Sankhyā : The Indian Journal of Statistics
1977, Volume 39, Series A, Pt. 1, pp. 63–81.

SEQUENCE-COMPOUND ESTIMATION WITH RATES IN NON-CONTINUOUS LEBESGUE-EXPONENTIAL FAMILIES

By R. S. SINGH[1]
Indian Statistical Institute

SUMMARY. This paper extends sequence-compound (SC) estimations in the normal family treated in Gilliland (1966) and Susarla (1974a), and in a restricted gamma family treated in Susarla (1974b) to the SC-estimations in Lebesgue-exponential families on the real line R.

Let X be a real valued random variable whose Lebesgue density conditional on $\omega \in \Omega$ is of the type (i) $p_\omega(x) = C(\omega)u(x) \exp(\omega x)$ or (ii) $p_\omega(x) = C(\omega)u(x) \exp(-x/\omega)$, and Ω is a bounded subset of $\{\omega \in R \mid \int u(x) \exp(\omega x)dx < \infty\}$ in case (i), and of $\{\omega > 0 \mid \int u(x) \exp(-x/\omega)dx < \infty\}$ in case (ii). The component problem is squared error loss estimation of ω based on an observation on X. In each of the two cases a sequence-compound estimator (SCE) is exhibited which is shown to be asymptotically optimal (a.o.)in the sense that the difference D_n between the average of risks up to stage n and the Bayes risk w.r.t. the empiric distribution of the parameters ω involved up to stage n converges to zero as $n \to \infty$. It is further shown that the SCE's in cases (i) and (ii) are a.o. with rates 1/5 and 1/3 respectively (i.e., D_n in (i) is $O(n^{-1/5})$, and in (ii) is $O(n^{-1/3})$). These asymptotic optimalities and their rates are uniform over the space of all parameter sequences. Examples of exponential families, including those whose Lebesgue-densities have infinitely many discontinuity points, are given where the aforesaid results hold good.

1. Introduction

Suppose the problem is squared-error loss estimation (SELE) of a real valued function $\theta(w)$ based on an observation of a random variable $X \sim P_w \in \mathcal{P} = \{P_w \mid w \in \Omega\}$, where $\mathcal{P}$ is a family of probability measures over a σ-field $\mathcal{B}$ of a sample space $\mathcal{X}$. Further, suppose this problem, to be called hereinafter the component problem, occurs repeatedly and independently, and we are to estimate the value of the θ function at each stage. Thus, at n-th stage we have an (unknown) vector $w_n = (w_1, \ldots, w_n) \in \Omega^n$ and corresponding vectors $X_n = (X_1, \ldots, X_n)$ $P_n = X_1^n P_j \in \mathcal{P}^n$ and $\theta_n = (\theta_1, \ldots \theta_n)$, where P_j and θ_j abbreviate P_{w_j} and $\theta(w_j)$. We consider here the sequence-compound version of the above problem, that is, at any particular stage i, an estimator φ_i of θ_i is allowed to depend on X_i and the loss is taken to be the average of the losses in the first i component problems. The vector $\varphi = (\varphi_1, \varphi_2, \ldots)$ is called *sequence-compound estimator* (SCE) of $\theta = (\theta_1, \theta_2, \ldots)$.

Let G_n be the *empiric distribution* function of $w_1, \ldots, w_n$. (Note that no assumption whatever on relationships among $w_1, \ldots, w_n$, and on the distributions governing $w_1, \ldots, w_n$ is made.) Let $R(G_n)$ be the *Bayes risk* w.r.t. G_n in

[1] Now at the University of Guelph.

the component problem. Let $P_j(Y)$ denote the expectation of Y w.r.t. P_j. The excess,

$$D_n(\theta, \varphi) = n^{-1} \Sigma_1^n P_j(\theta_j - \varphi_j)^2 - R(G_n), \qquad \dots \quad (1.1)$$

of the *compound risk* up to stage n over the Bayes risk, is called the *modified regret* of φ up to stage n. Such regret functions are often taken as standards for evaluating compound procedures, (e.g., Gilliland, 1966 and 1968; Hannan, 1956 and 1957; Hannan and Huang, 1972; Johns, 1967; Samuel, 1965; Singh, 1974; Susarla, 1974a and 1974b, of course with varying component problems). For a $\delta > 0$, and a subset $B \subseteq \Omega^\infty$, *we will say* φ *is asymptotically optimal* (a.o.) *with a rate* δ *uniformly on* B *if* $\sup_{w \in B} |D_n(\theta, \varphi)| = O(n^{-\delta})$ as $n \to \infty$.

When $\mathcal{P}$ is the family of normal distributions on the real line R with variance unity and means w, and Ω is a bounded interval of R and θ is the identity map, Gilliland (1966, chapter III) exhibits a SCE a.o. with a rate 1/5 uniformly on Ω^∞. Susarla (1974a, Section 3), extends Gilliland's work to the m-variate case. When the conditional density of X given w is $(\Gamma(\tau))^{-1} x^{\tau-1} w^{-\tau} \exp(-x/w)$, $x > 0$, $0 < c < w < 2c$, where c and τ are known constants and $\tau \geqslant 3$ satisfies certain conditions, Susarla (1974b, Section 2.1), exhibits SCE's of w which are a.o. with rates uniformly on $(c, 2c)^\infty$.

This paper extends the above work to Legesgue exponential families on R. (Sequence-compound estimations in certain discrete exponential families on R are already treated in Gilliland, (1968). No assumptions whatsoever on the smoothness of the Lebesgue densities involved are made and yet SCE's a.o. with rates uniformly on Ω^∞ are exhibited.

An explicit bound for $D_n(\theta, \varphi)$ without any assumption on the parametric form of P_w is obtained in Section 2. This bound is an extension of Gilliland's bound (Lemma 2.1 of Gilliland, 1968) where parameters are assumed to be uniformly bounded. In Section 3, some notations are introduced and the method of our analysis is explained.

In Section 4, SC-SELE of the natural parameters in Lebesgue exponential families is considered. Based on X_n, $\hat{\psi}_n$ are constructed such that $\hat{\psi} = (\hat{\psi}_1, \hat{\psi}_2, \dots)$ is a.o. for $w = (w_1, w_2, \dots)$. Sufficient conditions are given under which $\sup_{w \in \Omega^\infty} |D_n(w, \hat{\psi})| = O(n^{-1/5})$. Examples of exponential families such as normal, gamma and one with Lebesgue densities having infinitely many discontinuity points are given where conditions leading to the above rate uniform on Ω^∞ are satisfied.

In Section 5 SC-SELE of the scale parameters in Lebesgue exponential families is treated, and an a.o. SCE $\hat{\psi}$ (of w) is exhibited. Sufficient conditions leading to $\sup_{w \in \Omega^\infty} |D_n(w, \hat{\psi})| = O(n^{-1/3})$ are given. Examples of scale exponential families where such assumptions hold good are also given. The paper concludes with a few remarks in Section 6.

2. A bound for the modified regret

In this section we will prove two simple but useful lemmas. Special forms of both have been studied in Gilliland (1968) and Susarla (1974a) where parameters $w_1, w_2, \ldots$ are uniformly bounded.

With μ a σ-finite measure dominating $P_j \forall j = 1, 2, \ldots$, let f_j be a determination of $dP_j/d\mu$. Let $m_i \geqslant \max_{1 \leqslant j \leqslant i} f_j$ and $N_i \geqslant \max_{1 \leqslant j \leqslant i} |\theta_j|$ be such that m_i and N_i are non-decreasing. Recall that θ_j abbreviates $\theta(w_j)$. As the Bayes response against G_i in the component problem, we take the version of conditional expectation

$$\psi_{i+1} = \frac{\Sigma_1^i \theta_j f_j}{\Sigma_1^i f_j} I(\Sigma_1^i f_j > 0), \quad i = 1, 2, \ldots . \qquad \ldots \quad (2.1)$$

Thus $|\psi_{i+1}| \leqslant N_i$. For the purpose of this section only, take ψ_1 arbitrary real valued function on R, and for $j \geqslant 1$, define $\Delta_j = \psi_{j+1} - \psi_j$.

