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THE STABLE LAWS REVISITED

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SUMMARY. An alternative derivation is given for the familiar closed-form formulas for the stable characteristic functions.

We recall that a non-degenerate distribution function F on R^1 is called *stable* if, for every $n \in Z^+$ (the set of positive integers), the n-fold convolution of F with itself belongs to the same 'type' as F, i.e., if there exist $a_n > 0$ and real b_n such that

$$F^{n*} = F\left(\frac{\cdot - b_n}{a_n}\right) \qquad \text{for all } n \in Z^+. \qquad \ldots \quad (1)$$

If f be the characteristic function (c.f.) of F, then (1) is equivalent to

$$[f(t)]^n = f(a_n t)e^{ic_n t} \qquad \text{for all} \quad n \in Z^+, t \in R^1 \qquad \ldots \quad (2)$$

for some $a_n > 0$ and real c_n.

The sequence of relations (2) lead to the following explicit formula for $\phi = \log f$ (f is necessarily non-vanishing on R^1 and a continuous version of $\log f$ vanishing at the origin exists) :

$$\text{for } t > 0, \quad \phi(t) = \begin{cases} iat - ct^\alpha(1+i\theta) & \text{if} \quad \alpha \neq 1; \qquad \ldots \quad (3a) \\ iat - ct(1+i\lambda \log t) & \text{if} \quad \alpha = 1, \qquad \ldots \quad (3b) \end{cases}$$

the Hermitian property : $\overline{\phi(t)} = \phi(-t)$ taking care of values of $t < 0$. The parameters in (3) are subject to the following restrictions : $0 < \alpha \leqslant 2$, with $\alpha = 2$ corresponding to (and only to) the normal laws; a, θ and λ are real ; $c > 0$; $|\theta| \leqslant \tan(\pi\alpha/2)$ and $|\lambda| \leqslant 2/\pi$.

For a complete exhibition of the equivalence of (2) and (3), the usual—and only known— procedure is to first note that (2) implies the infinite divisibility of f (and hence in particular its non-vanishingness and the existence of ϕ on R^1); then identify the exact forms of the Levy functions in the Levy canonical representation for ϕ; and finally compute the integrals involved in that representation to obtain (3); also, conversely, a complex-valued function ϕ given on R^1 by (3), with the parameters restricted as stated, necessarily corresponds to an infinitely divisible c.f. f which satisfies (2) with $a_n = n^{1/\alpha}$ and a suitable real c_n. For details, refer for instance to Gnedenko and Kolmogorov (1968, Sections 33 and 34).

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The present Note has a modest aim : to point out a "quick" (and more transparent) derivation of (3) from (2). Instead of going into the infinite divisibility argument, we appeal to the fact (provable from first principles by a probabilistic-analytic argument as in Feller (1971, p. 170) that, if (2) holds, then $a_n = n^{1/\alpha}$ for some $\alpha > 0$. [or, denoting $|f|^2$ by g, so that $g \geqslant 0$ in particular, we have $g(a_{mn}\, t) = g(a_m . a_n t)$ for all t, from (2), so that $g \not\equiv 1$ implies that $a_{mn} = a_m . a_n$. Also, $g(a_{n+1}\, t) = g(a_n\, t) . g(t) \leqslant g(a_n\, t)$, also from (2), so that $g \not\equiv 1$ again implies that $a_n \leqslant a_{n+1}$ (if $a_{n+1}/a_n = \lambda < 1$, then $g(t) \geqslant g(\lambda t) \geqslant \ldots \geqslant g(\lambda^n t) \rightarrow g(0) = 1$ as $n \rightarrow \infty$). Now appeal to the fact that if the function h defined on Z^+, the set of positive integers, has the properties : (i) $h(mn) = h(m) . h(n)$, $m, n \in Z^+$ and (ii) h is non–decreasing, then ($h \not\equiv 0$ or $h \not\equiv 1$ or) there exists an $\alpha > 0$ such that $h(n) = n^{1/\alpha}$ for all $n \in Z^+$.]

Let us then take it for granted that $a_n = n^{1/\alpha}$ with $\alpha > 0$, in (2). If $\alpha > 2$, then $\log | f(t_n) | / t_n^2 \rightarrow 0$ as $n \rightarrow \infty$, and if $\alpha = 2$, then $\log | f(t_n) | / t_n^2 = \log |f(1)|$ where $t_n = n^{-1/\alpha}$. It follows from Ramachandran and Rao (1968) that F is degenerate (contrary to assumption) if $\alpha > 2$, so that $\alpha \leqslant 2$, and that F is normal if $\alpha = 2$. Our proof that (3) holds for $0 < \alpha < 2$ would then imply also that F is normal *only if* $\alpha = 2$. Incidentally, the above argument tacitly uses the Lévy-Cramér theorem on the decomposition of the normal law.

