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SOME FURTHER RESULTS ABOUT UNDOMINATED BAYESIAN EXPERIMENTS. [1]

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Abstract. A summary of the previous papers about undominated Bayesian experiments and some further results are presented in this article.
The main result concerns the case of a pair of statistical observations independent conditionally on the parameter studied in [10]. It allows to individuate (almost surely) the observations (t_1, t_2) which give rise to posterior distributions concentrated on the intersection of the supports of the singular parts of the posteriors related to the single observations t_1 and t_2 separately.
Finally we shall present two counterexamples and an example about the continuous time and homogeneous Markov chains.

1 Introduction.

The structure of *Bayesian experiment* is defined in [4] (page 27, Definition 1.2.1). Let us consider two measurable spaces $(A, \mathcal{A})$ (*parameter space*) and $(S, \mathcal{S})$ (*sample space*), a probability measure μ on $\mathcal{A}$ (*prior distribution*) and a family $(P^a : a \in A)$ of probability measures on $\mathcal{S}$ (*sampling distributions*) such that $(a \mapsto P^a(X) : X \in \mathcal{S})$ are measurable mappings w.r.t. $\mathcal{A}$; then a Bayesian experiment is the (unique) probability space

$$\mathcal{E} = (A \times S, \mathcal{A} \otimes \mathcal{S}, \Pi)$$

such that

$$\Pi(E \times X) = \int_E P^a(X) d\mu(a), \ \ \forall E \in \mathcal{A} \ and \ \forall X \in \mathcal{S}. \tag{1}$$

Furthermore the probability measure P on $\mathcal{S}$

$$X \in \mathcal{S} \mapsto P(X) = \Pi(A \times X) \tag{2}$$

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is called *predictive distribution* and $\mathcal{E}$ is said to be *regular* (see [4], page 27, Definition 1.2.2) if there exists a family $(\mu^s : s \in S)$ of probability measures on $\mathcal{A}$ such that

$$\Pi(E \times X) = \int_X \mu^s(E) dP(s), \ \ \forall E \in \mathcal{A} \ \ and \ \ \forall X \in \mathcal{S}. \tag{3}$$

Throughout this paper we shall think that $(A, \mathcal{A})$ and $(S, \mathcal{S})$ are Polish Spaces so that $\mathcal{E}$ is regular (see [4], page 31, Remark (i)) and the family $(\mu^s : s \in S)$ satisfying (3) is P a.s. unique. It will be useful to denote the two different desintegrations (1) and (3) of Π in a shorter way, i.e.

$$\Pi(da, ds) = P^a(ds)\mu(a) = \mu^s(da)P(ds). \tag{4}$$

Finally the *statistical experiment* $(P^a : a \in A)$ is said to be *dominated (by a σ-finite measure λ)* (see e.g. [1]) if all the sampling distributions are absolutely continuous w.r.t. a σ-finite measure λ, while the Bayesian experiment $\mathcal{E}$ is said to be *dominated* (see [4], page 30, Definition 1.2.4) if

$$\Pi \ll \mu \otimes P. \tag{5}$$

Several questions concerning Bayesian experiments in terms of domination and undomination were studied by the author. The main result shows how the Lebesgue decompositions of the conditional distributions w.r.t. the corresponding marginal distributions are deeply connected to the Lebesgue decomposition of the joint distribution Π w.r.t. the product of its marginal distributions μ and P. Then, before recalling this result, let us introduce the Lebesgue decomposition of Π w.r.t. $\mu \otimes P$:

$$C \in \mathcal{A} \otimes \mathcal{S} \mapsto \Pi(C) = \int_C g d[\mu \otimes P] + \Pi(C \cap D)$$

where $D \in \mathcal{A} \otimes \mathcal{S}$ and $[\mu \otimes P](D) = 0$; moreover we set

$$D(a, .) = \{s \in S : (a, s) \in D\}, \ \ \forall a \in A$$

and

$$D(., s) = \{a \in A : (a, s) \in D\}, \ \ \forall s \in S.$$

Then we have the following (see [7], Proposition 1):

Proposition 1. *μ a.s. the Lebesgue decomposition of P^a w.r.t. P is*

$$X \in \mathcal{S} \mapsto P^a(X) = \int_X g(a, s) dP(s) + P^a(X \cap D(a, .)); \tag{6}$$

P a.s. the Lebesgue decomposition of μ^s w.r.t. μ is

$$E \in \mathcal{A} \mapsto \mu^s(E) = \int_E g(a, s) d\mu(a) + \mu^s(E \cap D(., s)). \tag{7}$$

From now on we shall set:
$A_{\ll} = \{a \in A : P^a \ll P\}$; $A_{\perp} = \{a \in A : P^a \perp P\}$; $A_* = (A_{\ll} \cup A_{\perp})^c$;
$S_{\ll} = \{s \in S : \mu^s \ll \mu\}$; $S_{\perp} = \{s \in S : \mu^s \perp \mu\}$; $S_* = (S_{\ll} \cup S_{\perp})^c$.
All the sets are measurable as an immediate consequence of a Remark in [3] (page 58). However we can say that $S_{\ll}$, $S_{\perp}$ and S_* depend on the choice of the family of posterior

distribution but they are almost surely unique (w.r.t. P); thus, in particular, the probabilities $P(S_{\ll})$, $P(S_{\perp})$ and $P(S_{*})$ do not depend on the choice of the family $(\mu^s : s \in S)$ satisfying (3).

In Section 2 we shall present some relationship between these sets.

We stress that, if we have a statistical experiment $(P^a : a \in A)$ which is dominated by a σ-finite measure λ, for any prior distribution μ there exists a jointly measurable function f_λ such that

$$\mu(\{a \in A : X \in \mathcal{S} \mapsto P^a(X) = \int_X f_\lambda(a,s)d\lambda(s)\}) = 1$$

and we have

$$P(\{s \in S : E \in \mathcal{A} \mapsto \mu^s(E) = \frac{\int_E f_\lambda(a,s)d\mu(a)}{\int_A f_\lambda(a,s)d\mu(a)}\}) = 1$$

(this follows from Lemma 7.4 in [6], page 287). In conclusion, given a dominated statistical experiment $(P^a : a \in A)$, (5) holds whatever the prior distribution μ is; indeed, as a consequence of Proposition 1 (see Corollary in [7]), (5) and $P(S_{\ll}) = 1$ are equivalent conditions.

