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A UNIQUENESS PROBLEM FOR PROBABILITY MEASURES ON LOCALLY-COMPACT ABELIAN GROUPS

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SUMMARY. Suppose μ and ν are probability measures on a locally-compact abelian group G such that $\mu(E+x) = \nu(E+x)$ for every $x \in G$ and for a fixed set E with compact closure and positive Haar measure. We investigate the relation between ν and μ.

1. Introduction

Let $(X, \mathcal{B})$ be a measurable space and let μ, ν be two probability measures on $\mathcal{B}$. Let $\mathcal{S} \subset \mathcal{B}$ be a subclass of $\mathcal{B}$ such that the σ-algebra generated by $\mathcal{S}$ is $\mathcal{B}$. Let $\mu(A) = \nu(A)$ for every $A \in \mathcal{S}$. Then, by the well-known extension theory for measures, $\mu(A) = \nu(A)$ for every $A \in \mathcal{B}$. One can ask the question : what happens when $\mathcal{S}$ does not generate $\mathcal{B}$? Answers to this question are known when X is a 'nice' space and $\mathcal{S}$ is some 'nice' subclass of $\mathcal{B}$. For example, consider the situation when X is a metric space and $\mathcal{B}$ is the σ-algebra of Borel subsets of X. Let $B(x, r)$ denote the closed ball with center at x and radius r. X is said to be finite dimensional if every ball of radius r can be covered by a finite number of balls of radius $r/2$. Anderson (1971) showed that if X is a finite dimensional metric space and $\mu[B(x, r)] = \nu[B(x, r)]$ for every $x \in X$ and $r > 0$, then $\mu(A) = \nu(A)$ for every $A \in \mathcal{B}$. A measure m on a metric space X is said to be uniform if $0 < m[B(x, r)] < \infty$ for every $x \in X$, $r > 0$ and if $m[B(x, r)]$ is independent of x. Let X be a metric space on which there exists some uniform measure. Christenson (1970) showed that on such a metric space, if $\mu[B(x, r)] = \nu[B(x, r)]$ for every $x \in X$ and $r > 0$, then $\mu(A) = \nu(A)$ for every $A \in \mathcal{B}$. Consider the situation when $X = R^n$ and $\mathcal{B}$ is the σ-algebra of Borel subsets of R^n. Sapogov (1974) proved the following : let E be a fixed set of positive Lebesgue measure such that $\mu(E+x) = \nu(E+x)$ for every $x \in R^n$. If E has finite Lebesgue measure, then $\mu = \nu$. If the support of the Fourier transform of χ_E contains a non-empty open set, then $\mu = \nu$. In the present paper we investigate the following situation : let G be a locally-compact, second-countable abelian group and let $\mathcal{B}_G$ be the σ-algebra of Borel subsets of G. Let E be a fixed subset of G of positive Haar measure such that the closure of E is compact. Let μ and ν be two probability measures on

G such that $\mu(E+x) = \nu(E+x)$ for every $x \in G$. We ask the question : what is the relation between μ and ν ? When $G = R^n$, it is easy to see that $\mu = \nu$ (see Sapogov, 1974). However, in general, this is not true. For example consider the group $G = \mathscr{Z} \times K$, where $\mathscr{Z}$ denotes the integer group and K is some compact abelian group. Choose two probability measures μ_1 and μ_2 on K such that $\mu_1 \neq \mu_2$. Choose some probability measure λ on $\mathscr{Z}$. Put $\mu = \lambda \times \mu_1$ and $\nu = \lambda \times \mu_2$. Let $E = \{0\} \times K$. Then it is easy to check that $\mu(E+x) = \nu(E+x)$ for every $x \in \mathscr{Z} \times K$. Obviously $\mu \neq \nu$. However, if we put $\theta = \delta_0 \times \lambda_K$, where δ_0 denotes the probability measure degenerate at $0 \in \mathscr{Z}$: and λ_K denotes the normalized Haar measure of K, then it is easy to see that $\mu * \theta = \nu * \theta$.

