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HERMITE EXPANSIONS ON $\mathbb{R}^n$ FOR RADIAL FUNCTIONS

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ABSTRACT. It is proved that the Riesz means $S_R^\delta f$, $\delta > 0$, for the Hermite expansions on $\mathbb{R}^n$, $n \geq 2$, satisfy the uniform estimates $\|S_R^\delta f\|_p \leq C\|f\|_p$ for all radial functions if and only if p lies in the interval $2n/(n+1+2\delta) < p < 2n/(n-1-2\delta)$.

1. INTRODUCTION

Consider the Hermite polynomials $H_k(t)$ for $k = 0, 1, 2\dots$ on the real line defined by

$$H_k(t) = (-1)^k e^{t^2} \left(\frac{d}{dt}\right)^k (e^{-t^2}). \tag{1.1}$$

We define the normalized Hermite functions $h_k(t)$ by setting

$$h_k(t) = (2^k \sqrt{\pi} k!)^{-1/2} H_k(t) e^{-t^2/2}. \tag{1.2}$$

Then the family $\{h_k\}$ form a complete orthonormal system for $L^2(\mathbb{R})$. On $\mathbb{R}^n$, $n \geq 2$, we define the normalized multiple Hermite functions $\Phi_\nu(x)$, $x \in \mathbb{R}^n$, ν a multi-index by

$$\Phi_\nu(x) = \prod_{j=1}^n h_{\nu_j}(x_j). \tag{1.3}$$

These functions Φ_ν are then eigenfunctions of the Hermite operator $H = (-\Delta + |x|^2)$ with eigenvalues $(2|\nu| + n)$ and the family $\{\Phi_\nu\}$ forms a complete orthonormal system for $L^2(\mathbb{R}^n)$.

For a function f in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, we define the Fourier-Hermite coefficients $\hat{f}(\nu)$ of the function f by

$$\hat{f}(\nu) = \int_{\mathbb{R}^n} f(x) \Phi_\nu(x)\, dx. \tag{1.4}$$

We then have the Hermite series

$$f(x) = \sum \hat{f}(\nu) \Phi_\nu(x) \tag{1.5}$$

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where the sum is extended over all the multi-indices ν. For $p = 2$, the series converges to f in the norm, but for other values of p the series need not converge. Therefore, we are led to consider summability methods for the series (1.5). In this paper we are concerned with the Riesz summability of the above series.

In order to define the Riesz means it is convenient to introduce the projection operators P_k associated to the Hermite operator H. These are defined by

$$P_k f = \sum_{|\nu|=k} \hat{f}(\nu)\Phi_\nu \tag{1.6}$$

for $k = 0, 1, 2, \ldots$. We observe that P_k is an integral operator with kernel

$$\Phi_k(x, y) = \sum_{|\nu|=k} \Phi_\nu(x)\Phi_\nu(y). \tag{1.7}$$

For any $\delta > -1$ and $R > 0$ we define the Riesz means $S_R^\delta f$ of order δ of the functions f by

$$S_R^\delta f = \sum \left(1 - \frac{2k+n}{R}\right)_+^\delta P_k f. \tag{1.8}$$

The uniform boundedness of S_R^δ on L^p, $1 \le p \le \infty$, have been studied by the author in [9]. There it was proved that if $\delta > (n-1)/2$, $n \ge 2$, then one has the uniform estimates

$$\|S_R^\delta f\|_p \le c\|f\|_p, \qquad 1 \le p \le \infty, \tag{1.9}$$

and, moreover, if $p < \infty$, then $S_R^\delta f$ converges to f in the norm as $R \to \infty$.

We are interested in knowing what happens when $\delta \le (n-1)/2$. As a consequence of a transplantation theorem of Kenig-Stanton-Tomas [4] one has the conjecture that for $0 < \delta \le (n-1)/2$ the uniform estimates (1.9) hold iff $2n/(n+1+2\delta) < p < 2n/(n-1-2\delta)$. Of course, when $\delta = 0$ the Riesz means are bounded only on $L^2(\mathbb{R}^n)$ in view of Fefferman's celebrated theorem on the multiplier problem for the ball.