Lemma 2.1 : *With ψ_1 taking values in $[-N_n, N_n]$,*

$$\Sigma_1^n P_i |\Delta_i(X_i)| \leqslant 2N_n(1 + \log n) \int m_n \, d\mu.$$

Proof: Abbreviate, throughout this proof, N_n by N and m_n by m. From (2.1) it follows that, for $1 \leqslant i \leqslant n$,

$$\Delta_i = \frac{(\theta_i - \psi_i) f_i}{\Sigma_1^i f_j} \quad \text{a.e.} \ P_i.$$

Consequently, since $|\theta_i - \psi_i| \leqslant 2N$ for $\forall \ 1 \leqslant i \leqslant n$,

$$\Sigma_1^n P_i |\Delta_i(X_i)| \leqslant 2N \int \left\{ m \, \Sigma_1^n \frac{(f_i/m)^2}{\Sigma_1^i (f_j/m)} \right\} d\mu. \qquad \ldots \quad (2.2)$$

Since by Lemma 2.2 of Gilliland (1968), $\Sigma_1^n a_i^2 (\Sigma_1^i a_j)^{-1} \leqslant \Sigma_1^n i^{-1}$ for all $0 \leqslant a_i \leqslant 1$, $1 \leqslant i \leqslant n$ and $n \geqslant 1$, the r.h.s. of (2.2) is bounded above by $2N(\Sigma_1^n i^{-1}) \int m \, d\mu \leqslant 2N(1 + \log n) \int m \, d\mu$. Q.E.D.

Lemma 2.2 : *For a SCE* $\varphi = (\varphi_1, \varphi_2, \ldots)$ *with* φ_i *taking values in* $[-N_i, N_i]$,

$$|D_n(\theta, \varphi)| \leqslant 4n^{-1} \Sigma_2^n N_j P_j |\varphi_j(X_j) - \psi_j(X_j)| + 8n^{-1} N_n^2 (1 + \log n) \int m_n \, d\mu$$

provided the arbitrary ψ_1 *is taken as* φ_1.

Proof: Unless stated otherwise, sums in this proof are taken from 1 to n. Let the argument X_j in various summands below in this proof be abbreviated by omission. Inequalities (8.8) and (8.11) of Hannan (1957) which, in our case with our notations, can be stated as

$$\Sigma P_j |\psi_{j+1} - \theta_j|^2 \leqslant nR(G_n) \leqslant \Sigma P_j |\psi_j - \theta_j|^2,$$

and (1.1) here followed by the identity $b^2 - c^2 = (b-c)(b+c)$ give

$$\begin{aligned} \Sigma P_j((\varphi_j - \psi_j)(\varphi_j + \psi_j - 2\theta_j)) &\leqslant nD_n(\theta, \varphi) \\ &\leqslant \Sigma P_j((\varphi_j - \psi_{j+1})(\varphi_j + \psi_{j+1} - 2\theta_j)) \\ &= \Sigma P_j((\varphi_j - \psi_j) - \Delta_j)(\varphi_j + \psi_{j+1} - 2\theta_j)). \qquad \ldots \quad (2.3) \end{aligned}$$

Since for $j \geqslant 2$, φ_j, ψ_{j+1}, ψ_j and θ_j are in $[-N_j, N_j]$, and $\max_{1 \leqslant j \leqslant n} |\varphi_j + \psi_{j+1} - 2\theta_j| \leqslant 4N_n$, from (2.3).

$$-4 \Sigma_2^n N_j P_j |\varphi_j - \psi_j| \leqslant nD_n(\theta, \varphi) \leqslant 4(\Sigma_2^n N_j P_j |\varphi_j - \psi_j| + N_n \Sigma P_j |\Delta_j|).$$

The last inequalities and Lemma 2.1 now complete the proof. Q.E.D.

In the remainder of this paper we will exhibit an estimator $\hat{\psi}$ of w, in exponential families in R, and prove with the help of Lemma 2.2 the asymptotic optimality (with rates) of $\hat{\psi}$.

4. Notations and the method of analysis

Hereinafter, let θ introduced above be the identity map. To treat the cases of our interest, let μ above be a σ-finite measure dominated by the Lebesgue measure on R. With u a fixed determination of $d\mu/dx$, let there exist an $a \geqslant -\infty$ such that

$$u(x) > 0 \quad \text{iff} \quad x > a. \qquad \ldots \quad (3.1)$$

In the component problem, let $P_w \ll \mu$ for w in Ω, and f_w be a determination of $dP_w/d\mu$. Thus in the component problem f_w (and $p_w = uf_w$) are conditional μ (and Lebesgue)-densities of X given w.

To make the analysis simpler and help readers understand the material, we treat only the cases where $\Omega \subseteq [\alpha, \beta]$, $-\infty < \alpha < \beta < \infty$ and f_w on (a, ∞)

is positive for w in $[\alpha, \beta]$. (Readers interested in the cases where Ω are not necessarily bounded may look at Singh (1974, Ch. 2) apparently the only literature to date dealing sequence-compound problem involving unbounded parameters).

Let $0 < h_n \leqslant 1$ be a sequence of non-increasing numbers such that $h_n \to 0$ as $n \to \infty$. By Lemma 2.2 an idea of exhibiting an a.o. estimator $\hat{\psi} = (\hat{\psi}_1, \hat{\psi}_2, \ldots)$ of $w \in \Omega^\infty$ is to exhibit $\hat{\psi}_i(X_i)$ for $i = 2, 3, \ldots$ such that it approximates (at least in first mean) to $\psi_i(X_i)$. This is exactly what we have in mind. For each $i = 2, 3, \ldots$ we will exhibit $\hat{\psi}_i(X_i)$ by using h_{i-1} and X_i. Then we obtain a suitable bound for $P_i|\hat{\psi}_i(X_i) - \psi_i(X_i)|$ which will lead, with the help of Lemma 2.2, to a bound for $|D_n(w, \hat{\psi})|$ uniform in $w \in \Omega^\infty$. To this end, we *hereafter* fix i with $i = 1, 2, \ldots$ and drop the subscripts in h_i and X_{i+1}.

For $a_j \in R$, let $\bar{a} = i^{-1}\Sigma_1^i a_j$. Let $u_*(x)$ and $u^*(x)$ be, respectively, ess-inf and ess-sup (w.r.t. the Lebesgue measure), of the restriction to $[x, x+2h)$ of u. Abbreviate p_{w_j} and f_{w_j} to p_j and f_j respectively. Unless stated otherwise, arguments of f_j, u, u^* and u_* are in (a, ∞).

4. Sequence-compound estimation of natural parameters in Lebesgue exponential families

In this section we treat the cases where

$$f_w(x) = C(w)e^{wx} \text{ with } C(w) = \left(\int e^{wx}d\mu(x)\right)^{-1}. \qquad \ldots \quad (4.1)$$

Thus in the component problem the conditional Lebesgue density is of the form

$$p_w(x) = u(x)C(w)e^{wx}, \qquad \ldots \quad (4.2)$$

and $\Omega \subseteq [\alpha, \beta]$ is a subset of the natural parameter space $\{w \in R \mid C(w) > 0\}$. In this section we will consider SC-SELE of $w = (w_1, w_2, \ldots) \in \Omega^\infty$.

Since $f_j(x) = C(w_j)\exp(w_j x)$, by (2.1) $\psi_{i+1} = (\log \bar{f})^{(1)}$. Motivated by this expression, $\hat{\psi}_{i+1}$ to be introduced here will be based on a divided difference estimator of $(\log \bar{f})^{(1)}$. Define a real valued functional Q on the space of all real valued non-negative functions t on R by

$$Q(t)(x) = h^{-1} \left(\log \frac{t(x+h)}{t(x)}\right) I(t(x+h)+t(x) > 0). \qquad \ldots \quad (4.3)$$

For $j = 1, \ldots, i$, let $\delta_j(y) = \int_y^{y+h} f_j$ and $\hat{\delta}_j(y) = I(y \leqslant X_j < y+h)/u(X_j)$. Note that $\hat{\delta}$ is well defined with probability one. The proposed compound estimator of w is $\hat{\psi} = (\hat{\psi}_1, \hat{\psi}_2, \ldots)$ where $\hat{\psi}_1$ takes an arbitrary value in $[\alpha, \beta]$, and

$$\hat{\psi}_{i+1}(X) = (Q(\bar{\hat{\delta}})(X))_{\alpha, \beta} \qquad \ldots \quad (4.4)$$

where $(b)_{\alpha, \beta}$ is α, b or β according as $b < \alpha$, $\alpha \leqslant b \leqslant \beta$ or $b > \beta$.

