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TEST OF SYMMETRY OF A BIVARIATE DISTRIBUTION*

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SUMMARY. Suitable alternative hypotheses are defined for testing symmetry of a bivariate distribution against that one component is stochastically larger than the other. A sufficient condition for tests based on rank order statistics to be unbiased for these problems is obtained and explicit forms of the tests are given.

0. Introduction

The purpose of this paper is to find reasonable alternative hypotheses for testing the hypothesis of symmetry of a bivariate distribution and to construct unbiased tests for these problems.

For this purpose the definitions of stochastical largerness introduced by the authors (1972a) are reexamined and two new definitions are added. The properties of order statistics and rank order statistics in our set-up are discussed and their characteristics are found out. The similar tests in our problem are limited to the conditional sign tests. Theorems 3.1 and 3.2 give unbiased tests. The test statistics must have monotonicity described in Definition 3. Two classes of statistics of explicit form are given. Finally the normal approximation of test criteria is discussed.

This paper is a part of the authors' works on a unified approach to the nonparametric distribution theory.

1. Deficiency of conventional tests of symmetry

The problem of testing symmetry of a two-dimensional random variable (X, Y) is important for practical comparison of two treatments since this set-up assumes very little on the observations. The conventional tests for this hypothesis are the rank tests of symmetry of $X-Y$, the sign test as its special case, and the two sample tests of equality of the marginal distributions disregarding the association between the components X and Y. These tests, however, are vulnerable as discussed below.

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To study the tests we need adequate alternative hypotheses. The authors (1972a) introduced the notion that "the component X of a random variable (X, Y) is stochastically larger than the component Y in the sense $\mathcal{K}$", denoting it by $X \succ Y\ (\mathcal{K})$, for several $\mathcal{K}$'s of different strictness. They are $\mathcal{K}_{3A}$, $\mathcal{K}_3$, $\mathcal{K}_{2A}$, $\mathcal{K}_2$, $\mathcal{K}_{1A}$, $\mathcal{K}_1$, $\mathcal{K}_{II}$, $\mathcal{K}_I$, $\mathcal{K}_0$ and $\mathcal{K}_S$, and have a hierarchy of implication relation. See Fig. 1 of this section and Fig. 2 of (1972a).

The alternative hypothesis usually conceived for the sign test is $X \succ Y\ (\mathcal{K}_0)$, or $P(X > Y) \geqslant P(X < Y)$. That for the symmetry test of $X-Y$ is $X \succ Y\ (\mathcal{K}_I)$, or $P(X-Y > a) \geqslant P(X-Y < -a)$ for any $a > 0$. That for equality test of the marginal distribution is $X \succ Y\ (\mathcal{K}_1)$, or $P(X \leqslant a) \leqslant P(Y \leqslant a)$ for any real a.

(1) For $\mathcal{K} = \mathcal{K}_0$, $\mathcal{K}_1$, $\mathcal{K}_I$ and $\mathcal{K}_{II}$ $X \succ Y\ (\mathcal{K})$ and $X \prec Y\ (\mathcal{K})$ do not imply the symmetry of (X, Y). However $\mathcal{K} = \mathcal{K}_2$ or $\mathcal{K}_{1A}$ imply the symmetry.

(2) If the marginal distributions of (X, Y) are equal, $X \succ Y\ (\mathcal{K}_2)$ and $X \succ Y\ (\mathcal{K}_{1A})$ are contradictory conditions and they imply $X \succ Y\ (\mathcal{K}_0)$ and $X \prec Y\ (\mathcal{K}_0)$ respectively.

(3) $X \succ Y\ (\mathcal{K}_{II}$ or $\mathcal{K}_I)$ does not imply $X \succ Y\ (\mathcal{K}_{1A}$ or $\mathcal{K}_1)$ and vice versa.

We should therefore choose stronger condition as the alternative hypothesis. The tests for the alternatives $X \succ Y\ (\mathcal{K}_{II}$ or $\mathcal{K}_I)$ were studied thoroughly in Yanagimoto and Sibuya (1972b). More alternatives are necessary for our purpose.

