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Dedicated to the memory of R. C. Bose

CONSTRUCTION OF STRONGLY BALANCED UNIFORM REPEATED MEASUREMENTS DESIGNS: A NEW APPROACH

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SUMMARY. Strongly balanced uniform repeated measurements designs when the number of treatments is 0, 1 or 3 modulo 4 have been constructed. Also when the number of treatments is 2 modulo 4 and number of units, an even multiple of the number of treatments, the design is constructed. The methods used are method of linked differences and method of linked sum-differences.

1. Introduction

In repeated mesurements designs (RMD's) each experimental unit is exposed to a number of treatments applied sequentially over periods. If there are p periods $0, 1, \ldots, p-1$; t treatments and n experimental units, then an RMD(t, n, p) is an $n \times p$ array, say $D = ((d_{ij}))$ where d_{ij} denotes the treatment assigned to i-th unit in the j-th period, $i = 1, 2, \ldots, n$; $j = 0, 1, \ldots, p-1$. RMD's have been studied quite extensively. For a general review of such designs, one may refer to Hedayat and Afsarinejad (1975) and for an excellent review of the literature on optimal RMD's reference is made to Hedayat (1981).

An RMD is called uniform if in each period the same number of units is assigned to each treatment and on each unit each treatment appears in the same number of periods.

The underlying statistical model is called circular if in each unit the residuals in the initial period are incurred from the last period. Under the circular model an RMD is called strongly balanced if the collection of ordered pairs $(d_{i,j}, d_{i,j+1})$, $1 \leq i \leq n$, $0 \leq j \leq p-1$ (operation on the second suffix is modulo p), contains each ordered pair of treatments, distinct or not, the same number of times. Strongly balanced uniform RMD(t, n, p) will be abbreviated to SBURMD(t, n, p).

Throughout this paper, we assume the underlying model to be circular. The universal optimality of SBURMD(t, n, p) over the class of all RMD(t, n, p)'s was established by Magda (1980) under an additive set up and later on by Sen and Mukherjee (1987) under a non-additive set up.

Sen and Mukherjee (1987) also give a method for constructing SBURMD(t, n, p) if $t \mid n$ and pt^{-1} is an even integer. Roy (1988) gave a method of construction for $t \equiv 0, 1$ or $3 \pmod 4$. In this paper we give constructions for $t \equiv 0, 1$ or 3 (mod 4), but this method is considerably simpler than that of Roy (1988). Especially the case $t \equiv 0 \pmod 4$ is dealt by a method which is completely new. Also we give a method of construction of SBURMD(t,n,p) when $t \equiv 0$ or $2 \pmod 4$ and nt^{-1} is even.

2. Method of differences

Let G be a group with operation $+$, B be a subset of G and $g \in G$. Then $B+g$ is defined as follows:

$$B+g=\{b+g : b \in B\}$$

The proof of the following theorem, being trivial, is omitted.

Theorem 2.1: *Let G be a group with t elements. Consider the p-tuple B : $(a_0, a_1, \dots, a_{p-1})$ where $a_i \in G\ \forall i = 0,1,\dots,p-1$; each element of G occurs exactly s $(s = p/t)$ times in B and $\{a_i - a_{i+1} : i = 0,1,\dots,p-1\}$ (operation on the suffixes is modulo p) contains each element of G precisely s times. Then $\{B+g : g \in G\}$ arranged in t rows, forms* SBURMD(t,t,p).

B will be referred to as a difference vector. It is easy to see that the following are true.

Note 1: SBURMD(t,n,p) may be constructed by repeating $n \cdot t^{-1}$ SBURMD(t,t,p)'s vertically.

Note 2: If SBURMD$(t,t,2t)$ exists then SBURMD(t,t,mt), where m is even, can be constructed by repeating SBURMD$(t,t,2t)$'s horizontally.

Note 3: If SBURMD$(t,t,2t)$ and SBURMD$(t,t,3t)$ both have two columns of the form:

$$\begin{matrix} 0 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ t-1 & t-1 \end{matrix}$$

w.l.g. we may assume that the first and last columns of the SBURMD's are

$$\begin{matrix} 0 \\ 1 \\ \vdots \\ t-1 \end{matrix}$$

Then SBURMD(t,t,mt), where m is odd and $m \geq 3$, can be constructed by taking SBURMD$(t,t,3t)$ followed by SBURMD$(t,t,2t)$'s.

If SBURMD's are constructed using difference vectors, then the condition of Note 3 holds trivially.