The case of *statistical observations i.i.d. conditionally on the parameter* plays an important role in the literature. For some arbitrary integer $n \in \mathbb{N}$, this case can be described by considering the ensuing Bayesian experiment

$$\mathcal{E}_{(n)} = (A \times S^n, \mathcal{A} \otimes \mathcal{S}^{\otimes n}, \Pi_{(n)}),$$

where condition (4) is

$$\Pi_{(n)}(da, ds^{(n)}) = (P^a)^{\otimes n}(ds^{(n)})\mu(a) = \mu^{s^{(n)}}(da)P_{(n)}(ds^{(n)}).$$

The undominated case for this frame was studied in [8]. In particular, for a fixed integer $n \in \mathbb{N}$, the condition $P_{(n)}((S^n)_{\perp}) = 0$ was characterized by a condition of absolute continuity concerning $P_{(n)}$. Moreover it was presented an undominated Bayesian experiment $\mathcal{E}_{(1)}$ such that

$$P_{(n)}((S^n)_{\perp}) = 0, \ \ \forall n \in \mathbb{N}; \tag{8}$$

we stress that (8) holds when $\mathcal{E}_{(1)}$ is dominated because we have (see Proposition 3 in [8])

$$P_{(n)}((S^n)_{\ll}) = 1, \ \ \forall n \in \mathbb{N}.$$

In such undominated Bayesian experiment $\mathcal{E}_{(1)}$ we had $A = C[0,1]$ equipped with the smallest σ-algebra such that $(a \mapsto a(t) : t \in [0,1])$ are Borel mappings and, as shown in [12] (see Theorem 2.1, page 212), this is the Borel σ-algebra on $C[0,1]$ relative to the supremum norm. Hence there exist examples of undominated Bayesian experiment $\mathcal{E}_{(1)}$ satisfying (8) with a finite dimensional parameter space A; indeed any two Borel sets (contained in Polish Spaces) of the same cardinality are Borel-isomorphic (see Remark in [5], page 442) and, as a consequence of the Theorem of Alexandrov-Hausdorff (see [5], page 427), all the uncountable Borel subsets of Polish Spaces have the same cardinality (for instance the cardinality of $[0,1]$).

A different approach consists to consider recurrently the case of *a pair of observations independent conditionally on the parameter.* More precisely one can refer to the ensuing triplet of Bayesian experiments $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$:

$\mathcal{E}_1 = (A \times T_1, \mathcal{A} \otimes \mathcal{T}_1, \Pi_1)$ where $\Pi_1(da, dt_1) = P_1^a(dt_1)\mu(da) = \mu^{t_1,\bullet}(da)P_1(dt_1)$;

$\mathcal{E}_2 = (A \times T_2, \mathcal{A} \otimes \mathcal{T}_2, \Pi_2)$ where $\Pi_2(da, dt_2) = P_2^a(dt_2)\mu_(da) = \mu^{\bullet,t_2}(da)P_2(dt_2)$;
$\mathcal{E}_3 = (A \times (T_1 \times T_2), \mathcal{A} \otimes (\mathcal{T}_1 \otimes \mathcal{T}_2), \Pi_3)$ where $\underline{t} = (t_1, t_2)$ and
$\Pi_3(da, d\underline{t}) = P_1^a \otimes P_2^a(d\underline{t})\mu_(da) = \mu^{\underline{t}}(da)P_3(d\underline{t})$;
then, for $s^{(n)} = (s_1, \ldots, s_n)$, we can set
$\mathcal{E}_1 = \mathcal{E}_{(n-1)}$ where $t_1 = s^{(n-1)}$;
$\mathcal{E}_2 = \mathcal{E}_{(1)}$ where $t_2 = s_n$;
$\mathcal{E}_3 = \mathcal{E}_{(n)}$ where $\underline{t} = (t_1, t_2) = (s^{(n-1)}, s_n)$.

Some results concerning the triplet $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ have been proved in [10]; in particular Section 4 in [10] was devoted to illustrate how the results could be adapted to the frame studied in [8].

A completion of the results proved in [10] will presented in Section 3. In particular the main result (Proposition 5) allows to individuate (almost surely) the pairs $\underline{t} = (t_1, t_2)$ which give rise to posterior distributions concentrated on the intersection between the supports of the singular parts of the posteriors related to the single observations t_1 and t_2 separately.

Finally Section 4 is devoted to give two counterexamples while in Section 5 we shall present an example about the continuous time and homogeneous Markov chains.

Before concluding the Introduction, there are two other papers ([9] and [11]) in which the statistical experiment is fixed while one can consider different choices for prior distributions. The subjects in these two papers are presented below. To this aim the sets of all the probability measures on $\mathcal{A}$ and on $\mathcal{S}$ will be denoted by $\mathbb{P}(\mathcal{A})$ and $\mathbb{P}(\mathcal{S})$ respectively; moreover we shall use the symbol Π_μ in place of Π, the symbol P_μ in place of P and it will be useful to rewrite condition (4) as follows:

$$\Pi_\mu(da, ds) = P^a(ds)\mu(a) = \mu^s(da)P_\mu(ds).$$

In [9] the author studied some properties of the *dominating prior distributions* w.r.t. a fixed statistical experiment $(P^a : a \in A)$, i.e.

$$\mathbb{D} = \{\mu \in \mathbb{P}(\mathcal{A}) : \Pi_\mu \ll \mu \otimes P_\mu\}.$$

In [11] the definition of *maximal dominated subset* has been presented: a dominated subset B is said to be maximal if

$$E \in \mathcal{A}^* \Rightarrow \#(E \cap (B)^c) \leq \#\mathbb{N}$$

where $\mathcal{A}^*$ is the family the *dominated subset*, i.e.

$$\mathcal{A}^* = \{E \in \mathcal{A} : \exists Q_E \in \mathbb{P}(\mathcal{S}), a \in E \Rightarrow P^a \ll Q_E\}.$$

Some results relating the maximal dominated subsets and the dominating prior distributions are also presented in [11].

