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ON THE OPTIMALITY OF A CLASS OF MINIMAL COVERING DESIGNS

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Abstract: It is shown that the minimal covering designs for $v=6t+5$ treatments in blocks of size 3 are optimal w.r.t. a large class of optimality criteria. This class of optimality criteria includes the well-known criteria of A-, D- and E-optimality. It is conjectured that these designs are also optimal w.r.t. other criteria suggested by Takeuchi (1961).

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1. Introduction

Kiefer (1958) proved that a balanced incomplete block design (BIBD) is optimal w.r.t. each of the commonly used optimality criterion. Subsequently he established the property of universal optimality of the balanced designs (Kiefer (1975)). When a BIBD does not exist, the problem of finding an optimal design is somewhat difficult. Cheng (1978) commented on these difficulties and showed that certain partially balanced incomplete block designs (PBIBD's) which are group divisible and for which the off diagonal elements of the association matrix differ by at most one are optimal.

In this paper we consider incomplete block designs with blocks of size 3. If v, the number of treatments is 1 or 3 (mod 6), a Steiner Triple System (STS) can be obtained (Hanani (1961)). These have the property of each pair of treatments occurring precisely once in each block. Clearly, these designs are optimal.

Designs with the property that every pair of treatments occurs at least once in a block are called covering designs. A covering design with the smallest number of blocks is called a minimal covering design. In this paper we use the results of Cheng (1978) to show that when $v\equiv 5 \bmod 6$ a minimal covering design in blocks of size 3 is optimal w.r.t. a wide range of optimality criteria. These minimal covering designs have unequal number of replications. We note that some optimal designs givenby Cheng (1979) appear to be the first known instances of optimal designs with constant block size and unequal number of replications.

All optimality results in this paper refer to optimality within the class of designs $D_{b,v,k}$ consisting of the block designs in which v treatments are compared using b blocks each of size k.

2. Construction and analysis

Fort and Hedlund (1958) discussed the problem of the minimal covering of pairs by triplets. They first noted that for v treatments the number of blocks required must be at least $\phi(v)=\lceil \frac{1}{3}v\lceil \frac{1}{2}(v-1)\rceil\rceil$ where $\lceil x\rceil$ denotes the smallest integer greater than or equal to x. They then showed that for all v, designs with $\phi(v)$ triplets can in fact be constructed. When $v\equiv 5$ or 17 mod 18, a simpler construction may be obtained as follows.

Let $v\equiv 5$ mod 18 and let

$$A=\{a_1,a_2,\dots,a_{6t+1}\},\quad B=\{b_1,b_2,\dots,b_{6t+1}\}$$

and

$$C=\{c_1,c_2,\dots,c_{6t+1}\}.$$

Further, let $A'=A\cup\{\alpha,\beta\}$, $B'=B\cup\{\alpha,\beta\}$ and $C'=C\cup\{\alpha,\beta\}$. Let $\mathscr{B}_{A'}$, $\mathscr{B}_{B'}$, $\mathscr{B}_{C'}$ denote the blocks of the Steiner triple systems on A', B' C' respectively. Now consider a Latin Square of order $6t+1$ in which the rows are indexed by the a_i's the columns are indexed by the b_i's and the letters used are the c_i's.

Let $\mathscr{B}=\{(a_i,b_j,L(a_i,b_j));\ i,j=1,2,\dots,6t+1\}$ where $L(a_i,b_j)$ denotes the letters in the row a_i and the column b_j of the Latin square. It is easy to verify that $\mathscr{B}\cup\mathscr{B}_{A'}\cup\mathscr{B}_{B'}\cup\mathscr{B}_{C'}$ is the required covering design.

In the above design we used STS$(6t+3)$. If we use STS$(6t+1)$ we can get a covering design for $18t-1$ treatments. This construction was suggested by R. Mullin (University of Waterloo).

Let $N=((n_{ij}))$ denote the $v\times b$ incidence matrix for a design. In all cases when $v=6t+5$ the matrix NN' for the minimal covering design can be expressed as

$$NN'=\left[\begin{array}{cc|c} 3t+3 & 3 & J_{2,6t+3} \\ 3 & 3t+3 & \\ \hline J_{6t+3,2} & & (3t+1)I_{6t+3}+J_{6t+3,6t+3} \end{array}\right] \tag{2.1}$$

where J_{pq} denotes the $p\times q$ matrix of all 1's.

In the analysis of an incomplete block design, the so called C-matrix plays an important role. The C-matrix is the matrix of co-efficients in the equations for estimating the treatment effects when the block effects are eliminated. This matrix is related to the incidence matrix by the relation $C=\operatorname{diag}(r_1,r_2,\dots,r_v)-NN'/k$, where r_i, $i=1,2,\dots,v$, are the replication numbers for the treatments. Kempthorne (1956) showed that the average variance for the estimated paired comparisons is $2\sigma^2/\mu_{dH}$ where μ_{dH} is the harmonic mean of the $v-1$ non-zero eigenvalues of C_d, the C-matrix for design d.