In addition to $\mathcal{K}$'s mentioned above we introduce the following two definitions.

Definition 1 : (1) $X \succ Y\ (\mathcal{K}_{3I})$ iff (X, Y) has the same distribution with $(f(U), g(V))$, where (U, V) is a symmetric random variable and f and g are monotone increasing functions such that $f(s) \geqslant g(s)$, $-\infty < s < \infty$.

(2) $X \succ Y\ (\mathcal{K}_4)$ iff $P(S(a_2, a_3; b_1, b_2))\cdot P(s(b_2, b_3; a_1, a_2)) \geqslant P(S(b_1, b_2; a_2, a_3))\cdot P(S(a_1, a_2; b_2, b_3))$ for all $b_1 < b_2 < b_3 < a_1 < a_2 < a_3$, where $S(a; a'; b, b') = \{(x, y); a < x \leqslant a'$ and $b < y \leqslant b'\}$. If (X, Y) has the density function $f(x, y)$ this is equivalent to $f(a', b)\cdot f(b', a) \geqslant f(a, b')\cdot f(b, a')$.

Proposition 1.1 : (1) *The implication relations in Figure 1 and only these hold. The* $\mathcal{R}_i \to \mathcal{R}_j$ *indicates that if* $X \succ Y$ $(\mathcal{R}_i)$ *then* $X \succ Y$ $(\mathcal{R}_j)$.

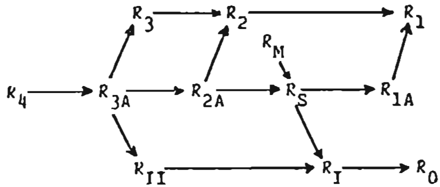

Fig. 1

(2) *If X and Y are statistically independent, then* $\mathcal{R}_M$, $\mathcal{R}_S$, $\mathcal{R}_{1A}$ *and* $\mathcal{R}_1$ *are equivalent.*

(3) *If* (X, Y) *degenerates on the line* $y = -x$, *then* $X \succ Y$ $(\mathcal{R}_M, \mathcal{R}_S$ *or* $\mathcal{R}_I)$ *is equivalent to* $X - Y \succ 0$ $(\mathcal{B}_1)$ *and* $X \succ Y$ $(\mathcal{R}_4)$ *to* $X - Y \succ 0$ $(\mathcal{B}_4)$, *where* $\mathcal{B}$*'s show positive biasedness introduced in* (1972b).

Proof : *Proof of* (2) : If X and Y are statistically independent, and $X \succ Y$ $(\mathcal{R}_M)$, then $X = f(U)$ and $Y = g(U')$, for some functions $f(s) \geqslant g(s)$, $-\infty < s < \infty$, and identically and independently distributed random variables U and U'. This means just $X \succ Y$ $(\mathcal{R}_1)$. (3) is clear from the discussion in (1972b). To prove (1), the implications are clear from the definition of $\mathcal{R}$'s. (2) and (3) mean the impossibility of the implications $\mathcal{R}_M \to \mathcal{R}_2$, $\mathcal{R}_M \to \mathcal{R}_{II}$ and $\mathcal{R}_{3A} \to \mathcal{R}_4$. It is necessary only to show that $\mathcal{R}_4 \to \mathcal{R}_M$ is impossible. We remark that if $X \succ Y$ $(\mathcal{R}_M)$ and the marginal distributions of $(f(X), g(Y))$ are equal for f and g in Definition 1, then $(f(X), g(Y))$ must be symmetry. The following distribution on 5×5 points satisfies $X \succ Y$ $(\mathcal{R}_4)$. $(g(X), Y)$ for $g(x) = x - 1$ has the same marginal distributions but is not symmetry.