2 Some results about the "general structure".

As shown in [9] (see Proposition 18) we can say that

$$P(S_\perp) \leq 1 - \mu(A_\ll),$$

where, w.r.t. a fixed statistical experiment $(P^a : a \in A)$, the right hand side is the distance of μ from the set of dominating prior distributions (w.r.t. the total variation metric); this was proved in [9] (see Section 2).

By reversing the role of A and S (and consequently the role of μ and P), we obtain an analogous result as an immediate consequence.

Proposition 2. *We have*

$$P(S_{\ll}) \leq 1 - \mu(A_{\perp}).$$

Furthermore we can prove the next

Proposition 3. *We have*

$$P(\{s \in S_{\perp} : \mu^s(A_{\ll}) > 0\}) = 0 \tag{9}$$

and

$$P(\{s \in S_{\ll} : \mu^s(A_{\perp}) > 0\}) = 0. \tag{10}$$

Proof. By (3) and (1) we can say that

$$\int_{S_{\perp}} \mu^s(A_{\ll})dP(s) = \Pi(A_{\ll} \times S_{\perp}) = \int_{A_{\ll}} P^a(S_{\perp})d\mu(a)$$

and, by the definition of $A_{\ll}$ and by the first part of Proposition 1, we obtain

$$\int_{S_{\perp}} \mu^s(A_{\ll})dP(s) = \int_{A_{\ll}} [\int_{S_{\perp}} g(a,s)dP(s)]d\mu(a).$$

Then (9) follows from Fubini Theorem, the definition of $S_{\perp}$ and the second part of Proposition 1; indeed we have

$$\int_{S_{\perp}} \mu^s(A_{\ll})dP(s) \leq \int_A [\int_{S_{\perp}} g(a,s)dP(s)]d\mu(a) = \int_{S_{\perp}} [\int_A g(a,s)d\mu(a)]dP(s) = 0.$$

Similarly we can prove (10). Indeed, by (3) and (1), we can say that

$$\int_{S_{\ll}} \mu^s(A_{\perp})dP(s) = \Pi(A_{\perp} \times S_{\ll}) = \int_{A_{\perp}} P^a(S_{\ll})d\mu(a)$$

and, by the definition of $A_{\perp}$ and by the first part of Proposition 1, we obtain

$$\int_{S_{\ll}} \mu^s(A_{\perp})dP(s) = \int_{A_{\perp}} P^a(S_{\ll} \cap D(a,.))d\mu(a).$$

Then (10) follows; indeed by construction we have

$$\int_{S_{\ll}} \mu^s(A_{\perp})dP(s) \leq \int_A P^a(S_{\ll} \cap D(a,.))d\mu(a) = \int_{S_{\ll}} \mu^s(D(.,s))dP(s) = 0.\Diamond$$

Now let us consider the set

$$S_{\equiv} = \{s \in S : \mu^s \equiv \mu\}$$

where $\equiv$ denotes the condition of mutual absolute continuity. The set $S_{\equiv}$ is measurable; this follows from a slight modification of the arguments used to explain the measurability of $S_{\ll}$. Then we have an immediate consequence of (10).

Proposition 4. *Assume that $\mu(A_\perp) > 0$. Then we have*

$$P(S_\equiv) = 0. \tag{11}$$

Proof. If $\mu(A_\perp) > 0$ holds, (11) immediately follows from (10) because

$$\{s \in S : \mu^s \equiv \mu\} \subset \{s \in S_\ll : \mu^s(A_\perp) > 0\}.\diamond$$

Before concluding this Section, we want to remark that the derivation of the predictive distribution by (2) can be easier than the derivation of a family of posterior distributions by (3). However we can compute $P(S_\ll)$, $P(S_\perp)$ and $P(S_*)$ even if we know the predictive distribution only; indeed we have

$$P(S_\ll) = P(\{s \in S : \int_A g(a,s)d\mu(a) = 1\}),$$

$$P(S_\perp) = P(\{s \in S : \int_A g(a,s)d\mu(a) = 0\})$$

and

$$P(S_*) = P(\{s \in S : \int_A g(a,s)d\mu(a) \in]0,1[\}),$$

where, as a consequence of the first part of Proposition 1, $g(a,s)$ can be seen as a jointly measurable version of $\frac{dP^a}{dP}(s)$.

Now let us consider the two positive measures Q and R defined as follows:

$$X \in \mathcal{S} \mapsto Q(X) = \int_X [\int_A g(a,s)d\mu(a)]dP(s);$$

$$X \in \mathcal{S} \mapsto R(X) = \int_X [1 - \int_A g(a,s)d\mu(a)]dP(s).$$

Then each possible case concerning the mutual Lebesgue decomposition between Q and R corresponds to a situation (in terms of zeros and non-zeros) for the triplet

$$(P(S_\ll), P(S_\perp), P(S_*));$$

thus the ensuing seven cases provide a qualitative classification for the Bayesian experiments.
Case A. $Q \perp R \Leftrightarrow (\geq 0, \geq 0, 0)$;
Case B. $R \ll Q \Leftrightarrow (\geq 0, 0, \geq 0)$;
Case C. $Q \ll R \Leftrightarrow (0, \geq 0, \geq 0)$;
thus we immediately have
Case D. $Q \perp R, Q \ll R, R \ll Q$ *are false* $\Leftrightarrow (> 0, > 0, > 0)$
and, by combining Case B and Case C,
Case E. $Q \equiv R \Leftrightarrow (0,0,1)$, where $Q \equiv R$ obviously means that $Q \ll R$ and $R \ll Q$.
In conclusion we stress that $P = Q + R$ by construction; moreover only the null measure is absolutely continuous and singular w.r.t. a positive measure. Then the last two cases are
Case F. $R = P \Leftrightarrow (0,1,0)$ (by combining Case A and Case C)
and
Case G. $Q = P \Leftrightarrow (1,0,0)$ (by combining Case A and Case B).