The main objective of this section is to prove the following theorem which gives sufficient conditions under which our SCE $\hat{\psi}$ is a.o. with rates uniformly in $w \in \Omega^\infty$. In the remainder of this section $c_0, c_1, \ldots$ denote absolute positive constants, $\eta = e^{2hc}$ where $c = |\alpha| \vee |\beta|$ and m_i and N_i introduced in Section 2 are taken as

$$m_i \equiv m \equiv \sup_{\alpha \leqslant \omega \leqslant \beta} f_\omega, \text{ and } N_i \equiv c \equiv |\alpha| \vee |\beta|. \qquad \ldots \quad (4.5)$$

Theorem 4.1 : *Let $h = h_i = c_0\ i^{-1/5}$ where c_0 is sufficiently small such that for a $0 < c_1 < 1$ and for all $i = 1, \ldots, n$,*

$$h\eta u^* \ m \leqslant 1 - c_1. \qquad \ldots \quad (4.6)$$

If for a $\gamma \epsilon (0, 1]$ and for $i = 1$

$$\int \left(\frac{u^*(x)}{u_*^2(x)}\right)^{\gamma/2} u(x)\{e^{x(\alpha-(\gamma\beta/2))}I(x \leqslant 0) + e^{x(\beta-(\gamma\alpha/2))}I(x > 0)\}dx < \infty \qquad \ldots \quad (4.7)$$

then

$$\sup_{w \in \Omega^\infty} |D_n(w, \hat{\psi})| = O(n^{-\gamma/5}). \qquad \ldots \quad (4.8)$$

To simplify the proof of the theorem we will first prove three useful lemmas. The first and third lemmas approximate respectively $Q(\bar{\hat{\delta}})$ to Q and $\hat{\psi}_{i+1}$ to ψ_{i+1} in γ-th mean, whereas the second one gives uniform bound for $|Q - (\log \hat{f})^{(1)}|$. Abbreviate $Q(\bar{\delta})(x)$ and $Q(\bar{\hat{\delta}})(x)$ to $Q(x)$ and $\hat{Q}(x)$. In Lemma 4.1 and in its proof Q, $\hat{Q}$, u_*, u^* and $\bar{f}$ all are evaluated at a fixed point $x > a$.

Lemma 4.1 : *For every $\gamma > 0$,*

$$P_i(|Q - \hat{Q}| \wedge 2c)^\gamma \leqslant k_0(\gamma)(ih^3\bar{f}u_*^2/u^*)^{-\gamma/2} \qquad \ldots \quad (4.9)$$

where $k_0(\gamma) = \gamma\Gamma(\gamma/2)(16\eta^3(1+\eta^2)/3k^+)^{\gamma/2}$ with $k = 1 - h\eta u^* m$.

Proof: The l.h.s. of (4.9) is

$$\int_0^{2c} P_i[|Q-\hat{Q}| > v]d(v^\gamma) = \int_0^{2c} (\xi_1(v)+\xi_2(v))d(v^\gamma), \quad \ldots \quad (4.10)$$

where $\xi_1(v) = P_i[(\hat{Q}-Q) > v]$ and $\xi_2(v) = P_i[(Q-\hat{Q}) > v]$. Our method of the proof here involves obtaining an appropriate upper bound for $\xi_1(v)+\xi_2(v)$ with $0 < v < 2c$.

Fix v in $(0, 2c)$ until stated otherwise. For $j = 1, \ldots, i$, let $Y_j = \hat{\delta}_j(x+h) - Re^{hv}\hat{\delta}_j(x)$, where $R = \hat{\delta}(x+h)/\hat{\delta}(x)$. Let $\nu_j = P_jY_j$ and $\sigma^2 = i \operatorname{var}(\bar{Y})$. Notice that $\nu_j = \delta_j(x+h)-Re^{hv}\delta_j(x)$. Hence $\bar{\nu} = (1-e^{hv})\bar{\delta}(x+h)$, and we get

$$-\eta\bar{\delta}(x+h) \leqslant \bar{\nu} \leqslant -hv\bar{\delta}(x+h). \quad \ldots \quad (4.11)$$

By independence of $Y_1, \ldots, Y_i$ and by c_r-inequality (see Loeve, 1963, p. 155) we have

$$i\sigma^2 \leqslant \Sigma_1^i P_jY_j^2 \leqslant 2\Sigma_1^i(P_j\hat{\delta}_j^2(x+h)+R^2e^{2hv}P_j\hat{\delta}_j^2(x)). \quad \ldots \quad (4.12)$$

Since $v < 2c$, $R = \bar{\delta}(x+h)/\bar{\delta}(x)$ and, for $y = x, x+h$, $P_j\hat{\delta}_j^2(y) \leqslant \delta_j(y)/u_*$, by (4.12) we get

$$\sigma^2 \leqslant \{2(1+R\eta)\bar{\delta}(x+h)/u_*\} = 2((\bar{\delta}(x+h))^{-1}+\eta(\bar{\delta}(x))^{-1})\bar{\delta}^2(x+h)/u_*.$$

Now, since, for $1 \leqslant j \leqslant i$, $w_j \in [-c, c]$,

$$h\eta^{-1} \leqslant \frac{\delta_j(y)}{f_j(x)} = \int_y^{y+h} e^{w_j(t-x)}\,dt \leqslant h\eta \text{ for } y = x,\ x+h. \quad \ldots \quad (4.13)$$

Therefore, weakening the final upper bound obtained above for σ^2 by the first inequality in (4.13) we get $u_*\bar{f}\sigma^2 \leqslant 2(1+\eta^2)\eta h^{-1}\bar{\delta}^2(x+h)$. This lest inequality and (4.11) give

$$\frac{(-\bar{\nu})^2}{\sigma^2} \geqslant \frac{h^3v^2\bar{f}u_*}{2(1+\eta^2)\eta}. \quad \ldots \quad (4.14)$$

Next we will obtain (4.17) below by obtaining appropriate lower bounds for $\sigma^2, \bar{\nu}, \nu_j$ and $-Y_j$. By independence of $Y_1, \ldots, Y_i$ and by the facts that $v > 0$, $P_j(\hat{\delta}_j(\cdot)) > 0$, and $\hat{\delta}_j(x+h)\,\hat{\delta}_j(x) = 0$ with probability one we get

$$\sigma^2 \geqslant i^{-1}\,\Sigma_1^i\,(\operatorname{var}(\hat{\delta}_j(x+h))+R^2\operatorname{var}(\hat{\delta}_j(x))). \quad \ldots \quad (4.15)$$

Now the definition of u^* and the second inequality in (4.13) yield for $y = x,\ x+h$,

$$\mathrm{var}\,(\hat{\delta}_j(y)) = \int_y^{y+h} (f_j/u) - \delta_j^2(y) \geqslant (\delta_j(y)(1-u^*\delta_j(y))^+/u^*)$$

$$\geqslant (\delta_j(y)(1-h\eta u^* f_j)^+/u^*) \geqslant k^+\delta_j(y)/u^*,$$

where k is as given in the lemma, and the last inequality follows from the definition of m given in (4.5). Consequently, from (4.15) we get

$$u^*\sigma^2 \geqslant (\bar{\delta}(x+h)+R^2\,\bar{\delta}(x))k^+ = (1+R)\bar{\delta}(x+h)k^+. \qquad \dots \quad (4.16)$$

Next observe that $-Re^{hv}\,\hat{\delta}_j(x) \leqslant Y_j \leqslant \hat{\delta}_j(x+h)$. Therefore, since for $y = x$, $x+h$, $\delta_j(y) \leqslant 1/u^*$ with probability one, $Y_j \leqslant 1/u^*$ and $-\nu_j \leqslant R\eta/u_*$ These upper bounds for Y_j and $-\nu_j$ together with (4.11) and (4.16) yield $(Y_j-\nu_j)(-\bar{\nu}/\sigma^2) \leqslant \{(1+\eta R)\eta u^*/(1+R)u^*k^*)\} \leqslant \eta^2u^*(k^+u_*)^{-1}$.