$y \diagdown x$	1	2	3	4	5	sum
5	0	0	0	0	0	0
4	0	0	1/6	0	0	1/6
3	0	1/6	0	1/6	0	1/3
2	0	0	1/6	1/6	0	1/3
1	0	0	0	0	1/6	1/6
sum	0	1/6	1/3	1/3	1/6	

2. Properties of some statistics

We define the order statistics of a random sample $(X, Y) = ((X_i, Y_i), i = 1, \ldots, n)$ as the ordered set of triples $((U_i, V_i, Z_i), i = 1, \ldots, n)$, where U_i is the i-th largest number of $\max(X_j, Y_j)$ and V_i is the value of $\min(X_j, Y_j)$ corresponding to $U_i = \max(X_j, Y_j)$. If $U_i = U_{i+1}$ then the order is determined so that $V_i > V_{i+1}$. For simplicity, we write the order statistics as (Z, T), where $Z = (Z_1, \ldots, Z_n)$ and $T = ((U_1, V_1), \ldots, (U_n, V_n))$.

Proposition 2.1 : (1) *(Z, T) is sufficient and complete for the family of all symmetric distributions $\mathcal{P}$.*

(2) *Let z be an n-vector with components -1, 0 and 1. $P(Z = z \mid T = t) = 2^{-s}$, $s = \Sigma z_i$, for any distribution of $\mathcal{P}$.*

(3) *Let $\phi(x, y)$ be a test function such that $\phi(X, Y) = \phi_0(Z, T)$. If $E_F[\phi(X, Y)] = \alpha$ for any distribution $F \in \mathcal{P}$, then $E_F[\phi(X, Y) \mid T = t] = \alpha$. That is, all similar test statistics are conditional sign test statistics.*

(4) *If $E_F[\phi(X, Y)] = \alpha$ for any absolutely continuous distribution F of $\mathcal{P}$ and for $\phi(x, y)$ of (3), then $E_F[\phi(X, Y) \mid T = t] = \alpha$ for a.e. t.*

Proof: The proof of (3) is similar to Lehmann (1959, p. 152). Others are simply proved.

Now we find maximal invariant statistics with respect to two invariant transformation groups.

Proposition 2.2 : (1) *Let f be a monotone increasing function. $X \prec Y\ (\mathcal{R})$ implies $f(X) \prec f(Y)\ (\mathcal{R})$ for $\mathcal{R} = \mathcal{R}_4, \mathcal{R}_{14}, \mathcal{R}_M$* or *$\mathcal{R}_S$.*

(2) *Let $r : R^2 \to R^1$ be a Borel measurable function such that $r(x', y) \geq r(x, y')$ for any $x' \geq x$ and $y' \leq y$. $X \succ Y\ (\mathcal{R})$ implies $r(X, Y) = r(Y, X)\ (\mathcal{R})$ for $\mathcal{R} = \mathcal{R}_{14}$ or $\mathcal{R}_S$. The statement is not always valid for $\mathcal{R} = \mathcal{R}_4$ or $\mathcal{R}_M$.*

Proof: (1) is clear from the definition $\mathcal{R}$'s. The first part of (2) is shown by direct computation. The counter examples for the last statement are as follows.

The following distribution on 5×5 points satisfies $X \succ Y\ (\mathcal{R}_4)$ but $r(X, Y) \succ r(Y, X)\ (\mathcal{R}_4)$ does not hold for $r(x, y) = x - y$.

$y \diagdown x$	1	2	3	4	5
5	1/8	0	0	0	0
4	0	1/8	0	0	0
3	0	0	0	0	0
2	1/8	0	0	1/8	0
1	0	1/4	0	0	1/4

Consider the uniform density on the segment between the two points (1/3, 2/3) and (1, 0), which satisfies $X \succ Y\ (\mathcal{H}_M)$. Let

$$r(x, y) = \begin{cases} 2x-y, & x \geqslant y, \\ x, & x < y. \end{cases}$$

$(r(X, Y), r(Y, X))$ has the uniform densities on the segments between (1/3, 1) and (1/2, 1/2) and between (1/2, 1/2) and (2, 0) and does not satisfy $r(X, Y) > r(Y, X)\ (\mathcal{H}_M)$.

