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On enumeration of catastrophic fault patterns ☆

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1. Introduction

Let $A = \{p_0, p_1, \ldots, p_N\}$ denote a one-dimensional array of processing elements (PEs). There exists a direct link (regular link) between p_i and p_{i+1}, $0 \leqslant i < N$. Any link connecting p_i and p_j where $j > i + 1$ is said to be a bypass link of length $j - i$. The bypass links are used strictly for reconfiguration purposes when a fault is detected. The links can be either unidirectional or bidirectional.

Given an integer $g \in [1, N]$, A is said to have link redundancy g, if for every $p_i \in A$ with $i \leqslant N - g$, there exists a link between p_i and p_{i+g}. Let $G = \{g_1, g_2, \ldots, g_k\}$, where $g_j < g_{j+1}$ and $g_j \in [1, N]$. The array A is said to have link redundancy G if A has link redundancy $g_1, g_2, \ldots, g_k$.

A fault pattern for A is a set of integers $F = \{f_0, f_1, \ldots, f_m\}$ where $m \leqslant N$, $f_j < f_{j+1}$ and $f_j \in [0, N]$. An assignment of a fault pattern F to A means that for every $f \in F$, p_f is faulty. The width W_F of a fault pattern $F = \{f_0, f_1, \ldots, f_{g-1}\}$ is defined to be the number of PEs between and including the first and the last fault in F, that is, $W_F = f_{g-1} - f_0 + 1$. At the two ends of the array two special PEs called I (for input) and O (for output) are responsible for I/O functions of the system. It is assumed that I is connected to $p_0, p_1, \ldots, p_{g_k-1}$ while O is connected to $p_{N-g_k}, p_{N-g_k-1}, \ldots, p_{N-1}$ so that all PEs in the system have the same degree and reliability bottlenecks at the borders of the array are avoided.

A fault pattern F is catastrophic for A with link redundancy g if the array cannot be reconfigured in the presence of such an assignment of faults. In other words, F is a cut-set of the graph corresponding to A.

Characterization of catastrophic fault patterns (CFPs) and its enumeration have been studied by several authors, e.g., in [3–6]. Enumeration of CFPs for $G = \{1, g\}$ has been done in [2] for bidirectional case and in [9] for unidirectional case. A method of enumeration of CFPs in the more general context is given in [8], but no closed form solution has been obtained. In this paper, we consider only bidirectional case and use random walk as a tool for such enumeration. We provide a simple proof for the case $G = \{1, g\}$ and then enumerate for $G = \{1, 2, \ldots, k, g\}$, $2 \leqslant k < g$.

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2. Preliminaries

For $G = \{g_1, g_2, \ldots, g_k\}$ with $g_1 = 1$, CFPs with exactly g_k faults are considered because of its minimality [6]. A fault pattern $F = \{f_0, f_1, \ldots, f_{g_k-1}\}$ is represented by a Boolean matrix [4] W of size $(W_F^+ \times g_k)$ where $W_F^+ = \lceil W_F/g_k \rceil$

$$W[i,j] = \begin{cases} 1 & \text{if } (ig_k + j) \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $W[0,0] = 1$ which indicates the location of the first fault. Let $W[h_{i-1}, i-1]$ and $W[h_i, i]$ both be 1 and define $m_i = h_{i-1} - h_i$.

Proposition 1 (Pagli and Pucci [7]). *Let $\{m_1, m_2, \ldots, m_{g-1}\}$ be a sequence of moves such that*
(1) $m_i = -1, 0$ *or* 1, *for* $1 \leqslant i \leqslant g-1$,
(2) $S_k = \sum_{i=1}^{k} m_i \leqslant 0$ *for any* $1 \leqslant k \leqslant g-2$,
(3) $S_{g-1} = \sum_{i=1}^{g-1} m_i = 0$.

Then, any such sequence corresponds to a minimal CFP and vice versa when $G = \{1, g\}$.

Definition 1 (Feller [1]). A random walk is a sequence $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\}$ where each $\varepsilon_i = +1$ *or* -1.

The sequence is normally represented by a polynomial line on a X-Y plane and whose kth side has slope ε_k and whose kth vertex has ordinate $S_k = \sum_{i=1}^{k} \varepsilon_i$; such lines are called paths. For example, the row $\{1, -1, -1, 1, -1, -1\}$ is represented by a path from $(0,0)$ to $(6,-2)$, with intermediate points $(1,1)$, $(2,0)$, $(3,-1)$, $(4,0)$, $(5,-1)$ in the given order.

Definition 2. A subsequence $\{\varepsilon_{s+1}, \varepsilon_{s+2}, \ldots, \varepsilon_{s+r}\}$ of $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, $r \geqslant 1$, is called a run of length r if $\varepsilon_s \neq \varepsilon_{s+1} = \varepsilon_{s+2} = \cdots = \varepsilon_{s+r} \neq \varepsilon_{s+r+1}$.

R is referred to as the number of runs in $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, ρ_1 and ρ_{-1} as the number of runs whose elements are 1 and -1, respectively ($R = \rho_1 + \rho_{-1}$).

Notations.

