Pak. J. Statist,
1993 Vol.9(2)A, pp 123-136

ON ESTIMATING THE VARIANCE OF HORVITZ-THOMPSON ESTIMATOR

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Abstract

In this paper a new model-based variance estimator is proposed. It is along the lines of those suggested by Hanif and Brewer(1980) and Kumar-Gupta-Agarwal(1985). Empirical investigations show that the amount of bias is fairly small for all values of g and that it is more stable than all other estimators considered.

Key Words

Sen-Yates-Grundy estimator; Horvitz-Thompson estimator; Super population model, Model-based and design-based estimator.

1. Introduction

Horvitz and Thompson (1952) suggested an unbiased estimator for population total Y i.e.

$$y'_{HT} = \sum_{i=1}^{n} \frac{y_i}{\pi_i} \tag{1.1}$$

The variance and variance estimator derived independently by Sen(1953) and Yates and Grundy(1953) were

$$Var_{SYG}(y'_{HT}) = \frac{1}{2} \sum\sum_{\substack{I,J=1 \\ I \neq J}}^{N} (\pi_I \pi_J - \pi_{IJ})\left(\frac{Y_I}{\pi_I} - \frac{Y_J}{\pi_J}\right)^2 \tag{1.2}$$

and

$$var_{SYG}(y'_{HT}) = \frac{1}{2}\sum\sum_{\substack{i,j=1 \\ j \neq i}}^{n} \frac{(\pi_i\pi_j - \pi_{ij})}{\pi_{ij}}(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j})^2, \qquad (1.3)$$

where π_i and π_{ij} are the inclusion probabilities for the unit i and a pair of units (i,j) in the sample size n. In order for (1.3) to hold, n must be fixed. A large number of selection procedures have been advanced for which $\pi_I = np_I$, $I = 1,2,\ldots N$, where p_I is the initial probability of selection of Ith population unit. For $\pi_I = n/N$ and $\pi_{IJ} = \frac{n(n-1)}{N(N-1)}$, (1.3) becomes

$$var_{SYG}(y'_{HT}) = \frac{N^2}{n} \; \frac{N-n}{N} \; \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 = var(y'_{ran}), \qquad (1.4)$$

which is an ordinary unbiased variance estimator as if simple random sample without replacement from a finite population has been selected.

Since π_{IJ} is involved in (1.3), the calculation becomes cumbersome for large n. In order to overcome these difficulties Hartley and Rao (1962) derived an asymptotic form of π_{IJ} for random systematic selection procedure. Two sets of approximate expressions of π_{IJ} were also suggested by Brewer and Hanif(1983), Herzel(1986) gave an approximate formula for π_{IJ}, Asok and Sukhatme (1976) also evaluated approximate expression for π_{IJ} correct to $O(N)^{-1}$ for Sampford's selection procedure, but all these have their limitations. Jessen (1978) derived an variance estimator which did not involve π_{IJ}. Hanif and Brewer(1980) and Kumar, Gupta and Agarwal(1985) derived model-based variance estimators.

(i) Jessen's variance estimator:

The variance estimator given by Jessen (1978) is

$$var_J(y'_{HT}) = \frac{n - \sum_{I=1}^{N} \pi_I^2}{2n(n-1)} \sum\sum_{\substack{i,j=1 \\ j \neq i}}^{n} (\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j})^2 \qquad (1.5)$$

(ii) Hanif-Brewer variance estimator:

Hanif and Brewer (1980) suggested model-based variance estimator i.e.

$$var_{HB}(y'_{HT}) = \frac{n}{n-1}\left[1 - \frac{\sum_{I=1}^{N} \pi_I^g}{\sum_{I=1}^{N} \pi_I^{g-1}}\right] \sum_{i=1}^{n}\left[\frac{y_i}{\pi_i} - \frac{y'_{HT}}{n}\right]^2 \qquad (1.6)$$