Definition 2 : (1) G_1 is the set of all transformations $\phi : R^2 \to R^2$ such that $\phi(x, y) = (f(x), f(y))$, where f as well as f^{-1} are monotone increasing functions.

(2) G_2 is the set of all transformations $\phi : R^2 \to R^2$ such that $\phi(x, y) = (r(x, y), r(y, x))$, where r as well as r^{-1} are functions satisfying $r(x', y) \geqslant r(x, y')$ for any $x' \geqslant x$ and $y' \geqslant y$.

G_1 is a proper subset of G_2. G_2 includes transformations like $r(x, y) = -y$ and $r(x, y) = x+h(x-y).c(x-y)$, where h is a continuous and monotone increasing with $h(0) = 0$, and $c(s)$ is the step function with the value 1 for $s \geqslant 0$ and 0 for $s < 0$. Both transformation groups look very natural. It means, from another point of view, that the role of G_1 and G_2 is not so definitive as other typical nonparametric test problems.

In the following throughout the paper we assume the distribution function of (X, Y) is continuous and $P(X = Y) = 0$.

Proposition 2.3 : (1) *Let R_i be the rank order of $U_i = max(X_j, Y_j)$ in the set $\{X_1, Y_1, \ldots, X_n, Y_n\}$ and S_i that of $V_i = min(X_j, Y_j)$. Write $R = ((R_1, S_1), \ldots, (R_n, S_n))$ for short. (Z, R) is a maximal invariant statistics with respect to G_1.*

(2) *Let $L = (L_{ij},\ i > j)$, where $L_{ij} = c(U_i - U_j) \times c(V_i - V_j) = c(V_i - V_j)$. L is equivalent to the rank order of $(V_1, \ldots, V_n)$ among them. (Z, L) is a maximal invariant statistics with respect to G_2.*

Proof : (1) can be proved in the same way as the two sample problem. To prove (2) let $r(x, y) = x+h(x-y).c(x-y)$ as mentioned above, assuming further $h(0) = 0$ and $h(\min(U_i - V_i)) + \min U_i - \max V_i > 0$. The function r belongs to G_2. The rank statistics R' of $\phi(X, Y) = ((r(X_i, Y_i), r(Y_i, X_i)), i = 1, \ldots, n)$ is $R'_i = 2n+1-i$ and $S'_i = 1 + \sum_{i > j} L_{ij}$. Thus $(X', Y') = \phi(X, Y)$ if (Z, L) for (X', Y') is equal to (Z, L) for (X, Y), proving ϕ to be maximal.

(Z, L) is related to the so-called layer rank statistics (see Bhattacharyya and Johnson (1970)). In the next section we deal only rank statistics (Z, R). The discussion for (Z, L) can be developed in parallel up to some point (Theorem 3.1 for example). We failed, however, to find effective test statistics of explicit form.

Proposition 2.4 : *Let z and z' be n-vectors with components 1 or -1 and $z_i \geqslant z_i'$, $i = 1, \ldots, n$. If $X \succ Y$ ($\mathcal{H}_{3A}$), then $P(Z = z,\ R = r) \geqslant P(Z = z',\ R = r)$ for any r.*

Proposition 2.5 : *Let z and z' be n-vectors with components 1 or -1 such that $z_j = z_k' = 1$, $z_k = z_j' = -1$ $(j < k)$ and $z_i = z_i'$, $i \neq j, k$. Let $r = ((r_i, s_i), i = 1, \ldots, n)$ be such that $r_j > r_k > s_k > s_j$. If $X \succ Y$ ($\mathcal{H}_4$), then $P(Z = z, R = r) \geqslant P(Z = z', R = r)$.*

3. Tests of symmetry

The purpose of this section is to construct some unbiased tests of the hypothesis of symmetry against suitable alternative hypotheses based on the previous sections. Firstly we introduce three monotonicity conditions on a critical function $\phi(z, r) = \phi((x_1, y_1), \ldots, (x_n, y_n))$. We are using the same notation ϕ for the functions of both (z, r) and (x_i, y_j)'s.