Hence

$$Y_j-\nu_j \leqslant \frac{\eta^2u^*}{k^+u_*}\left(-\frac{\sigma^2}{\bar{\nu}}\right). \qquad \dots \quad (4.17)$$

We will use (4.14) and (4.17) to obtain a suitable upper bound for $\xi_1(v)$. Note that the event in $\xi_1(v)$ is $[\bar{Y} > 0]$. Therefore, (4.17) and the Bernstein inequality stated in (2.13) of Hoeffding (1963) give

$$\xi_1(v) = P_i[\bar{Y}-\bar{\nu} > -\bar{\nu}] \leqslant \exp\left\{-\frac{i(-\bar{\nu})^2}{\sigma^2\,2(1+(\eta^2u^*/3k^+u_*))}\right\} \qquad \dots \quad (4.18)$$

$$\leqslant \exp\left\{-\frac{3ik^+h^3v^2 J v_*^2}{16\eta^3(1+\eta^2)u^*}\right\}$$

where the last inequality follows by (4.14) and by the fact that $\{1+(\eta^2u^*/(3k^+u_*)\}$ $\leqslant 4(3k^+u_*)^{-1}\eta^2u^*$, since $\eta \geqslant 1$, $k^+ \leqslant 1$ and $u^* \geqslant u_*$.

By interchanging x, $x+h$ in the definition of Y_j and by applying the techniques used for bounding $\xi_1(v)$, we see that $\xi_2(v)$ is also bounded above by the extreme r.h.s. in (4.18).

Now bounding above the integrand on the r.h.s. of (4.10) by the upper bound just obtained for $\xi_1(v)+\xi_2(v)$ and then performing the integration there after extending the range of integration from $(0, 2c)$ to $(0, \infty)$ we get the desired conclusion. Q.E.D.

Lemma 4.2 : *Let $f_j^{(\nu)}$ denote the ν-th derivative of f_j. Then*

$$\sup_{t>a}\left|\left(Q-\frac{\bar{f}^{(1)}}{\bar{f}}\right)(t)\right| \leqslant 4\,(c\eta)^2\,h. \qquad \dots \quad (4.19)$$

Proof: Since, for $1 \leqslant j \leqslant i$, $w_j \,\epsilon\, [-c, c]$, for each integer $\nu \geqslant 0$ and $\forall\, t \,\epsilon\, [\cdot\,, \cdot +2h]$ we have

$$\frac{|f_j^{(\nu)}(t)|}{f_j(\cdot)} = |w_j^\nu|\, e^{w_j(t-.)} \leqslant c^\nu \eta, \qquad \dots \quad (4.20)$$

and

$$\frac{f_j(t)}{f_j(\cdot)} = e^{w_j(t-.)} \geqslant \eta^{-1}. \qquad \dots \quad (4.21)$$

For the purpose of this proof only, let $g_j = w_j^{-1} f_j$. Since $\bar{\delta}(t) = \bar{g}(t+h)-\bar{g}(t)$, by Cauchy-mean value theorem (see Hoeffding, 1963, p. 81) for some ϵ in $(0, 1)$

$$\frac{\bar{\delta}(t+h)}{\bar{\delta}(t)} = \frac{\bar{g}^{(1)}(t+h+\epsilon h)}{\bar{g}^{(1)}(t+\epsilon h)} = \frac{\bar{f}(t+h+\epsilon h)}{\bar{f}(t+\epsilon h)}\,. \qquad \dots \quad (4.22)$$

Therefore, by (4.22) and by mean value theorem $Q(t) = h^{-1}\log(\bar{f}(t+h+\epsilon h)/\bar{f}(t+\epsilon h)) = \log \bar{f}(t'))^{(1)}_{t'=t+\gamma h}$ for some $\gamma\epsilon(0, 2)$. Making another use of mean value theorem at the third step below, we thus have, for some $\gamma', \gamma''\epsilon\,(0, \gamma h)$

$$\left|Q(t)-\left(\frac{\bar{f}^{(1)}}{\bar{f}}\,(t)\right| = \left|\left(\frac{\bar{f}^{(1)}}{\bar{f}}\right)(t+\gamma h)-\left(\frac{\bar{f}^{(1)}}{\bar{f}}\right)(t)\right|$$

$$\leqslant \frac{1}{\bar{f}(t+\gamma h)}\left(|\bar{f}^{(1)}(t+\gamma h)-f^{(1)}(t)| + \left|\left(\frac{\bar{f}^{(1)}}{\bar{f}}\right)(t)\right|\,|\bar{f}(t+\gamma h)-\bar{f}(t)|\right)$$

$$= \frac{\gamma h}{\bar{f}(t+h)}\left(|\bar{f}^{(2)}(t+\gamma')| + \left|\left(\frac{\bar{f}^{(1)}}{\bar{f}}\right)(t)\right|\,|\bar{f}^{(1)}(t+\gamma'')|\right) \leqslant 4h(c\eta)^2 \quad \dots \quad (4.23)$$

where the last inequality follows by applying (4.20) for $\nu = 2, 1$, (4.13) and the fact that $|\bar{f}^{(1)}/\bar{f}| \leqslant c$ and $\gamma < 2$. Since the r.h.s. of (4.23) is independent of t, the proof of the lemma is complete. Q.E.D.

Note that $\log C(w) = -\log \int e^{w\cdot} d\mu(\cdot)$ is concave on $[\alpha, \beta]$, and hence, so is $\log f_w(x) = wx + \log C(w)$ for each x. Thus $\inf\limits_{\alpha \leqslant w \leqslant \beta} f_w = f_\alpha \wedge f_\beta$ and for all $\gamma \geqslant 0$

$$q_\gamma = (m/(f_\alpha \wedge f_\beta)^{\gamma/2}) \geqslant f_{i+1}/(\bar{f})^{\gamma/2}. \qquad \dots \quad (4.24)$$

This observation is used in proving

Lemma 4.3 : *For each* $\gamma > 0$,

$$P_{i+1}|\psi_{i+1}(X)-\hat{\psi}_{i+1}(X)| \leqslant k_0'(\gamma)\{(ih^3)^{-\gamma/2}\int((u^*/u_*^2)^{\gamma/2}q_\gamma)d\mu+(c^2h)^\gamma\} \quad \ldots \quad (4.25)$$

where $k_0'(\gamma) = 2^{(\gamma-1)^+}\{(4\eta^2)^\gamma \vee k_0(\gamma)$ *with* k *in* $k_0(\gamma)$ *replaced by* $\inf_{x>a}(1-h\eta u^*(x)m(x))\}$. *(The inequality* (4.25) *is uniform in* $w_{i+1} \in \Omega^{i+1}$.*)*

Proof : Notice that $e^{ah} \leqslant (\delta_j(x+h)/\delta_j(x)) \leqslant e^{\beta h}$ for each $1 \leqslant j \leqslant i$. Therefore, $\alpha \leqslant Q \leqslant \beta$. Since $\psi_{i+1} = \bar{f}^{(1)}/\bar{f}$ and $c = |\alpha| \vee |\beta|$ by (4.4) and Lemma (4.2) we get

$$|\psi_{i+1}-\hat{\psi}_{i+1}| \leqslant |Q-\hat{\psi}_{i+1}|+|Q-\psi_{i+1}| \leqslant (|Q-\hat{Q}| \wedge 2c)+4h(c\eta)^2. \quad \ldots \quad (4.26)$$

Now (4.26) followed by c_r-inequality (see Loeve, 1963, p. 155), Lemma 4.1 and (4.24) lead to (4.25). Q.E.D.