3 Completion of the results in [10].

First of all we recall some notation and results in [10].
For $k = 1,2$ let us consider the Lebesgue decomposition of Π_k w.r.t. $\mu \otimes P_k$

$$C \in \mathcal{A} \otimes \mathcal{T}_k \mapsto \Pi_k(C) = \int_C g_k(a,t_k) d[\mu \otimes P_k](a,t_k) + \Pi_k(C \cap D_k)$$

where $[\mu \otimes P_k](D_k) = 0$ and, moreover, set

$$K = \{(a,t_1,t_2) \in A \times T_1 \times T_2 : (a,t_1) \in D_1, (a,t_2) \in D_2\};$$

it is easy to check (see [10], Proposition 2.1) that

$$K(.,\underline{t}) = D_1(.,t_1) \cap D_2(.,t_2)$$

where $\underline{t} = (t_1,t_2)$ and $D_k(.,t_k) = \{a \in A : (a,t_k) \in D_k\}$.
The main result proved in [10] is the following (see Proposition 2.4):

$$P_3(\{\underline{t} = (t_1,t_2) \in T_1 \times T_2 : E \in \mathcal{A} \mapsto \mu^{\underline{t}}(E) = \frac{G(E,\underline{t})}{G(A,\underline{t})}[1 - \mu^{\underline{t}}(K(.,\underline{t}))] + \mu^{\underline{t}}(E \cap K(.,\underline{t}))\}) = 1$$

where

$$G(E,\underline{t}) = \int_E g_1(a,t_1)g_2(a,t_2)d\mu(a) + \int_{E \cap D_1(.,t_1)} g_2(a,t_2)d\mu^{t_1,\bullet}(a) + \int_{E \cap D_2(.,t_2)} g_1(a,t_1)d\mu^{\bullet,t_2}(a);$$

for proving this, the ensuing formula is employed (see [10], Corollary 2.3):

$$X \in \mathcal{T}_1 \otimes \mathcal{T}_2 \mapsto P_3(X) = \int_X G(A,\underline{t})d[P_1 \otimes P_2](\underline{t}) + \int_X \mu^{\underline{t}}(K(.,\underline{t}))dP_3(\underline{t}). \tag{12}$$

Moreover (see [10], Proposition 3.1) we have

$$P_3((T_1 \times T_2)_\perp) \geq \max\{P_1((T_1)_\perp), P_2((T_2)_\perp)\}. \tag{13}$$

Finally, for concluding the presentation of the results in [10], set

$$B_1 = \{\underline{t} = (t_1,t_2) \in T_1 \times T_2 : \mu^{t_1,\bullet}(K(.,\underline{t})) > 0\},$$

$$B_2 = \{\underline{t} = (t_1,t_2) \in T_1 \times T_2 : \mu^{\bullet,t_2}(K(.,\underline{t})) > 0\},$$

$$B_0 = B_1 \cup B_2$$

and

$$B_3 = \{\underline{t} = (t_1,t_2) \in T_1 \times T_2 : \mu^{\underline{t}}(K(.,\underline{t})) = 1\}.$$

Then the following results hold (see [10], Lemma 3.3 and Proposition 3.4):

$$P_1 \otimes P_2(B_0) = 0; \tag{14}$$

$$P_3(B_0) > 0 \Rightarrow P_3(B_3|B_0) = 1.$$

Now one can expect that an inequality similar to (13) holds for absolutely continuous posterior distributions, i.e.

$$P_3((T_1 \times T_2)_{\ll}) \leq \min\{P_1((T_1)_{\ll}), P_2((T_2)_{\ll})\}. \tag{15}$$

It will be shown that (15) fails for Example 1 in the next Section even if

$$P_1((T_1)_{\ll}) = P_2((T_2)_{\ll}) = 0.$$

The main result in this paper consists to individuate (P_3 a.s.) the set B_3, i.e. the set of observations $\underline{t} = (t_1, t_2)$ which give rise to posterior distributions $\mu^{\underline{t}}$ concentrated on the intersection $K(.,\underline{t})$ of the supports of the singular parts of $\mu^{t_1,\bullet}$ and $\mu^{\bullet,t_2}$ (w.r.t. μ).

For doing this, we need to consider the Lebesgue decomposition of P_3 w.r.t. $P_1 \otimes P_2$:

$$X \in \mathcal{T}_1 \otimes \mathcal{T}_2 \mapsto P_3(X) = \int_X H(\underline{t})d[P_1 \otimes P_2](\underline{t}) + P_3(X \cap B) \tag{16}$$

where $[P_1 \otimes P_2](B) = 0$; moreover set

$$F = \{\underline{t} = (t_1, t_2) \in T_1 \times T_2 : G(A, \underline{t}) = 0\}.$$

Then we can prove the next

Proposition 5. *We have*

$$P_3(B_3 \triangle (F \cup B)) = 0. \tag{17}$$

Proof. We shall consider the case $P_3(B) \in]0,1[$ only; the cases $P_3(B_3) = 0$ and $P_3(B_3) = 1$ can be seen as two simplifications.

By taking into account (16), (12) can be rewritten as follows

$$X \in \mathcal{T}_1 \otimes \mathcal{T}_2 \mapsto P_3(X) = \int_X G(A,\underline{t})d[P_1 \otimes P_2](\underline{t}) +$$

$$+ \int_{X \cap B^c} \mu^{\underline{t}}(K(.,\underline{t}))dP_3(\underline{t}) + \int_{X \cap B} \mu^{\underline{t}}(K(.,\underline{t}))dP_3(\underline{t}) =$$

$$= \int_X [G(A,\underline{t}) + \mu^{\underline{t}}(K(.,\underline{t}))H(\underline{t})]d[P_1 \otimes P_2](\underline{t}) + \int_{X \cap B} \mu^{\underline{t}}(K(.,\underline{t}))dP_3(\underline{t});$$

then, by comparing the latter with (16) (in other words by the uniqueness of the Lebesgue decomposition), we obtain:

$$P_3(B_3|B) = 1 \tag{18}$$

and

$$P_3(\{\underline{t} = (t_1, t_2) \in T_1 \times T_2 : G(A,\underline{t}) + \mu^{\underline{t}}(K(.,\underline{t}))H(\underline{t}) = H(\underline{t})\}|B^c) = 1. \tag{19}$$