To formulate the notion of the efficiency factor for a design where bk is not divisible by v, we note that the average variance for a completely randomized design (CRD) is $2\sigma^2/\bar{r}_H$ where $\bar{r}_H$ is the harmonic mean of the r_i's. This is minimum when each r_i is $[bk/v]$ or $[bk/v]+1$, where $[x]$ denotes the greatest integer not exceeding x. The efficiency factor of a design d may be defined as

$$E_d = \frac{\bar{\mu}_{dH}}{\bar{r}_H}. \tag{2.2}$$

Thus, a design which maximizes $\bar{\mu}_{dH}$ is optimum. In the next section we shall show that E_d is maximized by the minimal covering design. We shall denote this design by d^*.

We note that the efficiency factor E_d is different from E_1 given in Shah (1960) which used the arithmetic mean of the r_i's.

For the design d^*, we may use (2.1) to show that a generalized inverse for C_{d^*} is given by

$$C_{d^*}^{-} = \frac{1}{B}\left[\begin{array}{cc|cccc} 3(6t+7) & 6 & 0 & 0 & \cdots & 0 \\ 6 & 3(6t+7) & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & & \\ \vdots & \vdots & & 3(6t+9)I_{v-2} & & \\ 0 & 0 & & & & \end{array}\right] \tag{2.3}$$

where $B=(6t+9)(6t+5)$.

From (2.3) it can be shown that the average variance for the estimated paired treatment comparisons is $2\sigma^2/m$ where

$$m = \frac{\gamma(2\gamma+1)(2\gamma+5)}{3(2\gamma^2+5\gamma-2)} \tag{2.4}$$

with $\gamma=3t+2$. Thus, m equals $\bar{\mu}_{d^*H}$, the harmonic mean of the non-zero eigenvalues of d^*. The efficiency factor for d^* can now be seen to be

$$E_{d^*} = \frac{(2\gamma+5)(2\gamma^2+3\gamma-1)}{3(\gamma+1)(2\gamma^2+5\gamma-2)}. \tag{2.5}$$

3. Optimality of minimal covering designs

We shall first establish a weak optimality result for the minimal covering designs. This result will be used in establishing stronger optimality results. Eccleston and Hadayat (1974) introduced the concept of (M, S)-optimality which is an extension of an optimality criterion introduced in Shah (1960). A design is said to be (M, S)-optimal if (i) tr C_d is maximum and (ii) C_d^2 is minimum among all designs which maximize tr C_d.

Since d^* is binary, tr C_d is maximized. To see that tr C_d^2 is minimized within the class of binary designs we note that

$$kC_{d^*}=\begin{bmatrix} 2(y+1) & -3 & -1 & & -1 \\ -3 & 2(y+1) & -1 & \cdots & -1 \\ -1 & -1 & 2y & \cdots & -1 \\ -1 & -1 & & \cdots\ -1 & 2y \end{bmatrix}. \tag{3.1}$$

For any binary design $\mathrm{tr}(kC)^2=D+E$ where $D=(k-1)^2\sum_{i=1}^{v}r_i^2$ and $E=\sum_{i\neq j}\lambda_{ij}^2$. We also note that $\sum_i r_i=bk$ and $\sum_{i\neq j}\lambda_{ij}=bk(k-1)$. Since for the covering design the r_i's differ by at most one, D is minimized. If one can find a design for which only 2 pairs λ_{ij} have value 2 and the rest have value one, E would be minimized. To see that such a design is impossible, without any loss of generality we assume that the pair (1,2) occurs in the two blocks say (1,2,3) and (1,2,4). We also assume that treatment 1 is not involved in the other repeated pair. Thus, treatment 1 must occur precisely once with each of the remaining $6t+1$ treatments. Since each block with treatment 1 gives two such pairs, this is impossible. It is easy to verify that any other set of values of λ_{ij} does not give a lower value for E. Thus the covering design minimizes each of D and E. Hence the design is seen to be (M,S)-optimal.

To establish stronger optimality results for d^*, the covering design, we introduce the following notation as in Cheng (1978). Let f be a real valued function defined $[0,b(k-1)]$ such that (i) f is continuous, strictly convex and strictly decreasing on $[0,b(k-1)]$. (ii) f is continuously differentiable on $(0,b(k-1))$ and f' is strictly concave on $(0,b(k-1))$. Let $\psi_f(C_d)=\sum_{i=1}^{v-1}f(\mu_{di})$ where $\mu_{d1},\dots,\mu_{d(v-1)}$ are the non-zero eigenvalues of C_d. This ψ_f gives a class of optimality criteria (called optimality criteria of type 1) which includes the A-optimality ($f(x)=1/x$) and D-optimality ($f(x)=\ln(1/x)$) as special cases. A generalized optimality criterion of type 1 is given by the pointwise limit of a sequence of such criteria. One such criterion is the well-known E-optimality criterion. Cheng (1978) showed that a design d^* satisfying the following conditions is optimal w.r.t. all generalized criteria of type 1 over all $d\in D_{b,v,k}$:

(i) C_{d^*} has two distinct non-zero eigenvalues $\mu>\mu'$, the multiplicity of μ being 1,

(ii) $\mathrm{tr}\ C_{d^*}=\mathrm{Max}_{d\in D_{b,v,k}}\ \mathrm{tr}\ C_d$,

(iii) $\mathrm{tr}\ C_{d^*}^2<(\mathrm{tr}\ C_{d^*})^2/(v-2)$,

(iv) C_{d^*} maximizes $\mathrm{tr}\ C_d-[(v-1)/(v-2)]^{1/2}[\mathrm{tr}\ C_d^2-(\mathrm{tr}\ C_d)^2/(v-1)]^{1/2}$ over all $d\in D_{b,v,k}$.