Proof of Theorem 4.1 : Fix $\gamma \in (0, 1]$ satisfying (4.7). Since $[\alpha, \beta]$ is a subset of the natural parameter space $\{\cdot|C(\cdot) > 0\}$, $C(w)$ is bounded away from 0 and ∞ on $[\alpha, \beta]$. Consequently $\exists c_2$ and c_3 such that

$$\int m d\mu \leqslant c_2 \int \{\exp(\alpha x)I(x \leqslant 0)+\exp(\beta x)I(x > 0)\}d\mu(x) < \infty \quad \ldots \quad (4.27)$$

and

$$q_\gamma(x) = (\sup_{\alpha \leqslant w \leqslant \beta} f_w(x)/(f_\alpha(x) \wedge f_\beta(x))^{\gamma/2}$$
$$\leqslant c_3[\exp\{x(\alpha-(\gamma\beta/2))\}I(x \leqslant 0)+\exp\{x(\beta-(\gamma\alpha/2))\}I(x > 0)].$$

Note that u^* and u_* depend on $h = c_0 i^{-1/5}$ and are respectively, decreasing and increasing in i. Thus if (4.7) holds for $i = 1$, then

$$\sup\left(\int((u^*/u_*^2)^{\gamma/2}q_\gamma)d\mu\right) < \infty. \quad \ldots \quad (4.28)$$

Next observe that by (4.6) k_0' in Lemma 4.3 is bounded in i. Also the trivial bound $|\hat{\psi}_{i+1}-\psi_{i+1}| \leqslant 2c$ gives $P_{i+1}|\psi_{i+1}(X)-\hat{\psi}_{i+1}(X)| \leqslant 2c^{1-\gamma}P_{i+1}|\psi_{i+1}(X)-\hat{\psi}_{i+1}(X)|^\gamma$. Thus, since $h = c_0 i^{-1/5}$, Lemma 4.3 gives finite $k_1 = k_1(\gamma)$ and $k_2 = k_2(\gamma)$ independent of i such that for all $i = 1, \ldots, n-1$ -

$$P_{i+1}|\psi_{i+1}(X)-\hat{\psi}_{i+1}(X)| \leqslant k_1 i^{-\gamma/5}(\text{lhs of } (4.28)+c^{2\gamma})$$
$$\leqslant k_2 i^{-\gamma/5}, \quad \text{by } (4.28). \quad \ldots \quad (4.29)$$

Since X abbreviates X_{i+1} and (4.29) holds for each $i = 1, 2, \ldots$, (k_1 and k_2 being independent of i), $n^{-1}\Sigma_{j=2}^n P_j|\psi_j(X_j)-\hat{\psi}_j(X_j)| \leqslant k_2 n^{-1}\Sigma_1^n i^{-\gamma/5}$. Thus the first term on the r.h.s. of the inequality in Lemma 2.2 with φ there replaced by $\hat{\psi}$ is bounded by $4c\ k_2 n^{-1}\Sigma_1^n i^{-\gamma/5} = O(n^{-1/5})$ uniformly in $w \in \Omega^\infty$, and so is the second term there by (4.27). Q.E.D.

We will now give examples of exponential families of distributions (including one whose Lebesgue densities have infinitely many discontinuity points) where conditions of the theorem are satisfied.

Example 1 (*Normal* $N(w, 1)$—*family*) : Suppose in the component problem the conditional Lebesgue density of X given w is $p_w(x) = (2\pi)^{-\frac{1}{2}} \exp(-(x-w)^2/2) I(-\infty < x < \infty)$. We can take $u(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2) I(-\infty < x < \infty)$. Then $a = -\infty$ and $C(w) = \exp(-w^2/2)$. Take $-\alpha = \beta = c > 0$.

Considering the upper and lower bounds for the ratio $u(t)/u(x)$ for $x \leqslant t < x+2h$, we get $u^*(x) \leqslant u(x)e^{2h|x|}$ and $u_*(x) \geqslant u(x)e^{-2h(|x|+h)}$. Therefore

$$u^*(x)f_w(x) \leqslant e^{2h|x|}u(x)f_w(x) = (2\pi)^{-\frac{1}{2}} \exp\{-(|x|-w \operatorname{sgn} x)^2 - 4h|x|)/2\}$$

$$\leqslant (2\pi)^{-\frac{1}{2}} \exp\{2h(h+w \operatorname{sgn} x)\}.$$

Thus, since $m = \sup_{\alpha \leqslant w \leqslant \beta} f_w$, $u^* m \leqslant \exp(2h^2+hc)$. Therefore by a suitable choice of c_0 in $h = c_0 i^{-1/5}$, (4.6) holds.

Moreover, bounds obtained above for u^* and u_* lead to

$$(u^*(x)/u_*^2(x))^{\frac{1}{2}} \leqslant (2\pi)^{1/4} \exp((x^2/4)+3h|x|+2h^2)$$

$$\leqslant (2\pi)^{1/4} \exp((x^2/4)+3|x|+2) \text{ since } h \leqslant 1.$$

Consequently

$$(u^*(x)/u_*^2(x))^{\frac{1}{2}}u(x) \leqslant (2\pi)^{-1/4} \exp((-x^2/4)+3|x|+2)$$

and (4.7) holds for $\gamma = 1$ and $-\alpha = \beta > 0$. We thus conclude the following corollary.

Corollary 4.1 : *If in the component problem the conditional Lebesgue density of X given w is $p_w(x) = (2\pi)^{-\frac{1}{2}} \exp(-(x-w)^2/2)$, $-\infty < x < \infty$ and $\Omega \subsetneq [-c, c]$, $0 < c < \infty$, then $\hat{\psi}$ given by* (4.4) *with $-\alpha = \beta = c$ satisfies*

$$\sup_{w \in \Omega^\infty} |D_n(w, \hat{\psi})| = O(n^{-1/5}).$$

This special result is obtained in Chapter III of Gilliland (1966) and in Section 3 of Susarla (1974a) where SC-SELE of uniformly bounded means of normal populations with unity variances is considered

Example 2 : (Gamma $\mathcal{G}(w, \tau)$-family). Suppose in the component problem the conditional density of X given w is $p_w(x) = (\Gamma(\tau))^{-1}(-w)^\tau x^{\tau-1} e^{wx}I(x>0)$ where $\tau \geqslant 1$ is known. Thus we can take $u(x) = x^{\tau-1}I(x>0)$. The natural parameter space is $(-\infty, 0)$ and $C(w) = (\Gamma(\tau))^{-1}(-w)^\tau$. Take $\beta < 0$, i.e., $\Omega \subseteq [\alpha, \beta]$ with $-\infty < \alpha < \beta < 0$.

Clearly $u^*(x) \leqslant (x+1)^{\tau-1}I(x>0)$, $u_*(x) = u(x)$. Inequality (4.6) is satisfied, since by c_r-inequality (Loeve, 1963, p. 155), $u^*(x) \leqslant 2^{\tau-2}(u(x)+1)$ and hence $u^*(x)/f_w(x) = 2^{\tau-2}(u(x)+1)(\Gamma(\tau))^{-1}(-w)^\tau e^{wx}$ is uniformly bounded in $w \in [\alpha, \beta]$ and in $x > 0$. Moreover, notice that $(u^*(x)/u_*^2(x))^{\gamma/2} \leqslant 2^{\tau-2}((u(x))^{-\gamma/2}+(u(x))^{-\gamma})$, again by c_r-inequality. Thus the l.h.s. of (4.7) is no more than a constant times

$$\int_0^\infty \{x^{(1-\gamma/2)(\tau-1)}+x^{(1-\gamma)(\tau-1)}\}e^{x(\beta-(\gamma\alpha/2))}\,dx < \infty$$

for all $0 < \gamma \leqslant 1$ such that $\gamma < 2\beta/\alpha$, since $\alpha < \beta < 0$. Hence for such γ (4.7) holds, and we get the following corollary.

Corollary 4.2 : *If in the component problem the conditional Lebesgue density of X given w is $p_w(x) = (\Gamma(\tau))^{-1}(-w)^\tau x^{\tau-1}e^{wx}$, $x > 0$, $\tau \geqslant 1$ and $\Omega \subseteq [\alpha, \beta]$, $-\infty < \alpha < \beta < 0$, then $\hat{\psi}$ given by (4.4) satisfies*

$$\sup_{w \in \Omega^\infty} |D_n(w, \hat{\psi})| = O(n^{-\gamma/5}) \forall 0 < \gamma \leqslant 1,\ \gamma < 2\beta/\alpha.$$

Example 4.3 : The Lebesgue density in the following corollary is an artificial one which has infinitely many discontinuity points. The proof of the corollary is similar to that of Corollary 4.2.