One now attempts to get difference vectors for SBURMD$(t,t,3t)$.

3. The Case $t = 2k + 1$

Consider the difference vector

$$D : (0, 2k, 1, 2k-1, \ldots, k-1, k+1, k, k, k+1, k-1, \ldots, 2k-1, 1, 2k, 0)$$

as constructed by Sen and Mukherjee (1987).

We replace *one* occurrence of i by the triplet $(i, 2k-i, i)$ for $i = 0, 1, \ldots, k-1$ and replace *one* occurrence of k by the ordered pair (k, k).

Consider this modification on D and call the resulting ordered $3t$-tuple D^*.

Lemma 3.1 : *$\{D^* + i : i = 0, 1, \ldots, 2k\}$ is an* SBURMD$(t, t, 3t)$.

Proof: Sen and Mukherjee (1987) have shown that D is a $2k$-tuple, each element of $\mathbb{Z}_{2k+1}$ occurs twice and the collection of linked differences in D contains each element of $\mathbb{Z}_{2k+1}$ precisely twice. Now introduction of $(i, 2k-i, i)$ instead of an i contributes the elements $2k-i, i$ to D^* and $2k-2i$ and $2i+1$ in the collection of linked differences in D^*, $i = 0, 1, \ldots, k-1$.

It is easy to observe that $\bigcup_{i=0}^{k-1}\{2k-i, i\} \cup \{k\} = \mathbb{Z}_{2k+1}$ and also $\bigcup_{i=0}^{k-1}\{2k-2i, 2i+1\} \cup \{0\} = \mathbb{Z}_{2k+1}$.

Hence D^* contains each element of $\mathbb{Z}_{2k+1}$ thrice and the collelction of linked differences in D^* also contains each element of $\mathbb{Z}_{2k+1}$ thrice. Thus D^* is a difference vector for SBURMD$(t, t, 3t)$. □

Illustration: $k = 3, t = 7$ and

$$D : (0\ 6\ 1\ 5\ 2\ 4\ 3\ 3\ 4\ 2\ 5\ 1\ 6\ 0)$$

One 0 is replaced by (0 6 0), one 1 is replaced by (1 5 1), one 2 is replaced by (2 4 2), one 3 is replaced by (3 3). Thus

$$D^* : (0\ 6\ 0\ 6\ 1\ 5\ 1\ 5\ 2\ 4\ 2\ 4\ 3\ 3\ 3\ 4\ 2\ 5\ 1\ 6\ 0).$$

In view of the previous lemma and notes the following theorem is immediate.

Theorem 3.2 : *Let $t \mid n$, $t \mid p$ and $p > t$. If t is an odd integer, then* SBURMD(t, n, p) *always exists.*

4. The Case $t = 2k$ and n an Even Multiple of t.

We define the following notations :

If A is an n-tuple $(a_1, a_2, \ldots, a_n)$ then A' is the tuple $(a_n, \ldots, a_2, a_1)$. If $A : (a_1, \ldots, a_n)$ and $B : (b_1, \ldots, b_m)$ then by AB we mean the $(n+m)$-tuple $(a_1, \ldots, a_n, b_1, \ldots, b_m)$. Consider $\mathbb{Z}_{2k}$.

Let $D_1 = (a_1, a_2, \ldots, a_k)$ and $D_2 = (a_{k+1}, a_{k+2}, \ldots, a_{2k})$ where

$$a_i = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd} \\ 2k - \frac{i}{2} & \text{otherwise} \end{cases}$$

Let $D = D_1 D_2$. Then $D' = D_2' D_1'$. Let $B = DD'D'$ and $C = D_1 D_1' D_2' D_2 D'$.

Lemma 4.1: *Let* $\mathcal{D}_1 = \{B + i : i \in \mathbb{Z}_{2k}\}$, $\mathcal{D}_2 = \{C + i : i \in \mathbb{Z}_{2k}\}$ *and* $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. *Then* $\mathcal{D}$ *is an* SBURMD$(t, 2t, 3t)$.

Proof: (i) The fact that both B and C have all the elements of $\mathbb{Z}_{2k}$ thrice follows from the ovservation that the set of elements in $\mathcal{D}$ of $\mathbb{Z}_{2k}$.

(ii) One can easily check that the collection of linked differences in B contains the element k four times, 0 twice and all other elements in $\mathbb{Z}_{2k}$ thrice.

Also the collection of linked differences in C contains the element k twice, element 0 four times and all other elements in $\mathbb{Z}_{2k}$ thrice.