Definition 3 : Condition A : $\phi(z, r) \geqslant \phi(z', r)$, for any r and any (z, z') of Proposition 2.4.

Condition B : $\phi(z, r) \geqslant \phi(z, r')$, for z, r and r' such that $r_i \geqslant r_i' > s_i' \geqslant s_i$ if $z_i = 1$ and $r_i' \geqslant r_i > s_i \geqslant s_i'$ if $z_i = -1$ for all $i = 1, \ldots, n$.

Condition C : $\phi((x_1, y_1), \ldots, (x_n, y_n)) \geqslant \phi((x_1', y_1'), \ldots, (x_n', y_n'))$, if $x_i \geqslant x_i'$ and $y_i \leqslant y_i'$, $i = 1, \ldots, n$.

In the following theorem we just write $X \succ Y$ ($\mathcal{H}$) to mean the alternative hypothesis that $X \succ Y$ ($\mathcal{H}$) and (X, Y) is asymmetry.

Theorem 3.1 : (1) *A sufficient condition for a test ϕ to be unbiased against $X \succ Y$ ($\mathcal{H}_{3A}$) is that ϕ satisfies Condition A.*

(2) *A sufficient condition for a test ϕ to be unbiased against $X \succ Y$ ($\mathcal{H}_{3I}$) is that ϕ satisfies Conditions A and B.*

(3) *A sufficient condition for a test ϕ to be unbiased against $X \succ Y$ ($\mathcal{H}_S$) is that ϕ satisfies Condition C.*

Proof : (1) Except for t in a set of probability zero (Z_i, T), $i = 1, \ldots, n$, given $T = t$ are mutually independent and $P(Z_i = 1 \mid T = t) \geqslant 1/2$, thus $E_F[\phi(Z, R)] = \Sigma \int_{R=r} E[\phi(Z, R) \mid T = t] dP_F^T$ is not less than the level of ϕ.

(2) We show $\phi((x_i, y_i),\ i = 1, \dots, n) \leqslant \phi((f(x_i), y_i),\ i = 1, \dots, n)$ for any monotone function f such that $f(s) \geqslant s$, $-\infty < s < \infty$. Without loss of generality we can assume that $f(x_i) \neq y_j$ for all $i, j = 1, \dots, n$ and that $x_1 > \dots > x_n$. Firstly we prove

$$\phi((f(x_1), y_1), (x_2, y_2), \dots, (x_n, y_n)) \geqslant \phi((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)).$$

Let B be the set of Y_i's which belong to the interval $(x_1, f(x_1))$. (a) If B is empty, then the value of (Z, R) is the same for (x_1, y_1) and $(f(x_1), y_1)$. (b) If B consists of a single element y_1, then the value of R is the same and $Z((f(x_1), y_1), (x_2, y_2), \dots, (x_n, y_n)) - Z((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n))$ is a vector with the value 1 for a component corresponding to x_1 and 0's for other components. The inequality holds from Condition A. (c) If the set B consists of a single element y_m, $m \neq 1$, then the value of Z is the same. Denote by z_j and z_k the components corresponding (x_1, y_1) and (x_m, y_m) respectively, and denote by $((r_1', s_1'), \dots, (r_n', s_n'))$ the rank order of $((f(x_1), y_1), (x_2, y_2), \dots, (x_n, y_n))$.

$$r_j' = s_k = r_j + 1 = s_k' + 1, \qquad \text{if } z_j = z_k = 1,$$

$$r_j' = r_k = r_j + 1 = r_k' + 1, \qquad \text{if } z_j = 1 \text{ and } z_k = -1,$$

$$s_j' = s_k = s_j + 1 = s_k' + 1, \qquad \text{if } z_j = -1 \text{ and } z_k = -1,$$

and

$$s_j' = r_k = s_j + 1 = r_k' + 1, \qquad \text{if } z_j = -1 \text{ and } z_k = -1.$$