In an earlier paper [10] we proved the above conjecture on $\mathbb{R}^{2n}$ for the radial functions. There we showed that when f is radial the Hermite series reduces to a Laguerre series. Here we do the same on $\mathbb{R}^n$ for radial functions. We prove the following theorem.

Theorem 1.1. *Let $f \in L^p(\mathbb{R}^n)$ be radial and $n \ge 2$. Then the uniform estimates*

$$\|S_R^\delta f\|_p \le c\|f\|_p$$

hold if and only if $2n/(n+1+2\delta) < p < 2n/(n-1-2\delta)$.

As in the case of even dimensions we show that for radial functions the Hermite series reduces to a Laguerre series. Then we can either apply known results for Laguerre series or follow the procedure used in the case of even dimensions. The procedure, essentially due to Fefferman-Stein and developed by Sogge [6], uses the $L^p - L^2$ restriction theorem for the projections P_k. For those projections we have proved in [11] the estimates

$$\|P_k f\|_2 \le ck^{(n-1)(1/p-1/2)/2}\|f\|_p, \qquad 1 \le p \le 2.$$

But on $\mathbb{R}^{2n}$, if f is radial we have improved the above estimates to give

$$\|P_k f\|_2 \leq c k^{n(1/p-1/2)-1/2}\|f\|_p, \qquad 1 \leq p \leq 4n/(2n+1). \tag{1.10}$$

This was done in [10] and was the main ingredient in proving the conjecture on the Riesz means. In this paper we get estimates for P_k on $L^p(\mathbb{R}^n)$.

Proposition 1.1. *Let $f \in L^p(\mathbb{R}^n)$ be radial. Then for $1 \leq p < 2n/(n+1)$ we have the estimates*

$$\|P_k f\|_2 \leq c k^{n(1/p-1/2)/2-1/2}\|f\|_p. \tag{1.11}$$

It would be interesting to see if the above estimates (1.11) are true for all functions. Once we have (1.11) we can combine that with certain kernel estimates to follow the recipe of Sogge to prove Theorem 1.1. This procedure is carried out in full detail in [10] and here we will not go into it.

In the next section we show that for radial functions the Hermite series reduces to Laguerre series. Actually we prove a formula for P_k analogous to the Hecke-Bochner formula for the Fourier transform. In the third section we prove our main theorem and the estimates for the spectral projections.

2. Hecke-Bochner type identity for the Hermite projection operators

Let f be a function on $\mathbb{R}^n$ of the form $f(x) = f_0(|x|)P(x)$ where P is a solid harmonic of degree m. Then the Hecke-Bochner identity for the Fourier transform says that $\hat{f}(x)$ is also of the same form. More precisely, $\hat{f}(x) = F_0(|x|)P(x)$ where

$$F_0(r) = 2\pi i^{-m} r^{-(n/2+m-1)} \int_0^\infty f_0(s) J_{n/2+m-1}(2\pi r s) s^{n/2+m}\, ds. \tag{2.1}$$

This was proved, e.g., in Stein-Weiss [7, §4, Theorem 3.10]. In this section we prove a similar formula for the action of the Hermite projection operators P_k on functions of the form $f_0(|x|)P(x)$. To state the identity we need to introduce some notation.

Let for $k = 0, 1, 2, \ldots$ and $\alpha > -1$, $L_k^\alpha(r)$ stand for Laguerre polynomials of type α. Define $\varphi_k^\alpha(r) = L_k^\alpha(r^2)e^{-r^2/2}$ and for functions on $[0, \infty)$ define $R_k^\alpha(f)$ by

$$R_k^\alpha(f) = 2\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \int_0^\infty f(r)\varphi_k^\alpha(r) r^{2\alpha+1}\, dr. \tag{2.2}$$

Now we can state and prove

Theorem 2.1. *Let $f(x) = f_0(|x|)P(x)$ where P is a solid harmonic of degree m. Then*

$$P_{2k+m} f(x) = F_k(|x|)P(x) \tag{2.3}$$

where $F_k(r)$ is given by the formula

$$F_k(r) = R_k^{n/2+m-1}(f_0)\varphi_k^{n/2+m-1}(r). \tag{2.4}$$

For other values of j, $P_j f = 0$.