Hence, a pair (a, a), $a \in \mathbb{Z}_{2k}$, appears twice in $\mathcal{D}_1$ and 4 times in $\mathcal{D}_2$, thus appearing 6 times in $\mathcal{D}$.

A pair (a, b), where $a - b = k$, appears 4 times in $\mathcal{D}_1$ and twice in $\mathcal{D}_2$, thus appearing 6 times in $\mathcal{D}$.

Any other ordered pair appears 3 times in $\mathcal{D}_1$ as well as in $\mathcal{D}_2$, thus appearing 6 times in $\mathcal{D}$.

Hence $\mathcal{D}$ is an SBURMD$(t, 2t, 3t)$ where $t = 2k$. □

Illustration: $k = 5$ i.e. $t = 10$,

$$D_1 = (0\ 9\ 1\ 8\ 2),$$
$$D_2 = (7\ 3\ 6\ 4\ 5).$$

Then

$$B = (0\ 9\ 1\ 8\ 2\ 7\ 3\ 6\ 4\ 5\ 5\ 4\ 6\ 3\ 7\ 2\ 8\ 1\ 9\ 0\ 5\ 4\ 6\ 3\ 7\ 2\ 8\ 1\ 9\ 0)$$
$$C = (0\ 9\ 1\ 8\ 2\ 2\ 8\ 1\ 9\ 0\ 5\ 4\ 6\ 3\ 7\ 7\ 3\ 6\ 4\ 5\ 5\ 4\ 6\ 3\ 7\ 2\ 8\ 1\ 9\ 0)$$

Observe that SBURMD$(t, 2t, 3t)$ constructed in the above method satisfies the condition of Note 3. Hence we have the following theorem.

Theorem 4.2: *Let* $t \mid n$, $t \mid p$, $p > t$, nt^{-1} *be even and* t *be an even integer. Then* SBURMD(t, n, p) *exists.*

5. The Case $t = 4k$

Consider the tuple

$$D : (1, 2, 3, \ldots, 4k-2, 4k-1, 0, 1, 2, \ldots, 2k, 0, 2k, 0)$$

Define tuples A_i's as follows:

$$A_i : (i, 2k-i, i), \quad i = 1, 2, \ldots, k-1$$
$$A_k : (k, k)$$
$$A_i : (i, 2k-i, i, 2k-i, i), \quad i = 2k+1, 2k+2, \ldots, 3k-1$$
$$A_{3k} : (3k, 3k, 3k)$$

Form a tuple D^* from D by replacing *one* occurrence of i by A_i, $i = 1, 2, \ldots, k$ and $i = 2k+1, \ldots, 3k$.

Let us define an operation $\bigoplus$ as follows:

Let $A = (a_1, a_2, \ldots, a_n)$, $a_i \in \mathbb{Z}_N$ and let $k \in \mathbb{Z}_N$. Then $A \bigoplus k$ is defined to be the tuple $C = (C_1, C_2, \ldots, C_n)$, where

$$C_i = \begin{cases} a_i + k & \text{if } a_i \text{ is even} \\ a_i - k & \text{otherwise} \end{cases}$$

Lemma 5.1 : $\{D^* \bigoplus i : i \in \mathbb{Z}_{4k}\}$ *is an* SBURMD$(t, t, 3t)$ *where* $t = 4k$.

Proof: (i) In D the elements 0 and $2k$ have occurred thrice, the elements $1, 2, \ldots, 2k-1$ have occurred twice and the elements $2k+1, 2k+2, \ldots, 4k-1$ have occurred once. Observe that the collection of elements in $\{\{2k-i, i\}, i = 1, 2, \ldots, k-1\}$ is $\{1, 2, \ldots, k-1, k+1, \ldots, 2k-1\}$, the collection of elements in $\{\{2k-i, i, 2k-i, i\} : i = 2k+1, \ldots, 3k-1\}$ is $2 \cdot \{2k+1, \ldots, 3k-1, 3k-1, \ldots, 4k-1\}$ where if X is a collection of elements then $2 \cdot X$ denotes the collection of elements in X, each occurring twice. Also we replace k by (k, k) and $3k$ by $(3k, 3k, 3k)$. Thus in D^* each element of $\mathbb{Z}_{4k}$ occurs thrice.

(ii) The collection of linked sums in D contains all the odd numbers in $\mathbb{Z}_{4k}$ and the number $2k$ precisely thrice.

(iii) The collection of linked differences in A_i, $i = 1, 2, \ldots, k$ and $i = 2k+1, \ldots, 3k$, contains all the even numbers in $\mathbb{Z}_{4k}$ except $2k$, precisely thrice.