Let λ_H denote a Haar measure of a locally-compact group H. We shall prove the following

Theorem 1 : *Let G be a locally-compact, second-countable abelian group and let $\mathscr{B}_G$ be the σ-algebra of Borel subsets of G. Let $E \in \mathscr{B}_G$ be a fixed set such that the closure of E is compact and $\lambda_G(E) > 0$. Let G_0 be the subgroup of G generated by E and let K be the maximal compact subgroup of G_0. If for any two probability measures μ and ν on $\mathscr{B}_G$, $\mu(E+x) = \nu(E+x)$ for every $x \in G$, then $\mu * \lambda_K = \nu * \lambda_K$.*

As an application of this theorem, we shall prove

Theorem 2 : *Let G be a locally-compact, second-countable abelian group and let $\mathscr{B}_G$ be the σ-algebra of Borel subsets of G. Let $E \in \mathscr{B}_G$ be a fixed set such that the closure of E is compact and $\lambda_G(E) > 0$. Let G_0 be the subgroup of G generated by E and let K be the maximal compact subgroup of G_0. Let μ_n, $n = 0, 1, 2, \ldots$ be probability measures on $\mathscr{B}_G$ such that $\mu_n(E+x) \to \mu_0(E+x)$ as $n \to \infty$ for almost all $x(\lambda_G)$. Then the following holds*

(i) *the set of limit points of $\{\mu_n\}$, $n = 1, 2, \ldots$ is non-empty;*

(ii) *for every limit point ν of $\{\mu_n\}$, $n = 1, 2, \ldots$, $\nu * \lambda_K = \mu_0 * \lambda_K$;*

(iii) *$\{\mu_n * \lambda_K\}$, $n = 1, 2, \ldots$ converges weakly to $\mu * \lambda_K$.*

2. Preliminaries

Throughout the discussion, G will stand for a locally-compact, second-countable abelian group. In particular G can be viewed as a complete and separable metric group. Let $\mathscr{B}_G$ be the σ-algebra of Borel subsets of G, $\mathscr{M}(G)$ the set of all probability measures on $(G, \mathscr{B}_G)$ with the weak topology and let $*$

be the convolution operation in $\mathcal{M}(G)$ (see Parthasarathy, 1967). For any measurable function f on $(G, \mathcal{B}_G)$ and $\mu \in \mathcal{M}(G)$, let $f \bullet \mu$ denote the function $f(x-y)\mu(dy)$, $x \in G$, whenever it is well defined.

Let H be any closed subgroup of G. A function f on G is said to be H-invariant if $f(x+y) = f(x)$ for every $x \in G$, $y \in H$. Let $I(H)$ denote the set of all H-invariant Borel functions on $(G, \mathcal{B}_G)$. Let $\mathcal{B}(G/H)$ denote the space of all Borel functions on the quotient group G/H with the natural quotient σ-algebra. Define the map $T : I(H) \to \mathcal{B}(G/H)$ by

$$(Tf)(x+H) = f(x),$$

for every $f \in I(H)$, $x+H \in G/H$. It is easy to see that T is a well-defined one-one map from $I(H)$ onto $\mathcal{B}(G/H)$. Further, for any $f_1, f_2 \in I(H)$, the following relation holds in the sense that whenever either side is well-defined, so is other and both are equal

$$T(f_1 * f_2) = (Tf_1) * (Tf_2) \qquad \dots \quad (1)$$

with the above notations and definitions, we have the following

Lemma : *Let G be a locally-compact, second-countable abelian group. Let $E \in \mathcal{B}_G$ be such that $\lambda_G(E) > 0$ and the closure of E is compact. Let G_0 be the subgroup of G generated by E and let K be the maximal compact subgroup of G_0. Let $f \in L_1(G)$ be a continuous function such that $\int_E f(x-y)\lambda_G(dy) = 0$ for every $x \in G$. Then $(f \bullet \lambda_K)(x) = 0$ for every $x \in G$.*

Proof : First note that G_0 is an open subgroup of G. Further, since G_0 is a locally-compact, compactly generated abelian group, by the structure theory, $G_0 = \mathcal{Z}^r \times R^n \times K$ where K is some compact abelian group and r, n are non-negative integers (see Hewitt and Ross, 1963). Further K is the maximal compact subgroup of G_0.

