for $\rho>0$. Hence we have

$3r^n+r^{n-2}\leq H_2(\rho)<H_2(1.324)=0.696$, since $\rho>1.324$. Therefore we have for al $n\geq 13$: $|3h_n-h_{n-2}|\leq H_1(\rho)H_2(\rho)<(0.65)(0.696)<0.5$ This completes the proof of the theorem.

We now consider the number of minimal path sets with known size in a circular conl2ln:F system. Suppose $\overset{=n,C}{r_2}$ and $\bar{r}_2^{n,C}$ denote the maximum and the minimum size of a minimal path set in a circular conl2ln:F system, respectively.

Lemma 10. We have:

(i) $\overset{=n,C}{r_2}=\overset{=n,L}{r_2}$

(ii) $$\bar{r}_2^{n,C}=\begin{cases}\bar{r}_2^{n,L}=[n/2] & \text{if } n \text{ is even}\\ \bar{r}_2^{n,L}+1=[n/2]+1 & \text{if } n \text{ is odd}\end{cases}$$

Proof.

(i) We consider two cases as follows:

• $n+1=3s$ for some integer $s \geq 1$.

By Lemma 3, we note that $R_1 = \{1,3,4,6,7,9, \ldots, 3(s-1), 3(s-1)+1\}$ is a minimal path set with maximum size in a linear con|2|n:F system and $|R_1| = 2(s-1)+1 = 2s-1$. From Lemma 7, we note that R_1 is also a minimal path set of a circular con|2|n:F system. We show that size of R_1 in a circular con|2|n:F system is also maximum. Let:

$C_1 = \{1,2\}$

$C_2 = \{3,4,5\}$

$C_3 = \{6,7,8\}$

$\vdots$

$C_s = \{3(s-1),\ 3(s-1)+1,\ 3(s-1)+2 = n\}$

Let P_C be a minimal path set of a circular con|2|n:F system. We note that $P_C \cap C_i$ is nonempty and by Remark 5, we also note that $|P_C \cap C_i| \leq 2$ for $i = 1, 2, \ldots, s$. We show that there exists $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 1$. Suppose $|P_C \cap C_i| = 2\ \forall i$. We then have: $\{1,2\} \subseteq P_C \Rightarrow 3 \notin P_C \Rightarrow \{4,5\} \subseteq P_C \Rightarrow 6 \notin P_C \Rightarrow \ldots \Rightarrow 3(s-1) \notin P_C \Rightarrow 3(s-1)+1 = n-1 \in P_C$ and $n \in P_C$. Hence we have: $\{1, 2, n-1, n\} \subseteq P_C$ that is, P_C is not a minimal path set, resulting in a contradiction. Therefore there exists $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 1$ It implies that:

$$|P_C| = \left|\bigcup_{i=1}^{s}(P_C \cap C_i)\right| = \sum_{i=1}^{s}|P_C \cap C_i| \leq 1 + 2(s-1) = 2s-1 = |R_1|.$$

That is R_1 is a minimal path set with maximum size in a circular con|2|n:F system.

• $n+1=3s+t,\ s \geq 1,\ 1 \leq t \leq 2$ and t and s are integers.

In view of Lemma 3, $R_2 = \{1,3,4,6,7,9,\ldots,3(s-1),\ 3(s-1)+1, 3s\}$ is a minimal path set with maximum size in a linear con|2|n:F system. $|R_2| = 2s$.

From Lemma 7, R_2 is also a minimal path set of a circular con|2|n:F system. We have $C_s = \{3(s-1),\ 3(s-1)+1,\ 3(s-1)+2\} = \{n-t-2, n-t-1, n-t\}$

We define $C_{s+1} = \{n-t+1, n\}$. Let P_C be a minimal path set of a circular conl2ln:F system.

By Remark 5, we note that $|P_C \cap C_i| \leq 2$ for $1, 2, \dots s+1$

If $t=1$ and $n \in P_C$ then we note that $|P_C \cap C_1| = |P_C \cap \{1, 2\}| = 1$ Hence we have

$$|P_C| = \left|\bigcup_{i=1}^{s-1}(P_C \cap C_i)\right| = \sum_{i=1}^{s+1}|P_C \cap C_i| \leq 1 + 2(s-1) + 1 = 2s = |R_2|.$$ Therefore

R_2 is a minimal path set with maximum size in a circular conl2ln:F system.

If $n \notin P_C$ we then have $|P_C| = \left|\bigcup_{i=1}^{s}(P_C \cap C_i)\right| = \sum_{i=1}^{s}|P_C \cap C_i| \leq 2s = |R_2|$ and the result is immediate.