(The first case does not occur actually for x_1 but for x_i's in later discussion.) Other components (r_i', s_i')'s are the same as (r_i, s_i)'s. Condition B shows that for these cases the inequality holds. (d) If the set B consists of more than one element, then order the element of B as $y_{(1)} < \dots < y_{(q)}$, and choose the numbers $\{f_i\}$ such that $x_1 = f_0 < y_{(1)} < f_1 < \dots < f_{q-1} < y_{(q)} < f_q = f(x_1)$. Apply the discussion of the above (b) and (c) replacing the interval $((x_1, f(x_1))$ by (f_i, f_{i+1}).

The similar argument is applied to prove recursively

$$\phi((f(x_1), y_1), \dots, (f(x_i), y_i), (x_{i+1}, y_{i+1}), \dots, (x_n, y_n))$$
$$\geqslant \phi((f(x_1), y_1), \dots, (f(x_{i-1}), y_{i-1}), (x_i, y_i), \dots, (x_n, y_n)).$$

(3) Put $F_0(x, y) = 1/2\,(F(x, y) + F(y, x))$, a symmetrized distribution function. The sets

$$S_i = \{(x, y);\ \phi((x_1, y_1), \dots, (x, y), \dots, (x_n, y_n)) > u\},$$

where the arguments of ϕ are fixed except for the i-th component, are all ones of the property that $(x, y) \in S_i$ implies $(x', y') \in S_i$ for all $x < x'$ and $y > y'$.

From the definition of $\mathcal{R}_S$,

$$\int \phi((x_1, y_1), \dots, (x_n, y_n))\, dF(x_1, y_1) \dots dF(x_n, y_n)$$
$$\geqslant \int \phi((x_1, y_1), \dots, (x_n, y_n))\, dF_0(x_1, y_1)\, dF(x_2, y_2) \dots dF(x_n, y_n)$$
$$\geqslant \dots$$
$$\geqslant \int \phi((x_1, y_1), \dots, (x_n, y_n))\, dF_0(x_1, y_1) \dots dF_0(x_n, y_n).$$

Although Condition C looks very natural and is expected to have nice properties, it is difficult to find an explicit statistic satisfying it because of the limitation of the similar test. This is true especially for the test statistic of the following Theorem 3.2.

To show more explicit form of critical functions we consider $\phi(z, r)$ such that $\phi = 1$ if $w(z, r) > c_\alpha$, $\phi = p_\alpha$ $(0 < p_\alpha \leqslant 1)$ if $w(z, r) = c_\alpha$ and $\phi = 0$ if $w(z, r) < c_\alpha$, where c_α and p_α depend on the level. The following tests of type (1) are proposed by P. K. Sen (1967) starting from the different model.

Theorem 3.2 : *Let $a(i)$, $i = 1, \dots, 2n$, be an increasing sequence.*

(1) *Put $w(Z, R) = \Sigma Z_i(a(R_i) - a(S_i))$. The critical function based on w satisfies Conditions A, B and that in Proposition 2.5.*

(2) *Put $w(Z, R) = \Sigma(Z_i + 1)\ \ a(R_i)$. The critical function based on w satisfies the above conditions as well as Condition C.*

Proof: It is clear that Condition A and the condition in Proposition 2.5 are satisfied.

$$(1)\ \sum_i z_i(a(r_i') - a(s_i')) - \sum_i z_i'(a(r_i') - a(s_i')) - \{\sum_i z_i(a(r_i) - a(s_i)) - \sum_i z_i'(a(r_i) - a(s_i))\}$$
$$= \sum_i (a(r_i') - a(s_i') - a(r_i) + a(s_i))(z_i - z_i') \geqslant 0.$$

Therefore

$$\Sigma z_i(a(r_i') - a(s_i)) > \Sigma z_i'(a(r_i') - a(s_i')),\ \text{if}\ \Sigma z_i(a(r_i) - a(s_i)) > \Sigma z_i(a(r_i) - a(s_i)).$$

The relation with the two inequalities reversed also holds. From these Condition B is shown.