It is useful to remark that

$$P_3(\{\underline{t} = (t_1, t_2) \in T_1 \times T_2 : H(\underline{t}) > 0\}|B^c) = 1;$$

then (19) can be rewritten as follows

$$P_3(\{\underline{t} = (t_1, t_2) \in T_1 \times T_2 : \mu^{\underline{t}}(K(.,\underline{t})) = \frac{H(\underline{t}) - G(A,\underline{t})}{H(\underline{t})}\}|B^c) = 1$$

and we have

$$P_3(B_3 \triangle F | B^c) = 0. \tag{20}$$

Now we can obtain (17) by proving that

$$P_3(B_3 \triangle (F \cup B) | B^c) = P_3(B_3 \triangle (F \cup B) | B) = 0.$$

Indeed by (20) we have

$$P_3(B_3 \triangle (F \cup B) | B^c) = P_3(B_3 \cap F^c \cap B^c | B^c) + P_3((F \cup B) \cap B_3^c | B^c) =$$
$$= P_3(B_3 \cap F^c | B^c) + P((F \cap B_3^c) \cup (B \cap B_3^c) | B^c) = P_3(B_3 \cap F^c | B^c) +$$
$$+ P_3(F \cap B_3^c | B^c) = P_3(B_3 \triangle F | B^c) = 0$$

and

$$P_3(B_3 \triangle (F \cup B) | B) = P_3(B_3 \cap F^c \cap B^c | B) + P_3((F \cup B) \cap B_3^c | B) =$$
$$= P_3((F \cup B) \cap B_3^c | B) = 0;$$

the last equality follows from (18) that is equivalent to $P_3(B_3^c | B) = 0$. ◇

By taking into account condition (14), one could wonder if the singular part of P_3 w.r.t. $P_1 \otimes P_2$ is concentrated on B_0. As we shall see in the next Section, this is not true for Example 2.

4 Two counterexamples.

In this Section we show two examples; for the first one (15) is false, the second one shows that, in general, we cannot say that the singular part of P_3 w.r.t. $P_1 \otimes P_2$ is concentrated on B_0.

For our aim in general we shall denote by the probability measure concentrated on the singleton x by δ_x and the usual Lebesgue measure on the real line by λ.

Example 1. *Let us consider the following positions:*
$A = T_1 = T_2 = [-\frac{1}{2}, \frac{1}{2}]$, $P_1^0 = P_2^0 = \lambda$ *and* $P_1^a = P_2^a = \delta_a$ *for* $a \neq 0$, $\mu = \frac{1}{2}[\delta_0 + \lambda]$.

Then, for $k = 1, 2$, we have ($\forall E \in \mathcal{A}$ and $\forall X_k \in \mathcal{T}_k$)

$$\Pi_k(E \times X_k) = \int_E P_k^a(X_k) d\mu(a) = \frac{1}{2}[\lambda(X_k) 1_E(0) + \lambda(E \cap X_k)]$$

whence it follows $P_k = \lambda$; thus $P_k((T_k)_*) = 1$ (and in particular $P_k((T_k)_{\ll}) = 0$) because

$$P_1(\{t_1 \in T_1 : \mu^{t_1, \bullet} = \frac{1}{2}[\delta_0 + \delta_{t_1}]\}) = 1$$

and

$$P_2(\{t_2 \in T_2 : \mu^{\bullet, t_2} = \frac{1}{2}[\delta_0 + \delta_{t_2}]\}) = 1.$$

Hence, if (15) holds, we should have $P_3((T_1 \times T_2)_{\ll}) = 0$. On the contrary we have

$$P_3((T_1 \times T_2)_{\ll}) = \frac{1}{2}$$

and

$$P_3((T_1 \times T_2)_\perp) = \frac{1}{2}.$$

We obtain this by noting that ($\forall E \in \mathcal{A}$, $\forall X_1 \in \mathcal{T}_1$ and $\forall X_2 \in \mathcal{T}_2$)

$$\Pi_3(E \times X_1 \times X_2) = \int_E P_1^a(X_1)P_2^a(X_2)d\mu(a) = \frac{1}{2}[\lambda(X_1)\lambda(X_2)1_E(0) + \lambda(E \cap X_1 \cap X_2)]$$

and

$$P_3(X_1 \times X_2) = \frac{1}{2}[\lambda(X_1)\lambda(X_2) + \lambda(X_1 \cap X_2)], \ \ \forall X_1, X_2 \in \mathcal{T}_1 = \mathcal{T}_2;$$

Then, if we set

$$U = \{\underline{t} = (t_1, t_2) \in T_1 \times T_2 : t_1 = t_2\}$$

and

$$V = \{\underline{t} = (t_1, t_2) \in T_1 \times T_2 : t_1 \neq t_2\},$$

we have

$$P_3(\{\underline{t} = (t_1, t_2) \in T_1 \times T_2 : \mu^{\underline{t}} = \delta_0\}|V) = 1$$

and

$$P_3(\{\underline{t} = (t_1, t_2) \in T_1 \times T_2 : \mu^{\underline{t}} = \delta_{t_1} = \delta_{t_2}\}|U) = 1$$

with

$$P_3(U) = P_3(V) = \frac{1}{2}.$$

Now let us consider the second example.

Example 2. *Set $I = [0,1]$, $A = T_1 = T_2 = I \times I \times I$ and $\mu = \lambda \otimes \lambda \otimes \lambda$. Moreover denot the Borel σ-algebra on I by $\mathcal{B}$. Finally set*
$P_1^a = P_1^{(a_1,a_2,a_3)} = \frac{1}{2}[\delta_{(a_1,a_1,a_1)} + \delta_{(a_1,a_2,a_2)}]$, $\forall a \in A$
and
$P_2^a = P_2^{(a_1,a_2,a_3)} = \frac{1}{2}[\delta_{(a_3,a_3,a_3)} + \delta_{(a_2,a_2,a_3)}]$, $\forall a \in A$.