To prove the lemma, let us first assume that $f \in I(K)$. We shall show that $f \equiv 0$. Choose $x_0 \in G$ arbitrarily and fix it. Put $f_{x_0}(x) = f(x_0+x)$, $x \in G$. Then $f_{x_0} \in I(K) \cap L_1(G)$, and from the given condition on f, we have

$$\int_E f_{x_0}(x-y)\lambda_G(dy) = 0 \quad \text{for every } x \in G.$$

Since the integration is only over a subset of G_0, we have

$$\int_E f_{x_0}(x-y)\lambda_{G_0}(dy) = 0 \text{ for every } x \in G_0.$$

i.e., $\qquad (f_{x_0} \bullet \chi_E)(x) = 0$ for every $x \in G_0$.

Thus

$$\begin{aligned}[f_{x_0} \bullet (\chi_E \bullet \lambda_K)](x) &= [(f \bullet \chi_E) \bullet \lambda_K](x) \\ &= 0 \text{ for every } x \in G. \qquad \dots \quad (2)\end{aligned}$$

Put $\varphi = \chi_E * \lambda_K$. Then $\varphi \in I(K)$ is a non-trivial bounded function with compact support. Equation (2) along with (1) gives

$$(Tf_{x_0}) * (T\varphi) = 0 \text{ on } G/K = \mathscr{T}^r \times R^n.$$

Taking Fourier transform, we have

$$(\hat{Tf_{x_0}}).(\hat{T\varphi}) = 0 \text{ on } \mathscr{T}^r \times R_n. \qquad \dots \quad (3)$$

Here $\mathscr{T}$ denotes the circle group. Since $T\varphi$ is a non-trivial bounded function with compact support, the set $\{\gamma \in \mathscr{T}^r \times R^n | (\hat{T\varphi})(\gamma) \neq 0\}$ is dense in $\mathscr{T}^r \times R^n$. Thus (3) gives $(\hat{Tf_{x_0}})(\gamma) = 0$ for all γ in a dense subset of $\mathscr{T}^r \times R^n$ Since the Fourier transform is a continuous function, we have

$$(\hat{Tf_{x_0}}) = 0 \quad \text{on} \quad \mathscr{T}^r \times R^n.$$

Hence $$f_{x_0}(y) = 0 \quad \text{a.e.} \quad y(\lambda_{G_0}).$$

Since f_{x_0} is a continuous function on G_0 we have $f_{x_0}(y) = f(x_0+y) = 0$ for every $y \in G_0$. Since this holds for every $x_0 \in G$, we have $f \equiv 0$.

To prove the lemma in the general case, put $\bar{f} = f * \lambda_K$. Then $\bar{f}$ is a continuous function and $\bar{f} \in I(K) \cap L_1(G)$. Further, since $\int_E f(x-y)\lambda_G(dy) = 0$ for every $x \in G$, we have

$$\int_E \bar{f}(x-y)\lambda_G(dy) = 0 \text{ for every } x \in G,$$

Thus $\bar{f} = f * \lambda_K$ satisfies all the conditions required and thus by the above discussion, $\bar{f} = f * \lambda_K \equiv 0$.

This proves the lemma completely.

3. Proof of Theorem 1

We are given that $\mu(E+x) = \nu(E+x)$ for every $x \in G$. Equivalently, we have

$$(\chi_{-E} * \mu)(x) = (\chi_{-E} * \nu)(x) \text{ for every } x \in G.$$

Now let f be any continuous function on G with compact support. Then

$$[f*(\chi_{-E} * \mu)](x) = [f*(\chi_{-E} * \nu)](x) \text{ for every } x \in G.$$

i.e., $$[(f*\mu) * \chi_{-E}(x)] = [(f*\nu) * \chi_{-E}](x) \text{ for every } x \in G.$$

Put $\bar{f} = f * \mu - f * \nu$. Then $\bar{f} \epsilon L_1(G)$ and

$$(\bar{f} * X_{-E})(x) = 0 \text{ for every } x \epsilon G.$$

Now applying the lemma and noting that groups generated by E and $-E$ are the same, we have

$$(\bar{f} * \lambda_K)(x) = 0 \text{ for every } x \epsilon G$$

i.e., $$[(f * \mu) * \lambda_K](x) = [(f * \nu) * \lambda_K](x) \text{ for every } x \epsilon G.$$

i.e., $$[f * (\mu * \lambda_K)](x) = [f * (\nu * \lambda_K)](x) \text{ for every } x \epsilon G.$$

Since this holds for every continuous function f with compact support, we have $\mu * \lambda_K = \nu * \lambda_K$. This proves Theorem 1.