Corollary. *If $f(x) = f_0(|x|)$ then $P_{2k+1} f = 0$ and*

$$P_{2k} f(r) = R_k^{n/2-1}(f_0)\varphi_k^{n/2-1}(r).$$

Proof of Theorem 2.1. We start with the generating function identity for the Hermite functions (see Folland [2] for a proof) for $|w| < 1$,

$$\sum_{k=0}^{\infty}\Phi_k(x, y)w^k = \pi^{-n/2}(1 - w^2)^{-n/2}e^{-(1/2)(1+w^2)/(1-w^2)(|x|^2+|y|^2)+(2wx\cdot y)/(1-w^2)}. \tag{2.5}$$

From this it follows that

$$\sum_{k=0}^{\infty}P_k f(x)w^k = \pi^{-n/2}(1 - w^2)^{-n/2} \times \int_{\mathbb{R}^n} e^{-(1/2)(1+w^2)/(1-w^2)(|x|^2+|y|^2)+(2wx\cdot y)/(1-w^2)}f(y)\,dy. \tag{2.6}$$

Let us write $x = rx'$, $y = sy'$ where $r = |x|$, $s = |y|$ so that $P(x) = r^m P(x')$; then we have

$$\sum_{k=0}^{\infty}P_k f(x)w^k = \pi^{-n/2}(1 - w^2)^{-n/2}e^{-(1/2)(1+w^2)r^2/(1-w^2)} \times \int_0^{\infty}\left[\int_{S^{n-1}} e^{(2wrsx'\cdot y')/(1-w^2)}P(y')\,dy'\right]g(s)s^{m+n-1}\,ds \tag{2.7}$$

where $g(s) = e^{-(1/2)(1+w^2)s^2/(1-w^2)}f_0(s)$. To evaluate the integral we proceed as follows.

One has

$$\int_{\mathbb{R}^n} e^{-2\pi i x\cdot y}g(|y|)P(y)\,dy = \int_0^{\infty}\left[\int_{S^{n-1}} e^{-2\pi irsx'\cdot y'}P(y')\,dy'\right]g(s)s^{m+n-1}\,ds.$$

In view of the Hecke-Bochner identity for the Fourier transform

$$\int_0^{\infty}\left[\int_{S^{n-1}} e^{-2\pi irsx'\cdot y'}P(y')\,dy'\right]g(s)s^{m+n-1}\,ds = 2\pi i^{-m}r^{-(n/2-1)}\left[\int_0^{\infty} g(s)J_{n/2+m-1}(2\pi rs)s^{n/2+m}\,ds\right]P(x'). \tag{2.8}$$

Since both sides are holomorphic functions of r, we can replace r by $iwr/(1-w^2)\pi$ to get

$$\begin{aligned}\int_0^{\infty}\left[\int_{S^{n-1}} e^{(2wrsx'\cdot y')/(1-w^2)}P(y')\,dy'\right]g(s)s^{n+m-1}\,ds &= 2\pi^{n/2}i^{-(n/2+m-1)}(1 - w^2)^{-(n/2-1)}r^{-(n/2+m-1)}\\ &\quad\times\left[\int_0^{\infty} g(s)J_{n/2+m-1}\left(\frac{2iw}{1-w^2}rs\right)s^{n/2+m}\,ds\right]P(x).\end{aligned} \tag{2.9}$$

Using this we have proved that

$$\begin{aligned}\sum_{k=0}^{\infty}P_k f(x)w^k &= 2i^{-(n/2+m-1)}(1 - w^2)^{-1}w^{-(n/2-1)}r^{-(n/2+m-1)}\\ &\quad\times\left[\int_0^{\infty} g(s)J_{n/2+m-1}\left(\frac{2iw}{1-w^2}rs\right)s^{n/2+m}\,ds\right]\\ &\quad\times P(x)e^{-(1/2)(1+w^2)r^2/(1-w^2)}.\end{aligned} \tag{2.10}$$

Now let us recall the following generating function identity for the Laguerre functions φ_k^α (see Szego [8]):