Then we have ($\forall E^1, E^2, E^3, X_1^1, X_1^2, X_1^3 \in \mathcal{B}$)

$$\Pi_1(E^1 \times E^2 \times E^3 \times X_1^1 \times X_1^2 \times X_1^3) =$$

$$= \frac{1}{2}[\lambda(E^1 \cap X_1^1 \cap X_1^2 \cap X_1^3)\lambda(E^2)\lambda(E^3) + \lambda(E^1 \cap X_1^1)\lambda(E^2 \cap X_1^2 \cap X_1^3)\lambda(E^3)]$$

whence we obtain

$$P_1(X_1^1 \times X_1^2 \times X_1^3) = \frac{1}{2}[\lambda(X_1^1 \cap X_1^2 \cap X_1^3) + \lambda(X_1^1)\lambda(X_1^2 \cap X_1^3)];$$

hence, for $t_1 = (t_1^1, t_1^2, t_1^3)$, if we set

$$U_1 = \{t_1 \in T_1 : t_1^1 = t_1^2 = t_1^3\}$$

and

$$V_1 = \{t_1 \in T_1 : t_1^1 \neq t_1^2 = t_1^3\},$$

we have

$$P_1(\{t_1 \in T_1 : \mu^{t_1,\bullet} = \delta_{t_1^1} \otimes \lambda \otimes \lambda\}|U_1) = 1$$

and

$$P_1(\{t_1 \in T_1 : \mu^{t_1,\bullet} = \delta_{t_1^1} \otimes \delta_{t_1^2} \otimes \lambda\}|V_1) = 1$$

with

$$P_1(U_1) = P_1(V_1) = \frac{1}{2}.$$

Similarly we have $(\forall E^1, E^2, E^3, X_2^1, X_2^2, X_2^3 \in \mathcal{B})$

$$\Pi_2(E^1 \times E^2 \times E^3 \times X_2^1 \times X_2^2 \times X_2^3) =$$

$$\frac{1}{2}[\lambda(E^1)\lambda(E^2)\lambda(E^3 \cap X_2^1 \cap X_2^2 \cap X_2^3) + \lambda(E^1)\lambda(E^2 \cap X_2^1 \cap X_2^2)\lambda(E^3 \cap X_2^3)]$$

and

$$P_2(X_2^1 \times X_2^2 \times X_2^3) = \frac{1}{2}[\lambda(X_2^1 \cap X_2^2 \cap X_2^3) + \lambda(X_2^1 \cap X_2^2)\lambda(X_2^3)];$$

hence, for $t_2 = (t_2^1, t_2^2, t_2^3)$, if we set

$$U_2 = \{t_2 \in T_2 : t_2^1 = t_2^2 = t_2^3\}$$

and

$$V_2 = \{t_2 \in T_2 : t_2^1 = t_2^2 \neq t_2^3\},$$

we have

$$P_2(\{t_2 \in T_2 : \mu^{\bullet,t_2} = \lambda \otimes \lambda \otimes \delta_{t_2^3}\}|U_2) = 1$$

and

$$P_2(\{t_2 \in T_2 : \mu^{\bullet,t_2} = \lambda \otimes \delta_{t_2^2} \otimes \delta_{t_2^3}\}|V_2) = 1$$

with

$$P_2(U_2) = P_2(V_2) = \frac{1}{2}.$$

For the Bayesian experiment $\mathcal{E}_3$ we have

$$\Pi_3(E^1 \times E^2 \times E^3 \times X_1^1 \times X_1^2 \times X_1^3 \times X_2^1 \times X_2^2 \times X_2^3) =$$
$$= \frac{1}{4}[\lambda(E^1 \cap X_1^1 \cap X_1^2 \cap X_1^3)\lambda(E^2)\lambda(E^3 \cap X_2^1 \cap X_2^2 \cap X_2^3)+$$
$$+\lambda(E^1 \cap X_1^1 \cap X_1^2 \cap X_1^3)\lambda(E^2 \cap X_2^1 \cap X_2^2)\lambda(E^3 \cap X_2^3)+$$
$$+\lambda(E^1 \cap X_1^1)\lambda(E^2 \cap X_1^2 \cap X_1^3)\lambda(E^3 \cap X_2^1 \cap X_2^2 \cap X_2^3)+$$
$$+\lambda(E^1 \cap X_1^1)\lambda(E^2 \cap X_1^2 \cap X_1^3 \cap X_2^1 \cap X_2^2)\lambda(E^3 \cap X_2^3)]$$

$(\forall E^1, E^2, E^3, X_1^1, X_1^2, X_1^3, X_2^1, X_2^2, X_2^3 \in \mathcal{B})$ whence we obtain

$$P_3(X_1^1 \times X_1^2 \times X_1^3 \times X_2^1 \times X_2^2 \times X_2^3) =$$

$$= \frac{1}{4}[\lambda(X_1^1 \cap X_1^2 \cap X_1^3)\lambda(X_2^1 \cap X_2^2 \cap X_2^3) + \lambda(X_1^1 \cap X_1^2 \cap X_1^3)\lambda(X_2^1 \cap X_2^2)\lambda(X_2^3)+$$
$$+\lambda(X_1^1)\lambda(X_1^2 \cap X_1^3)\lambda(X_2^1 \cap X_2^2 \cap X_2^3) + \lambda(X_1^1)\lambda(X_1^2 \cap X_1^3 \cap X_2^1 \cap X_2^2)\lambda(X_2^3)]$$

while we have

$$P_1 \otimes P_2(X_1^1 \times X_1^2 \times X_1^3 \times X_2^1 \times X_2^2 \times X_2^3) =$$

$$= \frac{1}{4}[\lambda(X_1^1 \cap X_1^2 \cap X_1^3)\lambda(X_2^1 \cap X_2^2 \cap X_2^3) + \lambda(X_1^1 \cap X_1^2 \cap X_1^3)\lambda(X_2^1 \cap X_2^2)\lambda(X_2^3) +$$

$$+\lambda(X_1^1)\lambda(X_1^2 \cap X_1^3)\lambda(X_2^1 \cap X_2^2 \cap X_2^3) + \lambda(X_1^1)\lambda(X_1^2 \cap X_1^3)\lambda(X_2^1 \cap X_2^2)\lambda(X_2^3)];$$

thus the singular part of P_3 w.r.t. $P_1 \otimes P_2$ is concentrated on

$$B = \{(t_1, t_2) \in T_1 \times T_2 : t_1^2 = t_1^3 = t_2^1 = t_2^2\}$$

with

$$P_3(B) = \frac{1}{4}.$$

Now let us consider the following sets:

$$W_1 = (U_1 \times T_2) \cap (T_1 \times U_2);$$

$$W_2 = (U_1 \times T_2) \cap (T_1 \times V_2);$$

$$W_3 = (V_1 \times T_2) \cap (T_1 \times U_2);$$

$$W_4 = (V_1 \times T_2) \cap (T_1 \times V_2) \cap B;$$

thus we have $P_3(W_k) = \frac{1}{4}$ (for $k = 1, 2, 3, 4$). Then, by recalling the notation

$$\underline{t} = (t_1, t_2) = ((t_1^1, t_1^2, t_1^3), (t_2^1, t_2^2, t_2^3)),$$

we have four cases:

Case 1. $P_3(\{\underline{t} \in T_1 \times T_2 : \mu^{\underline{t}} = \delta_{t_1^1} \otimes \lambda \otimes \delta_{t_2^3}\}|W_1) = 1$;
$K(.,\underline{t}) = (\{t_1^1\} \times I \times I) \cap (I \times I \times \{t_2^3\}) = \{t_1^1\} \times I \times \{t_2^3\}$
and in particular $\mu^{\underline{t}}(K(.,\underline{t})) = 1$;
$\mu^{t_1,\bullet}(K(.,\underline{t})) = \delta_{t_1^1} \otimes \lambda \otimes \lambda(\{t_1^1\} \times I \times \{t_2^3\}) = 0$;
$\mu^{\bullet,t_2}(K(.,\underline{t})) = \lambda \otimes \lambda \otimes \delta_{t_2^3}(\{t_1^1\} \times I \times \{t_2^3\}) = 0$.

Case 2. $P_3(\{\underline{t} \in T_1 \times T_2 : \mu^{\underline{t}} = \delta_{t_1^1} \otimes \delta_{t_2^2} \otimes \delta_{t_2^3}\}|W_2) = 1$;
$K(.,\underline{t}) = (\{t_1^1\} \times I \times I) \cap (I \times \{t_2^2\} \times \{t_2^3\}) = \{t_1^1\} \times \{t_2^2\} \times \{t_2^3\}$
and in particular $\mu^{\underline{t}}(K(.,\underline{t})) = 1$;
$\mu^{t_1,\bullet}(K(.,\underline{t})) = \delta_{t_1^1} \otimes \lambda \otimes \lambda(\{t_1^1\} \times \{t_2^2\} \times \{t_2^3\}) = 0$;
$\mu^{\bullet,t_2}(K(.,\underline{t})) = \lambda \otimes \delta_{t_2^2} \otimes \delta_{t_2^3}(\{t_1^1\} \times \{t_2^2\} \times \{t_2^3\}) = 0$.

Case 3. $P_3(\{\underline{t} \in T_1 \times T_2 : \mu^{\underline{t}} = \delta_{t_1^1} \otimes \delta_{t_1^2} \otimes \delta_{t_2^3}\}|W_3) = 1$;
$K(.,\underline{t}) = (\{t_1^1\} \times \{t_1^2\} \times I) \cap (I \times I \times \{t_2^3\}) = \{t_1^1\} \times \{t_1^2\} \times \{t_2^3\}$
and in particular $\mu^{\underline{t}}(K(.,\underline{t})) = 1$;
$\mu^{t_1,\bullet}(K(.,\underline{t})) = \delta_{t_1^1} \otimes \delta_{t_1^2} \otimes \lambda(\{t_1^1\} \times \{t_1^2\} \times \{t_2^3\}) = 0$;
$\mu^{\bullet,t_2}(K(.,\underline{t})) = \lambda \otimes \lambda \otimes \delta_{t_2^3}(\{t_1^1\} \times \{t_1^2\} \times \{t_2^3\}) = 0$.

Case 4. $P_3(\{\underline{t} \in T_1 \times T_2 : \mu^{\underline{t}} = \delta_{t_1^1} \otimes \delta_{t_1^3} \otimes \delta_{t_2^3}\}|W_4) = 1$;
$K(.,\underline{t}) = (\{t_1^1\} \times \{t_1^3\} \times I) \cap (I \times \{t_1^3\} \times \{t_2^3\}) = \{t_1^1\} \times \{t_1^3\} \times \{t_2^3\}$
and in particular $\mu^{\underline{t}}(K(.,\underline{t})) = 1$;

$$\mu^{t_1,\bullet}(K(.,\underline{t})) = \delta_{t_1^1} \otimes \delta_{t_1^3} \otimes \lambda(\{t_1^1\} \times \{t_1^3\} \times \{t_2^3\}) = 0;$$
$$\mu^{\bullet,t_2}(K(.,\underline{t})) = \lambda \otimes \delta_{t_1^3} \otimes \delta_{t_2^3}(\{t_1^1\} \times \{t_1^3\} \times \{t_2^3\}) = 0.$$

In conclusion we can say that $P_3(B_0) = 0$. We can also say that $P_3(B_3) = 1$ because we have $\mu^{\underline{t}}(K(.,\underline{t})) = 1$ in each one of these four cases.

5 An example about the continuous time and homogeneous Markov chains.

Let us start by recalling the terminology for a continuous time and homogeneous Markov chain (Z_t) with state space E. Under the continuity condition

$$\lim_{t \to 0} P(Z_t = j | Z_0 = i) = \delta_{ij} \ \ (i, j \in E),$$

we have (see [2], page 292)

$$\lim_{t \to 0} \frac{P(Z_t = j | Z_0 = i)}{t} = q_{ij} < +\infty \ \ (i \neq j \in E)$$

and

$$\lim_{t \to 0} \frac{1 - P(Z_t = i | Z_0 = i)}{t} = q_i \leq +\infty \ \ (i \in E).$$

Here we assume that (Z_t) is stable and conservative; then, for each $i \in E$, we have $q_i < +\infty$ and

$$q_i = \sum_{j \neq i} q_{ij}.$$

Moreover, for each $i \in E$, we shall use the convention

$$q_{ii} = -q_i$$

and the matrix $Q = (q_{ij})_{i,j \in E}$ is called *Q-matrix* of (Z_t).