Let us then consider the cases $0 < \alpha < 2$. Setting $\beta = 1/\alpha$, we may write (2) in the form

$$[f(t)]^n = f(n^{\beta} t) e^{itc_n} \qquad \ldots (4)$$

(4) is easily seen to imply that f is non-vanishing on R^1, so that we may speak of $\phi = \log f$ (chosen to be continuous on R^1 with $\phi(0) = 0$). Let $M(t) = | f(t) |$ and $\phi(t) = \log M(t) + iA(t)$, so that A is also continuous on R^1 with $A(0) = 0$. Since (4) implies that

$$[M(t)]^n = M(n^{\beta} t) \text{ for all } n \in Z^+,\ t \in R^1 \qquad \ldots (5)$$

we have for all $m, n \in Z^+$,

$$M(n^{\beta}) = [M(1)]^n \quad \text{and} \quad M(m^{\beta}/n^{\beta}) = [M(m^{\beta})]^{1/n} = [M(1)]^{m/n}$$

whence, by the continuity of M,

$$M(t) = [M(1)]^{t^{\alpha}} \qquad \text{for} \quad t > 0,$$

or,

$$M(t) = \exp(-c|t|^{\alpha}) \text{ for all } t \in R^1;\ c > 0 \qquad \ldots (6)$$

Also,

$$nA(t) = A(n^{\beta} t) + c_n t$$

so that

$$\begin{aligned} mnA(t) &= A(m^{\beta} n^{\beta} t) + c_{mn} t \\ &= mA(n^{\beta} t) - c_m . n^{\beta} t + c_{mn} t \\ &= m[nA(t) - c_n t] - c_m . n^{\beta} t + c_{mn} t \end{aligned}$$

so that $$c_{mn} = mc_n + c_m n^\beta = nc_m + c_n m^\beta \text{ by symmetry}$$

whence $$c_m(n-n^\beta) = c_n(m-m^\beta) \text{ for } m, n \in Z^+. \quad \ldots \quad (7)$$

In the case $\beta \neq 1$ (equivalently, $\alpha \neq 1$), we therefore have

$$c_n = a(n-n^\beta) \quad \text{for all } n \in Z^+$$

where a is a real constant. Setting $\psi(t) = A(t)-at$,

we have $$n\psi(t) = \psi(n^\beta t) \quad \text{for } t \in R^1 \text{ and } n \in Z^+,$$

which easily leads, as in the case of relation (5), to

$$\psi(t) = kt^\alpha \quad \text{for } t > 0 \quad (\alpha \neq 1). \quad \ldots \quad (8)$$

(3a) follows at once from (6) and (8).

If $\beta = \alpha = 1$, (7) does not help any; however, noting that now $nA(t) = A(nt)+c_n t$, we see easily that

$$A(mnt)-mA(nt)-nA(mt)+mnA(t) = 0.$$

Setting $\mathcal{L}(t) = A(t)/t$ and $\xi(u) = \mathcal{L}(e^u)$, we have for all real u, with $a_n = \log n$ for all $n \in Z^+$,

$$\xi(u+a_m+a_n)-\xi(u+a_m)-\xi(u+a_n)+\xi(u) = 0.$$

If we fix $m \in Z^+$ and set $\eta(u) = \xi(u+a_m)-\xi(u)$, then $\eta(u+a_n) = \eta(u)$ for all real u and $n \in Z^+$, whence $\eta(pa_r+qa_n) = \eta(0)$ whatever be the integers p, q, r, n, with r and n positive. Choosing and fixing r and n such that a_r/a_n is irrational, and noting that then the set $\{pa_r+qa_n : p, q \in Z\}$ is dense in R^1, we see that $\eta(u) = \eta(0) = \xi(a_m)-\xi(0)$, so that

$$\xi(u+a_m) = \xi(u)+\xi(a_m)-\xi(0)$$

for all $u \in R^1$ and $m \in Z^+$. Setting $\chi(u) = \xi(u)-\xi(0)$,

we have $$\chi(u+a_m) = \chi(u)+\chi(a_m) \text{ for all } u \in R^1,\ m \in Z^+$$

so that $$\chi(u+pa_m+qa_n) = \chi(u)+p\chi(a_m)+q\chi(a_n)$$
$$= \chi(u)+\chi(pa_m+qa_n)$$

which in turn implies that

$$\chi(u+v) = \chi(u)+\chi(v) \quad \text{for all real } u \text{ and } v.$$

This is the familiar Cauchy functional equation and, in view of the continuity of χ, leads to : $\chi(u) = ku$ or $\xi(u) = ku+l$, or

$$A(t) = at-ct.\log t. \quad \ldots \quad (9)$$

Relation (3b) follows from (6) and (9).

Remark: The restriction $|\theta| \leqslant \pi\alpha/2$ in formula (3a) can be obtained in the case $0 < \alpha < 1$ (taking $a = 0$ there without loss of generality) by expressing the condition that $p(0) \geqslant 0$, where $p = F'$ is the continuous version of the density function of F (the existence of which version is guaranteed by the absolute integrability of f). If γ is the unique value of $\tan^{-1}\theta$ in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, then

$$\pi p(0) = \text{Re} \int_0^\infty \exp(-c't^\alpha . e^{i\gamma})dt = \text{Re} \int_0^\infty \exp(-c'y^\alpha) . \quad e^{-i(\gamma/\alpha)}\, dy,$$

the last equality being easily justified via contour integration, and the last expression is non-negative if and only if $\cos(\gamma/\alpha) \geqslant 0$, i.e., $|\gamma/\alpha| \leqslant \frac{1}{2}\pi$, i.e., $|\theta| \leqslant \tan(\frac{1}{2}\pi\alpha)$. Such a simple argument does not appear available for the cases $\alpha \geqslant 1$. In any event, as stated at the outset, for proving the (necessity and) *sufficiency* of these restrictions in order that the ϕ given by (3) correspond to c.f.'s, there seems to be no alternative to the Levy representation approach. The derivation given in this Note does appear, however, to clarify better why the explicit formula for $\phi = \log f$ takes different forms for the cases $\alpha = 1$ and $\alpha \neq 1$.

References

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