(2) Firstly we prove that the critical function based on $w(Z, R)$ satisfies

$$\phi((x_1', y_1), (x_2, y_2), \dots, (x_n, y_n)) \geqslant \phi((x_1, y_1), \dots, (x_n, y_n))\ \ \text{for any}\ \ x_1' \geqslant x_1.$$

Let B be the set of numbers x_i's or y_i's which are included in the interval (x_1, x_1'). (a) If B is empty, then the equality holds in the above expression.

(b) If B consists of a single element y_1, then the inequality holds from Condition A. (c) Let B consists of a single element max (x_i, y_i), $i \neq 1$. If $z_1 = -1$, then (Z, R) is the same for both (x_1, y_1) and (x_1', y_1) and the equality holds. If $z_1 = 1$, then

$$\Sigma z_i a(r_i') - \Sigma z_i' a(r_i') - (\Sigma z_i a(r_i) - \Sigma z_i' a(r_i')) = a(r_1+1) - a(r_1) - z_1'(a(r_1+1) - a(r_1)).$$

Applying the same discussion as (1) to this fact we obtain the inequality. (d) Let B consist of a single element of min (x_i, y_i), $i \neq 1$. If $z_1 = -1$, then the value of (Z, R) is again the same. If $z_1 = 1$, then define

$$S_{11} = \{(z_1', \ldots, z_n') \,|\, \Sigma z_i a(r_i') > \Sigma z_i' a(r_i')\},$$

$$S_{12} = \{(z_1', \ldots, z_n') \,|\, \Sigma z_i a(r_i') < \Sigma z_i' a(r_i')\},$$

$$S_{21} = \{(z_1', \ldots, z_n') \,|\, \Sigma z_i a(r_i) > \Sigma z_i' a(r_i)\},$$

$$S_{22} = \{(z_1', \ldots, z_n') \,|\, \Sigma z_i a(r_i) < \Sigma z_i' a(r_i)\},$$

where $(r_1', \ldots, r_n') = (r_1+1, \ldots, r_{i-1}, \ r_1, \ r_{i+1}, \ldots, r_n)$, and we prove that the cardinalities of S_{11} and S_{21} are equal and those of S_{12} and S_{22} are so. Let c be a transformation of n-vectors of ± 1 components such that

$$c(z_1, \ldots, z_n) = (z_i, z_2, \ldots, z_{i-1}, z_1, z_{i+1}, \ldots, z_n)$$

and let

$$(z_1^*, \ldots, z_n^*) = c(z_1', \ldots, z_n').$$

As $\Sigma z_i' a(r_i) = \Sigma z_i^* a(r_i)$, the proof is completed.

(e) If B consists of more than one elements, then combine (a)-(d) in a chain to prove the inequality. The details are omitted.

The inequality $\phi((x_1, y_1'), (x_2, y_2), \ldots, (x_n, y_n)) \geqslant \phi((x_1, y_1), \ldots, (x_n, y_n))$, for any $y_1' < y_1$, is obtained by the dual argument.

Taking as $\{a(i)\}$ the normal scores, $a(i) = i$ or other scores we obtain test statistics analogous to usual ones. Remark that if we are dealing with distributions which degenerate on the line $x+y = 0$, then our test statistics reduce to those for testing the one-dimensional symmetry around the origin.

Practically the computation of criterion (c_α, p_α) for a given sample value is not easy. One way is to use normal approximate when the sample size is larger. Put

$$w_n = \Sigma z_i(a(r_i) - a(s_i)) / \sqrt{\Sigma(a(r_i) - a(s_i))^2},$$

where $a(i)$ depends on n also.