$E_{n,m}$: A path from $(0,0)$ to (n,m).

$E_{n,m}^{R}$: An $E_{n,m}$ path with R runs.

$E_{n,m}^{R+}$: An $E_{n,m}^{R}$ path starting with a positive step.

$E_{n,m}^{R-}$: An $E_{n,m}^{R}$ path starting with a negative step.

$E_{n,m}^{R+,t}$: An $E_{n,m}^{R+}$ path crossing the line $y = t$, $t > 0$ at least once.

$E_{n,m}^{R-,t}$: An $E_{n,m}^{R-}$ path crossing the line $y = t$, $t > 0$ at least once.

$N(A)$: The number of all A paths, e.g., $N(E_{n,m}) = \binom{n}{(n-m)/2}$.

Theorem 1 (Feller [1]). *Among the $\binom{2n}{n}$ paths joining the origin to the point $(2n, 0)$ there are exactly $\frac{1}{n+1}\binom{2n}{n}$ paths such that $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0$, $S_{2n} = 0$.*

Theorem 2 (Vellore [10]). *For $m \leqslant t < (n+m)/2$,*

$$N\left(E_{n,m}^{(2r-1)+,t}\right) = \binom{\frac{n-m}{2} + t - 1}{r-2}\binom{\frac{n+m}{2} - t - 1}{r-1},$$

$$N\left(E_{n,m}^{2r-,t}\right) = \binom{\frac{n-m}{2} + t - 1}{r-2}\binom{\frac{n+m}{2} - t - 1}{r}.$$

3. Main results

Theorem 3 (Nayak [2]). *For $G = \{1, g\}$, the number of CFPs for bidirectional links is given by*

$$\sum_{n=0}^{\lfloor (g-1)/2 \rfloor} \frac{1}{n+1}\binom{2n}{n}\binom{g-1}{2n}.$$

Proof. Number of catastrophic fault patterns is equal to the number of catastrophic sequences $\{m_1, m_2, \ldots, m_{g-1}\}$ satisfying conditions of Proposition 1. We take random walks from $(0,0)$ to $(2n, 0)$ such that $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0, S_{2n} = 0$ and "plug" $(g-1-2n)$ zeroes in the $2n+1$ "distinguishable places" (intermediate $2n-1$ places and two more places before and after the sequence) of each such path. Clearly for a given path there are $\binom{g-1}{2n}$ (negative binomial coefficient) ways of plugging zeroes. □

Proposition 2. *Necessary and sufficient conditions to have that $\{m_1, m_2, \ldots, m_{g-1}\}$ is the catastrophic sequence of a minimal CFP for a bidirectional linear array with link $G = \{1, 2, g\}$ are:*
(1) $m_{g-1} = 0$,
(2) $m_j = -1, 0, +1$ *for* $j = 1, 2, \ldots, g-2$,

(3) $\sum_{j=1}^{k} m_j \leqslant 0$ *for* $k = 1, 2, \ldots, g-3$,
(4) $\sum_{j=1}^{g-2} m_j = 0$,
(5) $m_i + m_{i+1} = -1, 0, +1$ *for* $i = 1, 2, \ldots, g-3$.
That is, two or more consecutive $+1$*'s or* -1*'s are not allowed.*

In general, we have the following characterization.

Proposition 3. *Necessary and sufficient conditions to have that* $\{m_1, m_2, \ldots, m_{g-1}\}$ *is the catastrophic sequence of a minimal CFP for a bidirectional linear array with link* $G = \{1, 2, 3, \ldots, k, g\}$ *are*:
(1) $m_{g-1} = m_{g-2} = \cdots = m_{g-k+1} = 0$,
(2) $m_j = -1, 0, +1$ *for* $j = 1, 2, \ldots, g-k$,
(3) $\sum_{j=1}^{k} m_j \leqslant 0$ *for* $k = 1, 2, \ldots, g-k-1$,
(4) $\sum_{j=1}^{g-k} m_j = 0$,
(5) $m_i + m_{i+1} + \cdots + m_{i+s} = -1, 0, +1$ *for* $s = 1, 2, \ldots, k-1$, *for* $i = 1, 2, \ldots, g-k-s$.

The characterizations described in Propositions 2 and 3 are easy to visualize and hence their proofs are omitted.

Lemma 1. *The number of paths from origin to the point* $(2n, 0)$ *such that* $S_1 \leqslant 0, S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0, S_{2n} = 0$ *and have* $2r$ *runs is*

$$\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}.$$

Proof. Clearly there exist exactly as many admissible paths as there are paths from $O_1 = (1, -1)$ to $N_1 = (2n, 0)$ which do not cross the X-axis and have $2r$ runs.