Corollary 4.3 : *Let in the component problem the Lebesgue density of X conditional on w be $p_w(x) = w(1-e^w)e^{wx}(\sum_0^\infty (j+1)[j<x\leqslant j+1])$, and $\Omega \subseteq [\alpha, \beta]$, $-\infty < \alpha < \beta < 0$. Then $\hat{\psi}$ given by (4.4) satisfies*

$$\sup_{w \in \Omega^\infty} |D_n(w, \hat{\psi})| = O(n^{-\gamma/5}) \forall\ 0 < \gamma \leqslant 1, \gamma < 2\beta/\alpha.$$

We have given sufficient conditions under which our SCE $\hat{\psi}$ is a.o. for w uniformly in $w \in \Omega^\infty$ with a rate 1/5. The existence of families of distributions where this rate can be achieved is also verified. This rate of asymptotic optimality is, however, not the best possible that can be achieved by our SCE $\hat{\psi}$. In fact it is shown in Section 2.5 of Singh (1974) that for any w in Ω^∞ with identical components $\hat{\psi}$ is a.o. with rates arbitrarily close to 2/5 in a number of exponential families. Nevertheless a rate better than 2/5 with $\hat{\psi}$ does not seem possible even when $\boldsymbol{w}$ has identical components (see Section 5 of Singh (1976) or Section 2.5 of Singh (1974).

5. Sequence-compound estimation of scale parameter in Lebesgue exponential families

In this section we treat the cases where in the component problem the conditional Lebesgue density p_w is of the form

$$p_w(x) = u(x)C(w)e^{-x/w}, \text{ with } C(w) = (\int e^{-x/w}\, d\mu(x))^{-1} \quad \ldots \quad (5.1)$$

and $\Omega \subseteq [\alpha, \beta]$ is a subset of $\{w > 0 \mid c(w) > 0\}$.

Thus
$$f_w(x) = c(w)e^{-x/w}. \quad \ldots \quad (5.2)$$

We will consider sequence-compound estimation of $\boldsymbol{w} = (w_1, w_2, \ldots) \in \Omega^\infty$, and exhibit a SCE which is a.o. with rates uniformly on Ω^∞.

A sequence-compound estimation, where the component problem is SELE of the scale parameter λ in $\Gamma(\lambda, \tau)$-family : $(\Gamma(\tau))^{-1}x^{\tau-1}\, \lambda^{-\tau}e^{-x/\lambda}I(x > 0)$; $\tau, \lambda > 0$, is an important example of our consideration in this section. This example of course includes the case of sequence-compound estimation where the component problem is SELE of σ^2 in $N(0, \sigma^2)$-family : $(2\pi\sigma^2)^{-\frac{1}{2}}$ exp $(-x^2/(2\sigma^2))$, $-\infty < x < \infty$, $\sigma > 0$; since X^2 is sufficient for σ^2, where $X \sim N(0, \sigma^2)$.

Since $f_j(x) = f_{w_j}(x) = C(w_j)e^{-x/w_j}$ and w_j are positive, $w_j f_j\,(x)$ can be written as $\int_x^\infty f_j\,(t)dt$. Thus by (2.1)

$$\psi_{i+1}(x) = (\int_x^\infty \bar{f}(t)dt)/\bar{f}(x) \quad \ldots \quad (5.3)$$

where
$$\bar{f} = i^{-1} \sum_{j=1}^{i} f_j.$$

As in Section 4, for $j = 1, \ldots, i$, let $\delta_j(y) = \int_y^{y+h} f_j$ and $\hat{\delta}_j\,(y) = I(y \leqslant X_j < y+h)/u(X_j)$. Then $\hat{\delta}_j$ is well defined with probability one, and is an unbiased estimator of δ_j. Let

$$h = h_i = i^{-1/3} \text{ and } H = H_i = \beta\, |\log h|. \quad \ldots \quad (5.4)$$

The proposed SCE of w is $\hat{\psi} = (\hat{\psi}_1, \hat{\psi}_2, \ldots)$ where $\hat{\psi}_1$ takes an arbitrary value in $[\alpha, \beta]$, and for $i = 1, 2, \ldots$,

$$\hat{\psi}_{i+1}(x) = \left(\frac{\int_X^{X+H} \hat{\bar{\delta}}(t)dt}{\hat{\delta}(X)} \right)_{\alpha, \beta} \qquad \ldots \quad (5.5)$$

where $(b)_{\alpha,\beta}$ is α, b or β according as $b < \alpha$, $\alpha \leqslant b \leqslant \beta$ or $b > \beta$.

The following theorem gives sufficient conditions under which $\hat{\psi}$ is a.o. with rates uniformly on Ω^{∞}. The symbols $c_0, c_1, \ldots$ below denote absolute positive constants. Denote $u_*(x)$, the ess-inf (w.r.t. the Lebesgue measure) of the restriction to $[x, x+2h)$ of u, by $u_{2h}(x)$.

Theorem 5.1 : *If for a $\gamma \in (0, 1]$ $\exists$ a $\xi \geqslant \gamma/2$ and a k_0 independent of i such that*

$$\int \left\{ exp\left(x\left(\frac{\gamma}{2\alpha} - \frac{1}{\beta}\right)\right) I(x \geqslant 0) + exp\left(x\left(\frac{\gamma}{2\beta} - \frac{1}{\alpha}\right)\right) I(x < 0)\right\}$$

$$\frac{u(x)}{(u_{h+H}(x))^{\gamma/2}}\, dx \leqslant k_0(1 \vee \log i)^{\xi - (\gamma/2)}, \qquad \ldots \quad (5.6)$$

then

$$\sup_{w \in \Omega^{\infty}} D_n(w, \hat{\psi})| = O(n^{-\gamma/3} (\log n)^{\xi}. \qquad \ldots \quad (5.7)$$

Lemma 5.2 below makes the proof of the theorem much simpler. This lemma is proved with the help of Lemma 5.1, which is of Singh (1974) and is found quite useful in obtaining rates of asymptotic optimality of some compound as well as empirical Bayes estimators. (For further applications of the lemma, see Singh (1977)).

Lemma 5.1 : *Let y, z and B be in R with $z \neq 0$ and $B > 0$. If Y and Z are real valued r.v.'s, then $\forall \gamma > 0$*

$$E(|(Y/Z) - (y/z)| \wedge B)^{\gamma} \leqslant 2^{\gamma + (\gamma-1)^+} |z|^{-\gamma}$$

$$\{E|y - Y|^{\gamma} + (|y/z|^{\gamma} + 2^{-(\gamma-1)^+} B^{\gamma}) E|z - Z|^{\gamma}\}. \qquad \ldots \quad (5.8)$$

Proof : Since $I(2|z - Z| \leqslant |z|) \leqslant I(2|Z| \geqslant |z|)$ the l.h.s. of (5.8) is exceeded by

$$E(|(Y/Z) - (y/z)|^{\gamma} I(2|Z| \geqslant |z|)) + B^{\gamma} E\{I(2|z - Z| \geqslant |z|)\} \qquad \ldots \quad (5.9)$$

By Markov-inequality, the second term in (5.9) is no more than $(2B)^{\gamma}|z|^{-\gamma} \times E|z - Z|^{\gamma}$. By triangle inequality with intermediate term y/Z and by

c_r-inequality(Love, 1963, . p. 155), the first term in (5.9) is bounced by $2^{r+(r-1)^+}|z|^{-r}(E|y-Y|^r+|y/z|^r E|z-Z|^r)$. Q.E.D.