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \varphi_k^\alpha(r)\varphi_k^\alpha(s) w^{2k} = (1-w^2)^{-1}(irsw)^{-\alpha} e^{-(1/2)(1+w^2)(r^2+s^2)/(1-w^2)} J_\alpha\left(\frac{2iw}{1-w^2} rs\right). \tag{2.11}$$

In view of this formula, we get

$$\begin{aligned} \sum_{k=0}^{\infty} P_k f(x) w^k &= P(x) \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+n/2+m)} \left[\int_0^\infty f_0(s)\varphi_k^{n/2+m-1}(s) s^{n+2m-1}\, ds\right] \\ &\quad \times \varphi_k^{n/2+m-1}(r) w^{2k+m} \\ &= P(x)\left[\sum_{k=0}^{\infty} R_k^{n/2+m-1}(f_0)\varphi_k^{n/2+m-1}(r) w^{2k+m}\right]. \end{aligned} \tag{2.12}$$

Comparing the coefficients on both sides we immediately get $P_{2k+m}f(s) = F_k(|x|)P(x)$ where

$$F_k(r) = R_k^{n/2+m-1}(f_0)\varphi_k^{n/2+m-1}(r).$$

This completes the proof of Theorem 2.1. The corollary is immediate.

3. Uniform boundedness of the Riesz means

The uniform boundedness of the Riesz means can now be easily deduced from a result concerning the summability of Laguerre series. The functions $\varphi_k^\alpha(r)$ form an orthogonal system in $L^2(\mathbb{R}_+, r^{2\alpha+1}dr)$ where $\mathbb{R}_+ = [0, \infty)$. For f in $L^p(\mathbb{R}_+, r^{2\alpha+1}dr)$ we have the formal Laguerre series

$$f(r) = \sum_{k=0}^{\infty} R_k^\alpha(f)\varphi_k^\alpha(r). \tag{3.1}$$

In [3] Gorlich-Markett proved that the partial sums of the above series are uniformly bounded on $L^p(\mathbb{R}_+, r^{2\alpha+1}dr)$ iff p lies in the interval

$$(4\alpha+4)/(2\alpha+3) < p < (4\alpha+4)/(2\alpha+1)$$

provided $\alpha \geq 0$.

When f is radial, $f(x) = f_0(|x|)$, the partial sums of the Hermite series become

$$S_R f(x) = \sum_{k=0}^{[R/2]} R_k^{n/2-1}(f_0)\varphi_k^{n/2-1}(r). \tag{3.2}$$

So we obtain the uniform estimates

$$\|S_R f\|_p \leq c\|f\|_p \tag{3.3}$$

for all p satisfying $2n/(n+1) < p < 2n/(n-1)$. Interpolating this with the result for $\delta > (n-1)/2$ we get Theorem 1.1.

We conclude this section with a proof of Proposition 1.1. Since for radial functions $P_{2k}f = R_k^{n/2-1}(f)\varphi_k^{n/2-1}$, we have

$$\|P_{2k}f\|_2^2 = c|R_k^{n/2-1}(f)|^2 \left[\int_0^\infty |\varphi_k^{n/2-1}(r)|^2 r^{n-1}\,dr\right]. \tag{3.4}$$

As the square of the norm of $\varphi_k^{n/2-1}$ in $L^2(\mathbb{R}_+, r^{n-1}dr)$ behaves like $k^{n/2-1}$, we get

$$\|P_{2k}f\|_2^2 \le ck^{-n/2+1}\left|\int_0^\infty f(r)\varphi_k^{n/2-1}(r)r^{n-1}\,dr\right|^2. \tag{3.5}$$

Applying Hölder's inequality we get

$$\|P_{2k}f\|_2 \le ck^{-n/4+1/2}\|f\|_{L^p(\mathbb{R}^n)}\|\varphi_k^{n/2-1}\|_{L^q(\mathbb{R}_+, r^{n-1}dr)}$$

where $p+q=pq$. An estimate for the L^q norm of $\varphi_k^{n/2-1}$ can be read off from Lemma 1 of Markett [5]. For $1 \le p < 2n/(n+1)$ it easily follows that the estimates (1.11) are valid.

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