(iv) The sum and difference of 0 and $2k$ are both $2k$.

(v) i and $2k - i$ are either both odd or both even.

(vi) Let us see that a pair (a, b), $a, b \in \mathbb{Z}_{4k}$ occurs in $\{D^* \bigoplus i : i \in \mathbb{Z}_{4k}\}$ precisely thrice.

Case 1 : $b = a$.
(a, a) occurs precisely once in $D^* \bigoplus (a-k)$ and twice in $D^* \bigoplus (a-3k)$ if k is even and occurs precisely once in $D^* \bigoplus (k-a)$ and twice in $D^* \bigoplus (3k-a)$ if k is odd.

Case 2 : $a - b = 2k$.
(a, b) occurs precisely twice in $D^* \bigoplus (a-k)$ and once in $D^* \bigoplus a$.

Case 3 : $a + b$ is odd or equivalently $a - b$ is odd.
In D, the collection of linked sums contains $a + b$ precisely thrice, say the corre-

sponding occurrences in D^* are: (u,v), (w,x) and (y,z). Then

$$(a,b) \text{ occurs once in } \begin{cases} D^* \oplus (a-u) & \text{if } u \text{ is even} \\ D^* \oplus (u-a) & \text{if } u \text{ is odd} \end{cases}$$

Similarly (a,b) occurs once corresponding to (w,x) and (y,z).

Case 4 : $a-b$ is even or equivalently $a+b$ is even and $a-b \neq 2k$. Due to (iii), we can find pairs (u,v), (w,x) and (y,z) in D^* such that $u-v = w-x = y-z = a-b$. Then corresponding to the pair (u,v),

$$(a,b) \text{ occurs once in } \begin{cases} D^* \oplus (a-u) & \text{if } u \text{ is even} \\ D^* \oplus (u-a) & \text{if } u \text{ is odd} \end{cases}$$

Similarly (a,b) occurs once corresponding to (w,x) and (y,z). □

Illustration: $k = 2, t = 8$.

$$\begin{array}{ll} D: & (1\,2\,3\,4\,5\,6\,7\,0\,1\,2\,3\,4\,0\,4\,0) \\ A_1: & (1\,3\,1) \\ A_2: & (2\,2) \\ A_5: & (5\,7\,5\,7\,5) \\ A_6: & (6\,6\,6) \end{array}$$

The method described gives the following SBURMD(8,8,24).

$$\begin{array}{cccccccccccccccccccccccc}
1&3&1&2&2&3&4&5&7&5&7&5&6&6&6&7&0&1&2&3&4&0&4&0 \\
0&2&0&3&3&2&5&4&6&4&6&4&7&7&7&6&1&0&3&2&5&1&5&1 \\
7&1&7&4&4&1&6&3&5&3&5&3&0&0&0&5&2&7&4&1&6&2&6&2 \\
6&0&6&5&5&0&7&2&4&2&4&2&1&1&1&4&3&6&5&0&7&3&7&3 \\
5&7&5&6&6&7&0&1&3&1&3&1&2&2&2&3&4&5&6&7&0&4&0&4 \\
4&6&4&7&7&6&1&0&2&0&2&0&3&3&3&2&5&4&7&6&1&5&1&5 \\
3&5&3&0&0&5&2&7&1&7&1&7&4&4&4&1&6&3&0&5&2&6&2&5 \\
2&4&2&1&1&4&3&6&0&6&0&6&5&5&5&0&7&2&1&4&3&7&3&7
\end{array}$$

One can see that the condition of Note 3 is satisfied by the design constructed by the method described in this section. Hence, we have the following theorem.

Theorem 5.2: *Let* $t\,|\,n$, $t\,|\,p$, $t > p$ *and* $t \equiv 0 \pmod 4$. *Then an* SBURMD(t,n,p) *always exists.*

6. Concluding Remarks

When $t \equiv 2 \pmod 4$ and $n \cdot t^{-1}$ is odd, the existence problem of SBURMD(t,n,p) is still unresolved. It has been shown by Roy (1988) that SBURMD(2,2,6) does not exist. The method of differences for constructing SBURMD$(t,t,3t)$ works only if the group with t elements has the property that the sum of all the elements in that group is the identity element. We could not get a group with $(4k+2)$ elements having the above mentioned property. The method of linked sum-difference as used in Section 5 uses the fact that $t \equiv 0 \pmod 4$ and could not be extended to the case $t \equiv 2 \pmod 4$.

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