The number of such paths is equal to

$$N\left(E_{2n,0}^{2r-}\right) - N\left(E_{2n,0}^{*2r-,0}\right), \tag{1}$$

where $E_{2n,0}^{*2r-,0}$ is an $E_{2n,0}^{2r-}$ path crossing the line $y = 0$ at least once (please note that $E_{2n,0}^{2r-,t}$ do not assume $t = 0$). It is known that

$$N\left(E_{2n,0}^{2r-}\right) = \binom{n-1}{r-1}^2 \tag{2}$$

(see Wald and Wolfowitz [11]). Now our aim is to enumerate $N(E_{2n,0}^{*2r-,0})$. Translating the origin to O_1, we now consider the paths from the new origin to the point N_1 (which has the new co-ordinates $2n-1$ and 1) which cross the line $y = 1$ (with respect to new X-axis) at least once and have $2r$ runs if the path starts with a negative step and have $(2r-1)$ runs if the path starts with a positive step. Number of such paths equal

$$N\left(E_{2n-1,1}^{2r-,1}\right) + N\left(E_{2n-1,1}^{(2r-1)+,1}\right).$$

It can be shown that there exists a $1:1$ correspondence between such paths and an $E_{2n,0}^{*2r-,0}$ path.

Take an $E_{2n-1,1}^{2r-,1}$ (or an $E_{2n-1,1}^{(2r-1)+,1}$) path and add a negative step before it. The resulting path is an $E_{2n,0}^{*2r-,0}$. Hence

$$\begin{aligned} &N\left(E_{2n,0}^{*2r-,0}\right) \\ &\quad = N\left(E_{2n-1,1}^{2r-,1}\right) + N\left(E_{2n-1,1}^{(2r-1)+,1}\right) \\ &\quad = \binom{n-1}{r-2}\binom{n-2}{r} + \binom{n-1}{r-2}\binom{n-2}{r-1} \\ &\quad = \binom{n-1}{r-2}\binom{n-1}{r}. \end{aligned} \tag{3}$$

The lemma follows from (1), (2) and (3). □

Theorem 4. *Let* $G = \{1, 2, g\}$. *Then the number of catastrophic fault pattern* $\gamma(1, 2, g)$ *for bidirectional link is given by*

$$\begin{aligned} &\gamma(1, 2, g) \\ &\quad = 1 + \sum_{n=1}^{\lfloor (g-2)/2 \rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}\right] \\ &\qquad \times \binom{g-2(n-r)-2}{2n}. \end{aligned}$$

Proof. Number of catastrophic fault patterns is equal to the number of catastrophic sequences $\{m_1, m_2, \ldots, m_{g-2}\}$ satisfying conditions of Proposition 2. Let the number of -1's (and so the number of $+1$'s) in the sequence be n. Clearly then the number of zeroes is $g - 2 - 2n$. We start with a path of length $2n$ such that $S_1 \leqslant 0,\ S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0,\ (S_{2n} = 0)$ and have $2r$ runs. R(run) $= 1 +$ number of change either of the type $(-1, +1)$ or $(+1, -1)$.

So, the number of paths having $(2r-1)$ changes either of the type $(-1, +1)$ or $(+1, -1)$ and satisfies $S_1 \leqslant 0,\ S_2 \leqslant 0, \ldots, S_{2n-1} \leqslant 0,\ (S_{2n} = 0)$ is

$$\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r}.$$

All the above paths have $2n-1-2r+1=2(n-r)$ identical pairs of the type $(+1,+1)$ or $(-1,-1)$. So, to satisfy condition (5) of Proposition 2, we have to plug in a zero between every two consecutive $+1$'s and every two consecutive -1's. So the number of zeroes plugged in are $2(n-r)$. The remaining positions $g-2-2n-2(n-r)=g-4n+2r-2$ are also to be filled up with 0's. There are $(2n+1)$ distinguishable positions in which $(g-4n+2r-2)$ 0's can be distributed in $\binom{g-2(n-r)-2}{2n}$ ways. Since n can vary from 1 to $\lfloor (g-2)/2 \rfloor$, the total number of such paths is

$$\sum_{n=1}^{\lfloor (g-2)/2\rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r} \right] \times \binom{g-2(n-r)-2}{2n}.$$

Note that these paths do not include the trivial path corresponding to the sequence $(0,0,\ldots,0)$. Hence the theorem. □

Theorem 5. *Let $G=\{1,2,3,\ldots,k,g\}$. Then, the number of catastrophic fault patterns $\gamma(1,2,3,\ldots,k,g)$ for bidirectional link is given by*

$$\gamma(1,2,3\ldots,k,g) = 1 + \sum_{n=1}^{\lfloor (g-k)/2\rfloor} \sum_{r=1}^{n} \left[\binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r} \right] \times \binom{g-k-2(n-r)(k-1)}{2n}.$$

Proof. The number of catastrophic fault patterns is equal to the number of catastrophic sequences $\{m_1, m_2, \ldots, m_{g-k}\}$ satisfying conditions (2)–(5) of Proposition 3. Proof is similar to the proof of Theorem 4. Here to satisfy condition (5) of Proposition 3, we have to plug in $(k-1)$ 0's between every two consecutive $+1$'s and between every two consecutive -1's. □

4. Conclusion

A method of enumeration of CFPs for an arbitrary link configuration G was discussed in [8], but no closed form solution was obtained. In this paper, we used the random walk as a tool for such enumeration. We provided a simple proof for the case $G=\{1,g\}$ and a closed form expression for $G=\{1,2,\ldots,k,g\}$, $2 \leqslant k < g$ in the case of bidirectional links.

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