After these preliminaries, we shall present the example we want to study. Let us suppose that:
$Z = (Z_t)$ defined above is irriducible;
$P(Z_0 = z_0) = 1$ for some $z_0 \in E$;

$$Z_t = (X_t, Y_t)$$

where $X = (X_t)$ and $Y = (Y_t)$ are also Markov chains with the same hypothesis. More precisely we have

$$E = E_X \times E_Y$$

where E_X is the state space of X and E_Y is the state space of Y.
Furthermore the Q-matrix of X will be denoted by

$$Q^X = (q^X_{x_0 x_1})_{x_0, x_1 \in E_X}$$

and the Q-matrix of Y will be denoted by

$$Q^Y = (q^Y_{y_0 y_1})_{y_0, y_1 \in E_Y};$$

thus we shall use the notation

$$Q = \big(q_{(x_0,y_0)(x_1,y_1)}\big)_{(x_0,y_0),(x_1,y_1)\in E}.$$

Thus, by referring to the previous terminology, we shall consider X as the parameter (*signal*) and Y as the sample (*observation*); thus the law of Z is Π, the law of X is μ and the law of Y is P. Moreover we remark that Z is also a Markov chain under the law $\mu \otimes P$ and we shall denote by

$$Q^* = \big(q^*_{(x_0,y_0)(x_1,y_1)}\big)_{(x_0,y_0),(x_1,y_1)\in E}$$

the correspondent intensity matrix; in this case the condition of Bayesian domination (5) is equivalent to

$$q^*_{(x_0,y_0)(x_1,y_1)} = 0 \Rightarrow q_{(x_0,y_0)(x_1,y_1)} = 0, \quad ((x_0,y_0) \neq (x_1,y_1) \in E). \tag{21}$$

Let us start by considering the following result which illustrates the link between Q^X, Q^Y and Q^*.

Proposition 6. *We have:*

$$x_0 \neq x_1, y_0 \neq y_1 \Rightarrow q^*_{(x_0,y_0)(x_1,y_1)} = 0; \tag{22}$$

$$x_0 = x_1 = x, y_0 \neq y_1 \Rightarrow q^*_{(x,y_0)(x,y_1)} = q^Y_{y_0 y_1}; \tag{23}$$

$$x_0 \neq x_1, y_0 = y_1 = y \Rightarrow q^*_{(x_0,y)(x_1,y)} = q^X_{x_0 x_1}. \tag{24}$$

Proof. Let us consider $(x_0,y_0) \neq (x_1,y_1)$. Then we have

$$q^*_{(x_0,y_0)(x_1,y_1)} = \lim_{t\to 0} \frac{\mu \otimes P(Z_t = (x_1,y_1)|Z_0 = (x_0,y_0))}{t} =$$

$$= \lim_{t\to 0} \frac{\mu(X_t = x_1|X_0 = x_0)P(Y_t = y_1|Y_0 = y_0) - \delta_{(x_0,y_0)(x_1,y_1)}}{t} =$$

$$= \lim_{t\to 0} \frac{\mu(X_t = x_1|X_0 = x_0)P(Y_t = y_1|Y_0 = y_0) - \delta_{x_0x_1}P(Y_t = y_1|Y_0 = y_0)}{t}$$

$$\frac{+\delta_{x_0x_1}P(Y_t = y_1|Y_0 = y_0) - \delta_{x_0x_1}\delta_{y_0y_1}}{t} =$$

$$= \lim_{t\to 0}[P(Y_t = y_1|Y_0 = y_0)\frac{\mu(X_t = x_1|X_0 = x_0) - \delta_{x_0x_1}}{t} + \delta_{x_0x_1}\frac{P(Y_t = y_1|Y_0 = y_0) - \delta_{y_0y_1}}{t}] =$$

$$= \delta_{y_0y_1}\lim_{t\to 0}\frac{\mu(X_t = x_1|X_0 = x_0) - \delta_{x_0x_1}}{t} + \delta_{x_0x_1}\lim_{t\to 0}\frac{P(Y_t = y_1|Y_0 = y_0) - \delta_{y_0y_1}}{t}.$$

This completes the proof, as one can directly check by considering all the possible cases. ◊

As an immediate consequence we have the following

Proposition 7. *(21) holds if and only if we have*

$$x_0 \neq x_1, y_0 \neq y_1 \Rightarrow q_{(x_0,y_0)(x_1,y_1)} = 0. \tag{25}$$

Proof. If (21) holds, (25) follows from (22).
Then now assume that (25) holds. First of all by (22) we can say that

$$q^*_{(x_0,y_0)(x_1,y_1)} = 0 \Rightarrow q_{(x_0,y_0)(x_1,y_1)} = 0$$

when $x_0 \neq x_1$ and $y_0 \neq y_1$. Then we obtain (21) by proving that

$$q^*_{(x,y_0)(x,y_1)} = 0 \Rightarrow q_{(x,y_0)(x,y_1)} = 0, \ \ (x \in E_X, y_0 \neq y_1 \in E_Y). \tag{26}$$

and

$$q^*_{(x_0,y)(x_1,y)} = 0 \Rightarrow q_{(x_0,y)(x_1,y)} = 0, \ \ (x_0 \neq x_1 \in E_X, y \in E_Y). \tag{27}$$

Let us start with (26). For $y_0 \neq y_1 \in E_Y$ we have

$$\begin{aligned} q_{(x,y_0)(x,y_1)} &= \lim_{t\to 0} \frac{\Pi(Z_t = (x,y_1)|Z_0 = (x,y_0))}{t} \\ &= \lim_{t\to 0} \frac{\Pi(X_t = x, Y_t = y_1, X_0 = x, Y_0 = y_0)}{t\Pi(X_0 = x, Y_0 = y_0)} = \\ &= \lim_{t\to 0} \frac{\Pi(X_t = x, Y_t = y_1, X_0 = x, Y_0 = y_0)}{t\Pi(Y_0 = y_0)} \frac{\Pi(Y_0 = y_0)}{\Pi(X_0 = x, Y_0 = y_0)} \leq \\ &\leq \frac{\Pi(Y_0 = y_0)}{\Pi(X_0 = x, Y_0 = y_0)} \lim_{t\to 0} \frac{\Pi(Y_t = y_1, Y_0 = y_0)}{t\Pi(Y_0 = y_0)} = \frac{\Pi(Y_0 = y_0)}{\Pi(X_0 = x, Y_0 = y_0)} q^Y_{y_0 y_1} \end{aligned}$$

and (26) follows from (23). In a similar way (27) follows from (24). ◊

In conclusion the condition "the probability to have common jumps for $(Z_t)_{t\in[0,T]}$ is null (for any arbitrary $T > 0$)" is equivalent to the Bayesian domination; this follows from the equivalence between (5) and (21) and by taking into account the interpretation of the entries of Q (see [2], page 293, equation (8.15)).

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