Now suppose $t=2$. We show that there exist

$1 \leq i^* \leq s+1,\ 1 \leq j^* \leq s+1,\ i^* \neq j^*$ such that $|P_C \cap C_{i^*}| = |P_C \cap C_{j^*}| = 1$

We note that $\{1, 2, n-1, n\} \not\subset P_C$ that is $|P_C \cap C_1| = 1$ or $|P_C \cap C_{s+1}| = 1$ Without loss of generality we assume that $|P_C \cap C_{s+1}| = 1$ We have :

$|P_C \cap C_i| = 2$ for $i = 1, 2, \dots, s$

$\{1, 2\} \subseteq P_C \Rightarrow 3 \notin P_C \Rightarrow \{4, 5\} \subseteq P_C \Rightarrow 6 \notin P_C \Rightarrow \cdots \Rightarrow 3(s-1) = n-4 \notin P_C \Rightarrow \{n-3, n-2\} \subseteq P_C \Rightarrow n-1 \notin P_C \Rightarrow n \in P_C$

Therefore we have $\{1, 2, n\} \subset P_C$ resulting in a contradiction. Hence there exists $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 1$ We then have: $|P_C| = \left|\bigcup_{i=1}^{s+1}(P_C \cap C_i)\right|$

$$= 1 + \sum_{i=1}^{s+1}|P_C \cap C_i| = 1 + \sum_{i \neq i^*}^{s}|P_C \cap C_i| + 1 \leq 2 + 2(s-1) = 2s = |R_2|.$$

This completes the proof of part (i).

(ii) We consider two cases as follows:

- $n = 2s$ for some integer $s \geq 1$.

Let $R_3 = \{2, 4, 6, ..., 2s\}$ From Lemma 3, we know that R_3 is a minimal path set with minimum size in a linear con|2|n:F system and by Lemma 7, R_3 is also a minimal path set of a circular con|2|n:F system. We have $|R_3| = s = [n/2]$ We show that size of R_3 is minimum. Suppose P_C is a minimal path set of a circular con|2|n:F system. We note that $C_i = \{2i-1, 2i\}$, $i = 1, 2, ..., [n/2]$, are $[n/2]$ disjoint minimal cut sets. We also note that $|P_C \cap C_i| \geq 1$ for $i = 1, 2, ..., [n/2]$.

Hence $|P_C| = \left|\bigcup_{i=1}^{[n/2]}(P_C \cap C_i)\right| \geq [n/2] = |R_3|$ Therefore the result is immediate.

• $n = 2s+1$ for some integer $s \geq 1$.

Let $R_4 = \{1, 2, 4, 6, 8, ..., 2s\}$. We know that R_4 is a minimal path set with minimum size in a linear con|2|n:F system and is also a minimal path set of a circular con|2|n:F system. We show that size of R_4 is minimum. We have: $|R_4| = s + 1 = [n/2] + 1$ and $|P_C \cap C_i| \geq 1$ for $i = 1, 2, ..., s$ Hence:

$$|P_C| = \left|\bigcup_{i=1}^{s}(P_C \cap C_s) \cup (P_C \cap \{n\})\right| \geq \left|\bigcup_{i=1}^{s}(P_C \cap C_i)\right| = \sum_{i=1}^{s}|P_C \cap C_i| \geq s$$

If $n \in P_C$ then $|P_C| \geq s+1 = |R_4|$ and the required result follows.

If $n \notin P_C$ we show that there exists $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 2$ Suppose $|P_C \cap C_i)| = 1$ for $i = 1, 2, ..., s$ We then have:
$n \notin P_C \Rightarrow 1 \in P_C \Rightarrow 2 \notin P_C \Rightarrow 3 \in P_C \Rightarrow 4 \notin P_C \Rightarrow ... \Rightarrow 2s-1 \in P_C \Rightarrow 2s \notin P_C \Rightarrow 2s+1 = n \in P_C$ and this gives arise to a contradiction. Therefore, there exists $1 \leq i^* \leq s$ such that $|P_C \cap C_{i^*}| = 2$. Hence:

$$|P_C| = \left|\bigcup_{i=1}^{s}(P_C \cap C_i)\right| = \sum_{i=1}^{s}|P_C \cap C_i| \geq (s-1) + 2 = s + 1 = |R_4|$$

This completes the proof of the lemma.

Lemma 11. We have:

$$p_n^{r,C} = 2p_{n-3}^{r-2,L} + p_{n-6}^{r-4,L}; \; n \geq 6, \; \bar{r}_2^{n,C} \leq r \leq \bar{\bar{r}}_2^{n,C}$$

In view of Lemma 8, the proof of this lemma is easy and omitted.

Lemma 12. We have:

$$p_n^{r,C} = p_{n-2}^{r-1,C} + p_{n-3}^{r-3,C}; \; n \geq 3, \; r_2^{n,C} \leq r \leq \bar{\bar{r}}_2^{n,C}$$

That is, the recurrence relation for the number of minimal path sets of given size r in both a linear and a circular conl2ln:F system is the same.

Now using Theorem 2 and Lemma 11 we give a closed formula for $p_n^{r,C}$ We have:

$$p_n^{r,C} = 2p_{n-3}^{r-2,L} + p_{n-6}^{r-4,L} = 2\binom{n-3-(r-2)+1}{2(n-3)-3(r-2)} + \binom{n-6-(r-4)+1}{2(n-6)-3(r-4)}$$

$$= 2\binom{n-r}{2n-3r} + \binom{n-r-1}{2n-3r} = \frac{n}{2r-n}\binom{n-r-1}{2n-3r}$$

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