4. Proof of theorem 2

Since G is locally-compact and second-countable, it is σ-compact. Choose a sequence K_n of compact subsets of G such that $K_1 \subset K_2 \subset \dots \subset K_n \subset \dots$, $\bigcup_{n=1}^{\infty} K_n = G$ and $K_n = -K_n$ for every n.

To prove (i), we shall show that the sequence $\{\mu_n\}$, $n = 1, 2, \dots$ is uniformly tight. Let $\varepsilon > 0$ be given. Let $\varepsilon' = \varepsilon.\lambda_G(E)/1+\lambda_G(E)$. We choose integers n_1, n_2 and n_3 as follows

Choose n_1 so large such that

$$E \subset K_{n_1} \text{ and } \mu_0(K_{n_1}) > 1-\varepsilon'. \qquad \dots \quad (4)$$

Choose $n_2 > n_1$ such that $K_{n_1}+K_{n_1} \subset K_{n_2}$.

Choose $n_3 > n_2$ such that $K_{n_1}+K_{n_2} \subset K_{n_3}$.

Finally choose an integer N such that for every $n \geqslant N$,

$$\int_{K_{n_2}} \mu_n(E+x)\lambda_G(dx) > \int_{K_{n_2}} \mu_0(E+x)\, \lambda_G(dx) - \varepsilon'. \qquad \dots \quad (5)$$

Define $A_0, A_1, A_2 \subset G \times G$ as follows

$$A_0 = \{(x, y) \mid x \epsilon K_{n_2},\ y-x \epsilon K_{n_1}\},$$

$$A_1 = \{(x, y) \mid y \epsilon K_{n_1},\ x-y \epsilon K_{n_1}\},$$

$$A_2 = \{(x, y) \mid y \epsilon K_{n_3},\ x-y \epsilon K_{n_1}\}.$$

It is easy to check that $A_1 \subset A_0 \subset A_2$. Thus for any probability measure ρ on G

$$\int_{A_1} \chi_E(y-x)\rho(dy)\lambda_G(dx) \leqslant \int_{A_0} \chi_E(y-x)\rho(dy)\lambda_G(dx)$$

$$\leqslant \int_{A_2} \chi_E(y-x)\rho(dy)\lambda_G(dx).$$

But

$$\int_{A_1} \chi_E(y-x)\rho(dy)\lambda_G(dx) = \int_{y \in K_{n_1}} \Big(\int_{x-y \in K_{n_1}} \chi_E(y-x)\lambda_G(dx)\Big)\rho(dy).$$

$$= \int_{y \in K_{n_1}} \Big(\int_{K_{n_1}} \chi_E(x)\lambda_G(dx)\Big)\rho(dy)$$

$$= \int_{K_{n_1}} \lambda_G(E)\rho(dy) = \lambda_G(E)\rho(K_{n_1}).$$

Similarly

$$\int_{A_2} \chi_E(y-x)\rho(dy)\lambda_G(dx) = \lambda_G(E)\rho(K_{n_3}).$$

Finally

$$\int_{A_0} \chi_E(y-x)\rho(dy)\lambda_G(dx) = \int_{x \in K_{n_2}} \Big(\int_{y \in K_{n_1}+x} \chi_{E+x}(y)\rho(dy)\Big)\lambda_G(dx)$$

$$= \int_{K_{n_2}} \rho(E+x)\lambda_G(dx).$$

Thus for any probability measure ρ on G, we have

$$\lambda_G(E)\rho(K_{n_1}) \leqslant \int_{K_{n_2}} \rho(E+x)\lambda_G(dx) \leqslant \lambda_G(E)\rho(K_{n_3}).$$

In particular take $\rho = \mu_n$, $n = 0, 1, 2, \ldots$. Then we have for $n = 0, 1, 2, \ldots$

$$\lambda_G(E)\mu_n(K_{n_1}) \leqslant \int_{K_{n_2}} \mu_n(E+x)\lambda_G(dx) \leqslant \lambda_G(E)\mu_n(K_{n_3}). \qquad \ldots (6)$$