Lemma 5.2 : *For every $0 \leqslant \gamma \leqslant 1$ there is a finite $k_1 = k_1(\gamma)$ such that*

$$P_{i+1}|\psi_{i+1}(X)-\hat{\psi}_{i+1}(X)| \leqslant k_1 h^\gamma\{1+(|\log h|^{\gamma/2}+1)\int(q_\gamma/u_{h+H})^{\gamma/2}d\mu\} \quad \dots \quad (5.10)$$

where $$q_\gamma = (\sup_{\alpha \leqslant w \leqslant \beta} f_w)/(\inf_{\alpha \leqslant w \leqslant \beta} f_w)^{\gamma/2}$$

Proof : First of all we prove that $\exists$ a k_2 independent of i such that

$$P_i|h^{-1}\bar{\hat{\delta}}-\bar{f}| \leqslant k_2 h\bar{f}(1+(\bar{f}u_h)^{-\frac{1}{2}}). \quad \dots \quad (5.11)$$

Note that $P_j(\hat{\delta}_j) = \delta_j = w_j(1-e^{h/w_j})f_j$. Thus, since $0 < \alpha \leqslant w_j \leqslant \beta$, $|h^{-1}P_i(\bar{\hat{\delta}})-\bar{f}| \leqslant (h/\alpha).\bar{f}\exp(h/\alpha)$. Also, since $X_1,\dots,X_i$ are independent var$(\bar{\hat{\delta}})$ =variance $(\bar{\hat{\delta}}) \leqslant i^{-2}\Sigma_1^i P_i(\hat{\delta}_j^2)$. But since $u_h(x) \leqslant u(t)$ for $x \leqslant t < x+h$ a.e. (Lebesgue-measure) and $\delta_j = w_j(1-e^{h/w_j})f_j \leqslant hf_je^{h/\alpha}$, $P_j(\hat{\delta}_j^2) \leqslant (\delta_j/u_h) \leqslant (hf_j e^{h/\alpha})/u_h$. Consequently, var$(\bar{\hat{\delta}}) \leqslant (i^{-1}h\bar{f}e^{h/\alpha})u_h$. By the Schwarz inequality $P_i|\bar{\hat{\delta}}-P_i(\bar{\hat{\delta}})| \leqslant (\text{var}(\bar{\hat{\delta}}))^{\frac{1}{2}}$. Thus, since $|\bar{\hat{\delta}}-\bar{f}| \leqslant |P_i(\bar{\hat{\delta}})-\bar{f}|+|\bar{\hat{\delta}}-P_i(\bar{\hat{\delta}})|$, and $h = i^{-1/3}$, we conclude (5.11).

Now for $j = 1, \dots, i$, $\int_{x+H}^{\infty} f_j = w_j e^{-H/w_j} f_j(x) \leqslant \beta h f_j(x)$, since $H = -\beta\log h$ and $0 < \alpha \leqslant w_j \leqslant \beta$. Therefore,

$$\int_{x+H}^{\infty} \bar{f} \leqslant \beta h\bar{f}(x). \quad \dots \quad (5.12)$$

Now Tonelli-Theorem followed by (5.11), the inequality $u_{h+H}(x) \leqslant u_h(t)$ $\forall x \leqslant t < x+H$ and Schwartz inequality gives

$$P_i \int_x^{x+H} |\bar{f}-h^{-1}\bar{\hat{\delta}}| \leqslant k_2 h\{\int_x^{x+H}\bar{f}+\{H(\int_x^{x+H}\bar{f})/u_{h+H}(x)\}^{\frac{1}{2}}\}$$

$$\leqslant k_2 h\beta\bar{f}(x)\{1+|\log h|^{\frac{1}{2}}(\bar{f}(x)u_{h+H}(x))^{-\frac{1}{2}}\} \quad \dots \quad (5.13)$$

since $\int_x^{\infty}\bar{f} \leqslant \beta\bar{f}(a)$ and $H = -\beta\log h$. Thus Liapunov's inequality, (5.12), (5.13) and c_r-inequality (Loeve, 1963, p. 155) give

$$P_i|\int_x^{\infty}\bar{f}-\int_x^{x+H}h^{-1}\bar{\hat{\delta}}|^\gamma \leqslant \{\int_{x+H}^{\infty}\bar{f}+P_i\int_x^{x+H}|\bar{f}-h^{-1}\bar{\hat{\delta}}|\}$$

$$\leqslant (h\beta\bar{f}(x))^\gamma\{1+k_2^\gamma+|\log h|^{\gamma/2}(\bar{f}(x)u_{h+H}(x))^{-\gamma/2}\}. \quad \dots \quad (5.14)$$

Since $\alpha \leqslant \psi_{i+1}, \hat{\psi}_{i+1} \leqslant \beta$, $|\psi_{i+1}-\hat{\psi}_{i+1}| \leqslant \beta$. Therefore, (5.3) and (5.5) followed by a proper use of Lemma 5.1 give

$$P_i\,|\psi_{i+1}(x)-\hat{\psi}_{i+1}(x)|^\gamma \leqslant 2^\gamma(f(x))^{-\gamma}\{\text{lhs of }(5.14)+2\beta^\gamma P_i|\bar{f}(x)-h^{-1}\bar{\hat{\delta}}(x)|^\gamma\}. \quad \ldots \quad (5.15)$$

For $0<\gamma\leqslant 1$ Holder's inequality implies $P_i(|\cdot|^\gamma) \leqslant P_i(|\cdot|))^\gamma$.

Therefore, since $u_{h+H} \leqslant u_h$, (5.11) followed by c_r-inequality and (5.15) gives a $k_1 = k_1(\gamma)$ independent of i such that

$$\text{r.h.s. of }(5.15) \leqslant k_1 h^\gamma\{1+(\bar{f}(x)u_{h+H}(x))^{-\gamma/2}(|\log h|)^{\gamma/2}+1). \quad \ldots \quad (5.16)$$

Since $X \sim P_{i+1}$ has μ-density f_{i+1} and $f_{i+1}/(\bar{f})^{\gamma/2}) \leqslant q_\gamma$ in the lemma, (5.15), followed by (5.16) leads to (5.10). Q.E.D.

Proof of Theorem 5.1 : Since $C(w) = (\int e^{-x/w}d\mu(x))^{-1}$, and α, β are in $\{w>0 \mid C(w)<0\}$, $C(w)$ are bounded away from 0 and ∞ on $[\alpha, \beta]$. Consequently, $\mu(m) = \int\{\sup_{\alpha\leqslant w\leqslant\beta}(C(w)e^{-x/w})d\mu(x)\} < \infty$. And also, since $(q_\gamma(x)u(x)/u_{h+H}(x))$ by the definition of q_γ in Lemma 5.2 is no more then the integrand in (5.6); by $\alpha \leqslant \psi_i, \hat{\psi}_i \leqslant \beta$ and by (5.6) for each $i = 1, 2, \ldots$

$$\begin{aligned} P_{i+1}|\psi_{i+1}(X)-\hat{\psi}_{i+1}(X)| &\leqslant \beta^{1-\gamma}P_{i+1}|\hat{\psi}_{i+1}(X)-\psi_{i+1}(X)|^\gamma \\ &\leqslant \beta^{1-\gamma}k_1h^\gamma\{1+(|\log h|^{\gamma/2}+1)k_0(1\vee\log i)\} \\ &\leqslant k_3 i^{-\gamma/3}(1\vee(\log i)^\xi) \text{ since } h = i^{-1/3}. \quad \ldots \quad (5.17)\end{aligned}$$

Since X abbreviates X_{i+1} and (5.17) holds for each $i \geqslant 1$ ($k_0, k_1, \ldots$ being independent of i) $n^{-1}\Sigma_{i=2}^n P_i|\hat{\psi}_i(X_i)-\psi_i(X_i)| \leqslant k_3 (\log n)^\xi\Sigma_1^{n-1}i^{-\gamma/3}$. Thus, since N_i introduced in Section 2 are $\leqslant \beta$, the first term on the r.h.s. of (2.5) with φ there replaced by $\hat{\psi}$ is $O(n^{-\gamma/3}(\log n)^\xi)$, and so is the second term there as $\mu(m) < \infty$. Q.E.D.

Now we will give examples where (5.6) of the theorem is satisfied for every $0<\gamma\leqslant 1$ and $0 \leqslant \xi \leqslant \gamma/2$.

Example 5.1 : ($\Gamma(w, \tau)$-family). Let in the component problem the conditional Lebesgue density of X given w be $p_w(x) = (\Gamma(\tau))^{-1}x^{\tau-1}w^{-\tau}e^{-x/w}I(x>0)$, $\tau>0$, $w>0$. Thus $a=0$, $C(w) = (\Gamma(\tau))^{-1}w^\tau$ and $u(x) = x^{\tau-1}I(x>0)$.