Let d^* denote the covering design. Clearly, d^* satisfies (ii). We shall now show that it satisfies the remaining conditions. From (3.1) it is easily verified that $(u+5)$ is an eigenvalue of kC_d with multiplicity one and $(u+1)$ is an eigenvalue with multiplicity $(v-2)$ where $u=2y=v-1$. Thus (i) is satisfied. Also, $\mathrm{tr}(kC_{d^*})=u^2+u+4$ and $\mathrm{tr}(kC_d)^2=u^3+2u^2+9u+24$. Condition (iii) can now easily be verified.

Let $A_d=\mathrm{tr}(kC_d)$, $B_d=\mathrm{tr}(kC_d)^2$ and $P_d^2=B_d-A_d^2/(v-1)$. To establish (iv) we have to show that d^* maximizes $A_d-(u/(u-1))^{1/2}P_d$. For design d^*, $A_{d^*}=u^2+u+4$ and $(u/(u-1))^{1/2}P_{d^*}=4$. Thus we have to show that for any design $d\in D_{b,v,k}$,

$$A_{d^*}-A_d\geq 4-(u/(u-1))^{1/2}P_d. \tag{3.2}$$

If $A_{d^*}-A_d\geq 4$, (3.2) clearly holds. We note that

$$A_{d^*}-A_d=\mathrm{tr}(N_dN_d')-\mathrm{tr}(N_{d^*}N_{d^*}')=\sum_i\sum_j n_{dij}^2-\sum_i\sum_j n_{d^*ij}^2.$$

Since $\sum_i n_{ij}=3$, $A_{d^*}-A_d=0$ for all binary designs. Thus if d is binary, the (M, S)-optimality of d^* implies that $B_{d^*}\leq B_d$ and hence $P_{d^*}\leq P_d$. Thus in this case (iv) holds. We thus restrict our attention to designs d which are not binary and for which $A_{d^*}-A_d=1,2$ or 3. A block with one treatment occurring twice contributes 2 to $A_{d^*}-A_d$ and a block with a treatment occurring thrice contributes 6 to $A_{d^*}-A_d$. Thus $A_{d^*}-A_d$ cannot take values 1 or 3 and it is enough to consider designs in which one treatment occurs twice in one block and no other block has a repeated treatment.

Since for such a design $A_d=u^2+u+2$, (3.2) simplifies to

$$B_d\geq u^3+2u^2+5u+8. \tag{3.3}$$

Now, $B_d=$ sum of squares of diagonal terms of kC_d + sum of squares of off-diagonal terms of kC_d, where the sum of the diagonal terms + the sum of the off-diagonal terms is zero. Also, the diagonal elements are non-negative integers whereas the off-diagonal elements are non-positive integers. Since the sum of the $u+1$ diagonal elements $=u^2+u+2$, it is easily seen that their sum of squares must be at least $(u-1)u^2+2(u+1)^2=u^3+u^2+4u+2$. Similarly, the sum of the $u(u+1)$ off-diagonal elements is $-(u^2+u+2)$ and their sum of squares is at least $\{u(u+1)-2\}(-1)^2+2(-2)^2=u^2+u+6$ and hence $B_d\geq u^3+2u^2+5u+8$. In fact, for B_d to attain this value one block must have a repeated treatment say 1 in block (1, 1, 2) which gives $\lambda_{12}=2$ and all other λ_{ij}'s must be 1. Since there are $6t+3$ treatments remaining, it is not possible to pair treatment 1 precisely once with each of these and hence $B_d>u^3+2u^2+5u+8$. This completes the verification of all the conditions. Our result can now be stated as follows:

Theorem 3.1. *The covering design on $v=6t+5$ treatments in blocks of size 3 is optimal w.r.t. any generalized criterion of type* 1.

Takeuchi (1961) considered the criterion of minimizing the maximum variance of a paired treatment comparison. He also gave a lower bound for this maximum variance and showed that when $k\geq 3$ a necessary and sufficient condition for this lower bound to be attained is that the design is a group divisible partially balanced incomplete block design with $\lambda_2=\lambda_1+1$. For the minimal covering design there are three possible values of the variances of the paired comparisons. These are

$$\frac{6}{(2y+5)}, \quad \frac{12y+24}{(2y+1)(2y+5)} \quad \text{and} \quad \frac{6}{(2y+1)}.$$

so that the maximum is $6/(2y+1)$ which is very close to $6/((2y+1)+4/(2y+1))$, the bound given by Takeuchi. Since a design for which this bound is attained does not exist, we are led to believe that the minimal covering design is optimal in this sense also.

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