Let $n \geqslant N$. Then

$$\lambda_G(E)\mu_n(K_{n_3}) \geqslant \int_{K_{n_2}} \mu_n(E+x)\lambda_G(dx) \qquad \text{(by (6))}$$

$$> \int_{K_{n_2}} \mu_0(E+x)\lambda_G(dx) - \varepsilon' \quad \text{(by (5))}$$

$$\geqslant \lambda_G(E)\mu_0(K_{n_1}) - \varepsilon' \qquad \text{(by (6))}$$

$$\geqslant \lambda_G(E)[1-\varepsilon'] - \varepsilon'. \qquad \text{(by (4))}$$

Since $\lambda_G(E) > 0$ we have

$$\mu_n(K_{n_3}) > 1 - \varepsilon' \left(\frac{1+\lambda_G(E)}{\lambda_G(E)}\right) = 1 - \varepsilon, \quad \text{for} \quad n \geqslant N.$$

This shows that $\{\mu_n\}$, $n = 1, 2, \ldots$ is uniformly tight and thus the set of limit points of μ_n, $n = 1, 2, \ldots$ is non-empty (see Parthasarathy, 1967). This proves (i).

To prove (ii), let ν be any limit point of $\{\mu_n\}$, $n = 1, 2, \ldots$ Let $\{\mu_{n_k}\}$ be a sub-sequence of $\{\mu_n\}$, $n = 1, 2, \ldots$ such that $\{\mu_{n_k}\}$ converges weakly to ν as $k \to \infty$. Since the Fourier transform is a continuous operation on $\mathcal{M}(G)$ (see Parthasarathy, 1967) we have $\hat{\mu}_{n_k}(\gamma) \to \hat{\nu}(\gamma)$ for every $\gamma \in \hat{G}$, the character group of G. Thus

$$\hat{X}_E(\gamma)\hat{\mu}_{n_k}(\gamma) \to \hat{X}_E(\gamma)\hat{\nu}(\gamma) \text{ for every } \gamma \in \hat{G}. \qquad \ldots (7)$$

On the other hand

$$\mu_{n_k}(E+x) \to \mu_0(E+x) \text{ for every } x \in G.$$

Thus

$$\int <x, \gamma> \left(\int X_E(x+y)\mu_{n_k}(dy)\right)\lambda_G(dx) \to \int <x, \gamma> \left(\int X_E(x+y)\mu_0(dy)\right)\lambda_G(dx),$$

for every $\gamma \in \hat{G}$. Thus

$$\hat{X}_E(\gamma). \quad \hat{\mu}_{n_k}(\gamma) \to \hat{X}_E(\gamma)\hat{\mu}_0(\gamma) \text{ for every } \gamma \in \hat{G}. \qquad \ldots (8)$$

From (7) and (8) it follows that

$$\hat{X}_E(\gamma)\hat{\mu}_0(\gamma) = \hat{X}_E(\gamma)\hat{\nu}(\gamma) \text{ for every } \gamma \in \hat{G}.$$

i.e.,

$$(\widehat{X_E * \mu_0})(\gamma) = (\widehat{X_E * \nu})(\gamma) \text{ for every } \gamma \in \hat{G}.$$

Thus

$$(X_E * \mu_0)(x) = (X_E * \nu)(x) \text{ for a.e. } x(\lambda_G).$$

Now let f be any continuous function with compact support on G. Then for every $y \in G$

$$\begin{aligned}[f*(X_E*\mu_0)](y) &= \int f(y-x)(X_E*\mu_0)(x)\lambda_G(dx)\\ &= \int f(y-x)(X_E*\nu)(x)\lambda_G(dx)\\ &= [f*(X_E*\nu)](y).\end{aligned}$$

Thus

$$[f*(X_E*\mu_0)](x) = [f*(X_E*\nu)](x) \text{ for every } x \in G$$

and for every continuous function f with compact support on G. Now proceeding as in the proof of Theorem 1 we have $\mu_0 * \lambda_K = \nu * \lambda_K$. This proves (ii).

To prove (iii) we have only to note that the sequence $\{\mu_n * \lambda_K\}$, $n = 1, 2, \ldots$ has one and only one limit point, namely $\mu_0 * \lambda_K$. This proves Theorem 2 completely.

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