By c_r-inequality (Loeve, 1963, p. 155), $u_{h+H}(x) \geq \{x^{\tau-1}I(\tau \geq 1)+(h+H)^{1-\tau}I(0 < \tau<1)\}^{-1} \forall x > 0$. Thus (5.6) holds $\forall\, 2\xi(1+(1-\tau)\, I(0< \tau<1))^{-1} = \gamma \in (0, 1]$ with $\gamma < 2\alpha/\beta$; and we have the following corollary.

Corollary 5.1 : *Let in the component problem the family of distributions be as given in Example* 5.1. *Let* $\Omega \subseteq [\alpha, \beta]$, *where* $0<\alpha < \beta<\infty$, *and* $\hat{\psi}$ *be as given in* (5.5). *Let* $\gamma \in (0, 1]$ *be such that* $\gamma < 2\alpha/\beta$. *Then*

$$\sup_{w \in \Omega^\infty} |D_n(w, \hat{\psi})| = O(n^{-\gamma/3}) \qquad \textit{if} \quad \tau \geq 1,$$
$$= O(n^{-\gamma/3}(\log n)^{\gamma(2-\tau)/2}) \qquad \textit{if}\ 0 < \tau < 1.$$

Let us consider an artificial example just to emphasize the point that our SCE could be a.o. with rates even when the Lebesgue-densities involved contain infinitely many discontinuity points.

Example 5.2 : Let in the component problem the conditional Lebesgue density of X given w be $p_w(x) = w^{-1}(1-e^{-1/w})\left(\sum_0^\infty (j+1)I(j\leq x< j+1)\right) e^{-x/w}$ $I(x>0)$. (Thus $u(x) = \sum_0^\infty (j+1)I(j \leq x < j+1)$ and $a = 0$). Clearly (5.6) is satisfied $\forall\ 2\xi = \gamma \in (0, 1]$ such that $\gamma < 2\alpha/\beta$ since $u(x) \geq 1$ uniformly in x ; and we get

Corollary 5.2 : *Let in the component problem the family of distributions be as given in Example* 5.2. *Let* $\Omega \subseteq [\alpha, \beta]$, $0 < \alpha < \beta < \infty$. *If* $\hat{\psi}$ *be given by* (5.5), *then* $\forall\ 0 < \gamma \leq 1$ such that $\gamma < 2\alpha/\beta$,

$$\sup_{w \in \Omega^\infty} |D_n(w, \hat{\psi})| = O(n^{-\gamma/3}).$$

Notice that through Example 5.1 we have covered the case of sequence-compound estimation where the component problem is SELE of σ^2 in the normal $N(0, \sigma^2)$-family, for X^2 is sufficient for σ^2, where $X \sim N(0, \sigma^2)$.

Susarla (1974b) deals with sequence-compound estimation only in $\Gamma(w, \tau)$-family (described in our Example 5.1). Further, it is not known whether his SCE's are even a.o. if $0 < \alpha < \beta < 2\alpha$ does not hold or if $\tau < 2$; and thus limiting the areas of applications of his estimators. Contrary to his conclusion in his final remark, his estimation, in view of his restriction on τ, does not cover the case of sequence-compound SELE of variance σ^2 in normal $N(0, \sigma^2)$-family, unless he makes at least four observations at each stage.

6. Remarks

In Section 4, $g_j(x) = u(x)f_j(x) = u(x)C(w_j)e^{w_j x}$, therefore on $(0, \infty)$

$$\psi_{i+1} = \frac{\sum_1^i w_j f_j}{\sum_1^i f_j} (\log \bar{f})^{(1)} = (\log \bar{g})^{(1)} - (\log u)^{(1)}$$

where $\bar{f} = i^{-1} \sum_1^i f_j$. If it is known that u is *continuously twice differentiable* on (a, ∞), then taking, as an estimate of ψ_{i+1},

$$\psi^*_{i+1} = (Q(\bar{\delta}^*) - (\log u)^{(1)})_{a,\beta}$$

in (4.4) (instead of $\hat{\psi}_{i+1}(X)$ there), where $\delta^*_j(X) = I(X \leqslant X_j < X+h)$, it it expected that the analysis would become simpler, and perhaps (4.6) could be eliminated (provided a suitable lower bound for "σ^2" in (4.16) is used), and (4.7) could be weakened to

$$\int (u(x))^{1-\gamma/2}\{e^{x(\alpha-(\gamma\beta/2))}I(x \leqslant 0) + e^{x(\beta-(\gamma\alpha/2))}I(x > 0)\}dx \leqslant \infty.$$

Nevertheless, no rate of asymptotic optimality with ψ^* is ensured if the second derivative of u is not continuous on (a, ∞).

In Section 5, $g_j(x) = u(x)f_j(x) = u(x)\, C(w_j)e^{-x/w_j}$, and

$$\psi_{i+1}(x) = \frac{\sum_1^i w_j f_j(x)}{\sum_1^i f_j(x)} = \frac{\int_x^\infty \bar{f}(t)dt}{\bar{f}(x)}$$

where $\bar{f} = i^{-1} \sum_1^i f_j$. Therefore, no matter how somooth is u it does not seem possible to express ψ_i in terms of $\bar{g}$ unless we work with some special form of u and take the help of some auxiliary r.v.'s (see Susarla, 1974b).

The scope of applications of sequence-compound procedures is wide. Situations involving sequences of similar but independent decision problems arise in many areas of applications. Routine bioassay (Chase, 1966) and lot by lot acceptance sampling are typical examples of such situations. In the highly illustrative paper by Neyman (1962) various examples, where compound decision theory or empirical decision theory are applicable, have been noted.

ACKNOWLEDGMENT

The author wishes to thank Professor James Hannan for his guidance, encouragement and valuable comments.

REFERENCES

CHASE, G. R. (1966): An empirical Bayes approach in routine bioassay. Technical report No. 13, Department of Statistics, Stanford University.

GILLILAND, DENNIS C. (1966): Approximation to Bayes risk in sequences of non-finite decision problems. RM-162. Department of Statistical Probability, Michigan State University

——— (1968): Sequential compound estimation. *Ann. Math. Statist.*, 39, 1890-1905.

Graves, Lawrence M. (1956): *The Theory of Functions of Real Variables*, (2nd ed.), Macmillan, New York.

HANNAN, JAMES F. (1956): The dynamic statistical decision problem when the component problem involves a finite number, *m*, of distributions (Abstract). *Ann. Math. Statist.*, 27, 212-

——— (1957): Approximation to Bayes risk in repeated play. *Contributions to the Theory of Games*, 3, 97-139.

HANNAN, JAMES F. and HUANG, J. S. (1972): Equivariance procedures in the compound decision problem with finite state component problem. *Ann. Math. Statist.*, 43, 102-112.

HOEFFDING, Wassily (1963): Probabilities inequalities for sums of bounded random variables *J. Amer. Stat. Assc.*, 58, 13-30.

JOHNS, M. V. JR. (1967): Two-action compound decision problems. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.*, 1, University of California Press.

LOEVE, MICHEL (1963): *Probability Theory*, (3rd Ed.), Van Nostrand, Princeton.

NEYMAN, J. (1962): Two breakthroughs in the theory of Statistical decision making. *Rev. Inter. Statist. Inst.*, 30, 11-27.

SAMUEL, ESTER (1965): On simple rules for the compound decision Problem, *J. Roy. Statist. Soc.*, Ser. B, 27.

——— (1974): Estimation of derivatives of average of -densities and sequence compound estimation in exponential families. RM-381, Department of Statistical Probability, Michigan State University.

SINGH, R. S. (1976): Empirical Bayes estimation with convergence rates in non-continuous Lebesgue exponential families. *Ann. Statist*, 4, 431-439.

——— (1977a): Improvement on some known nonparametric uniformly consistent estimators of derivatives of a density. *Ann. Statist.* 5, 394-399.

——— (1977b): Applications of estimators of a density and its derivatives to certain statistical problems. *J. Royal Statist. Soc.*, Ser. B, 39, Pt. 3.

SUSARLA, V. (1974b): Rate of convergence in the sequence-compound squared-distance loss estimation problem for a family of *m*-variate normal distributions. *Ann. Statist.*, 2, 118-133.

——— (1974b): Rates of convergence in the sequence-compound squared distance loss estimation and linear loss two-action problems for a family of scale parameter exponential distribution. *Inst. Statist. Math.*, 26, 53-67.

Paper received : June, 1974.
Revised : October, 1975.