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AN ELEMENTARY PROOF OF THE BOREL ISOMORPHISM THEOREM

In this note we present a very elementary proof of the Borel isomorphism theorem (Corollary 6). The traditional and more well known proof of this theorem uses the first separation principle for analytic sets. A proof of this avoiding the first separation principle is also known ([1, p. 450]). Our proof is perhaps the simplest.

A *Polish space* is a second countable, completely metrizable topological space. The Borel σ-field of a metrizable space X will be denoted by $\mathcal{B}(X)$. The space $\{0,1\}^\omega$ of sequences of 0's and 1's will be denoted by **C**. Equipped with the product of discrete topologies on $\{0, 1\}$, it is a compact metrizable space. A *bimeasurable map* from a measurable space $(X, \mathcal{A})$ to a measurable space $(Y, \mathcal{B})$ is a measurable map $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ such that $f(A) \in \mathcal{B}$ for every $A \in \mathcal{A}$. A Borel subset of a Polish space will be called a *standard Borel set*. It is assumed that a standard Borel set is always equipped with its Borel σ-field. Two standard Borel sets X and Y are called *isomorphic* if there is a bijection $f : X \longrightarrow Y$ which is bimeasurable.

Lemma 1 *([1, page 348, Theorem 3]) If X is a metrizable space, then $\mathcal{B}(X)$ is the smallest class $\mathcal{B}$ of subsets of X such that*

i) every open set in X belongs to $\mathcal{B}$;

ii) if $B_0, B_1, \ldots$ are pairwise disjoint and belong to $\mathcal{B}$, then so does $\bigcup_n B_n$; and

iii) if $B_0, B_1, \ldots$ belong to $\mathcal{B}$, then so does $\bigcap_n B_n$.

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PROOF. If $\mathcal{C} = \{A \in \mathcal{B} : X \setminus A \in \mathcal{B}\}$, then $\mathcal{C}$ satisfies conditions i) – iii). Hence $\mathcal{C}$ is closed under complementation and so equals $\mathcal{B}(X)$. This completes the proof.

The next result can be found in ([1, page 448, Theorem 1]). However, our proof is significantly simpler than the one given in ([1]).

Proposition 2 *If X is a Polish space, then for every Borel set B in X there is a Polish space Z and a continuous bijection $f : Z \to B$. Moreover, for every Borel set A in Z, $f(A)$ is Borel in B.*

PROOF. Let $\mathcal{B}$ be the class of all Borel sets in X satisfying the above property.

i) Let U be an open set in X. As U is Polish we take $Z = U$ and f the identity map. This shows that $U \in \mathcal{B}$.

   Let $B_0, B_1, \ldots$ belong to $\mathcal{B}$. For each n, fix a Polish space Z_n and a continuous bijection $f_n : Z_n \to B_n$ which is bimeasurable.

ii) Set $Z = \{(z_0, z_1, \ldots) \in \prod_n Z_n : f_0(z_0) = f_1(z_1) = \cdots\}$ and define $f : Z \to X$ by $f(z_0, z_1, \ldots) = f_0(z_0)$, $(z_0, z_1, \ldots) \in Z$. Then Z is Polish and $f : Z \to X$ is a continuous injection such that $f(Z) = \bigcap_n B_n$. It is also clear that f is bimeasurable. Thus, $\bigcap_n B_n \in \mathcal{B}$.

iii) If, moreover, $B_0, B_1, \ldots$ are pairwise disjoint, then let Z be the direct sum of $Z_0, Z_1, \ldots$ and $f : Z \to X$ be defined by $f(z) = f_i(z)$ if $z \in Z_i$, $i \in \omega$. This shows that $\bigcup_n B_n \in \mathcal{B}$. We get the result from Lemma 1.

The following result is a measurable analogue of the Schröder-Bernstein theorem and is a part of folklore. A sketch of the proof is given for the sake of completeness.

Proposition 3 *(Schröder–Bernstein) : If there exist injective bimeasurable maps $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ and $g : (Y, \mathcal{B}) \to (X, \mathcal{A})$, then there is a bimeasurable bijection $h : (X, \mathcal{A}) \to (Y, \mathcal{B})$.*

PROOF. Inductively we define sets $A_0, A_1, \ldots$ in $\mathcal{A}$ by $A_0 = \emptyset$ and $A_{n+1} = X \setminus g(Y \setminus f(A_n))$. Set $A = \bigcup_n A_n$. Then $A \in \mathcal{A}$ and $A = X \setminus g(Y \setminus f(A))$. Now, define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in X \setminus A. \end{cases}$$

Clearly h is a desired bimeasurable bijection.

We shall need one more well known result for our proof.

Proposition 4 *([1, p.444, Theorem]) Every uncountable Polish space Z contains a homeomorph of* $\mathbf{C}$.

Theorem 5 *If B is an uncountable standard Borel set, then B is isomorphic to* $\mathbf{C}$.

PROOF. Let D be the set of all dyadic rationals (including 0 and 1) in $I = [0,1]$ and E the set of all eventually constant sequences $(x_n) \in \mathbf{C}$. Define $f: I \to \mathbf{C}$ by $f|D$ to be any bijection from D to E and for $x \in I \setminus D, f(x) = (x_n)$ where $x = \sum_0^\infty x_n \cdot 2^{-n-1}$. Note that $f|(I \setminus D)$ is a homeomorphism from $I \setminus D$ onto $\mathbf{C} \setminus E$. Thus I is isomorphic to $\mathbf{C}$. It follows that the Hilbert cube $H = I^\omega$ is isomorphic to $\mathbf{C}^\omega$ which is homeomorphic to $\mathbf{C}$. Since B is homeomorphic to a Borel subset of H, it is isomorphic to a Borel subset of $\mathbf{C}$.

On the other hand, by Proposition 2, there is a Polish Z and a continuous bijection $g : Z \to B$. Since B is uncountable, so is Z. By Proposition 4, Z contains a homeomorph of $\mathbf{C}$ and, hence, so does B.

Our result follows from Proposition 3.

Corollary 6 *(The Borel Isomorphism Theorem): Two standard Borel sets X and Y are isomorphic iff they are of the same cardinality.*

References

[1] K. Kuratowski, *Topology, Vol I*, Academic press, New York, San Francisco, London, 1966.