Proposition 3.3: *The statistic w_n is asymptotically normally distributed if there exists $N = N(\varepsilon_1, \varepsilon_2)$ for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that*

$$P(R = r \text{ and } \max \frac{a(r_i)-a(s_i)}{\sqrt{\Sigma(a(r_i)-a(s_i))^2}} < \varepsilon_1) > 1-\varepsilon_2$$

for all $n > N$ and for any symmetric distributions.

Proof: Let the characteristic function of w_n be $\phi_n(t)$ and that of $N(0, 1)$ be $\phi(t)$. Let Z_i, $i = 1, 2, \ldots$, be mutually independent random variables taking ± 1 with probability 1/2. If $\psi_n(t)$ is the characteristic function of $\sum_{i=1}^{n} Z_i/\sqrt{n}$, then $\psi_n(t) \to \phi(t)$ as well known.

$\Sigma a_i Z_i$, with $a_i > 0$ and $\Sigma a_i^2 = 1$, has the characteristic function $\prod_{i=1}^{n} \cos a_i t$. It is shown that

$$n'' \log \cos \frac{t}{\sqrt{n''}} \geqslant \Sigma \log \cos a_i t \geqslant n' \log \cos \frac{t}{\sqrt{n'}},$$

if

$$\frac{1}{\sqrt{n''}} \leqslant a_i \leqslant \frac{1}{\sqrt{n'}} \quad \text{and} \quad \left|\frac{t}{\sqrt{n'}}\right| < \frac{\pi}{2}.$$

We show that for given T and ε there exists N such that

$$\max_{|t|<T} |\phi_n(t)-\phi(t)| < \varepsilon \text{ for all } n > N.$$

Let N_1 be such a number that

$$\max_{|t|<T} |\psi_n(t)-\phi(t)| < \varepsilon/2 \text{ for all } n > N_1,$$

and put $\varepsilon_2 < \varepsilon/3$ and $\varepsilon_1 = \min(\varepsilon_2, 1/n')$. Let

$$S = \left\{((x_1, y_1), \ldots, (x_n, y_n));\ R = r \text{ and } \max_i \frac{a(r_i)-a(s_i)}{\sqrt{\Sigma(a(r_i)-a(s_i))^2}} < \varepsilon_1\right\}.$$

There exists $N(\varepsilon_1, \varepsilon_2)$ such that $P_F(S) > 1-\varepsilon_2$ for $n > N(\varepsilon_1, \varepsilon_2)$ by the hypothesis of the theorem.

$$|\phi_n(t)-\phi(t)| \leqslant \left|\int_S (e^{itw_n}-\phi(t))dF\right| + \left|\int_{R^{n}-S} (e^{itw_n}-\phi(t))dF\right|$$

$$\leqslant \varepsilon_1(1-\varepsilon_2)+2\varepsilon_2 \leqslant 3\varepsilon_2 < \varepsilon.$$

Remark that the approximation and propositions similar to Proposition 3.3 can be applied to w in Theorem 3.2 (2) and to conditional sign test statistics based on order statistics (Z, T). For example normal approximate of $\Sigma Z_i(U_i - V_i) = \Sigma(X_i - Y_i)$ is $\Sigma(X_i - Y_i)/\sqrt{\Sigma(X_i - Y_i)^2}$ or $\Sigma(X_i - Y_i)/\sqrt{\Sigma(X_i - Y_i - \bar{X} + \bar{Y})^2}$. This is the test statistic for equality of the means of bivariate normal distribution of common variances, and asymptotically normally distributed.

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Addendum

While the paper was in print, Dr. Tom Snijders pointed out a fallacy in Definition 2(2) and Proposition 2.3(2). A function r satisfying $r(x', y) \geqslant r(x, y')$ whenever $x' \geqslant x$ and $y' \geqslant y$ does not have the inverse, and G_2, the set of transformations $\phi(x, y) = (r(x, y), r(y, x))$, is not a group. The statistic L is invariant under the transformations in G_2. It is an open problem, however, whether L is a maximal invariant statistic for some transformation group.

The following paper is closely related to this one. T. Snijders, "Tests for the problem of bivariate symmetry", University of Groningen.