Putting $g = 2$,

$$var_{HB}(y'_{HT}) = \frac{n}{n-1}\left[1 - \frac{\sum_{I=1}^{N}\pi_I^2}{\sum_{I=1}^{N}\pi_I}\right]\sum_{i=1}^{n}\left(\frac{y_i}{\pi_i} - \frac{y'_{HT}}{n}\right)^2.$$

and using the result

$$\sum_{i=1}^{n}\left(\frac{y_i}{\pi_i} - \frac{y'_{HT}}{n}\right)^2. = \sum\sum_{\substack{i,j=1\\ j<i}}^{n}\left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2 \tag{1.7}$$

we get

$$var_{HB}(y'_{HT}) = \frac{n - \sum_{I=1}^{N}\pi_I^2}{2n(n-1)}\sum\sum_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2 \equiv var_J(y'_{HT}).$$

Hence Jessen's variance estimator is the special case of Hanif-Brewer variance estimator for $g = 2$

(iii) Kumar-Gupta-Agarwal variance estimator:

Kumar, Gupta and Agarwal (1985) proposed another model-based variance estimator

$$var_{KGA}(y'_{HT}) = \frac{\sum_{I=1}^{N}P_I^{g-1}(1-nP_I)}{2(n-1)\sum_{I=1}^{N}P_I^{g-1}}\sum\sum_{\substack{i,j=1\\ j\neq i}}^{n}\left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2 \tag{1.8}$$

Using $\pi_I = nP_I$ and the result (1.7) we get

$$var_{KGA}(y'_{HT}) = \frac{n}{n-1}\left[1 - \frac{\sum_{i=1}^{N}\pi_I^g}{\sum_{i=1}^{N}\pi_I^{g-1}}\right]\sum_{i=1}^{n}\left(\frac{y_i}{\pi_i} - \frac{y'_{HT}}{n}\right)^2 \equiv var_{HB}(y'_{HT}). \tag{1.9}$$

Hence variance estimator proposed by Kumar, Gupta and Agarwal is identical with the estimator proposed by Hanif and Brewer. For simple random sampling without replacement this turns to be

$$var_{HB}(y'_{HT}) = \frac{N^2}{n}\frac{N-n}{N}\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 = var(y'_{ran})$$

This necessarily holds also for the Kumar-Gupta-Agarwal variance estimator (with which it is identical) and for the Jessen's variance estimator (which is a special case).

2. New Variance Estimator

Consider the following super-population model as in Royall and Cumberland (1981), Kumar et al (1985),

$$Y_I = \beta P_I + e_I, \quad I = 1, 2, 3, \cdots, N$$

where

$$\varepsilon(e_I|P_I) = 0 = \varepsilon(e_I, e_J|P_I, P_J), J \neq I$$
$$\varepsilon(e_I^2|P_I) = aP_I^g \quad \text{for } a > 0 \quad g \geq 0 \tag{2.1}$$

then (Rao, 1966)

$$\varepsilon[Var_{SYG}(y'_{HT})] = \frac{a}{n}\sum_{I=1}^{N} P_I^{g-1}(1 - nP_I) \tag{2.2}$$

It also follows that $var_{KGA}(y'_{HT})[\equiv Var_{HB}(y'_{HT})]$ is design-model-unbiased for any πps - design under the model (2.1) i.e.

$$E_p\left[\varepsilon\{var_{KGA}(y'_{HT})\}\right] \equiv E_p\left[\varepsilon\{var_{HB}(y'_{HT})\}\right] = E_p\left[\varepsilon\{Var_{SYG}(y'_{HT})\}\right],$$

where E_p denotes expectation with respect to sampling design.

It makes good sense to use the quantities $(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j})^2$ for $J \neq I$ as building blocks for constructing an estimator of $Var_{SYG}(y'_{HT})$. If we divide each of these terms by its model expectation (which is a function of P_I and P_J) and take their sum T_S its model expectation will be a constant for all samples. T_S will provide a basis for providing an estimator of (2.2). The variance estimator (1.8) is design-model unbiased but its model expectation is not a constant for all samples. It appears, therefore, that the proposed estimator based on T_S would be more stable than the one given by Kumar, Gupta-Agarwal and by Hanif - Brewer. We note first that

$$\varepsilon(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j})^2 = \frac{a}{n^2}[P_I^{g-2} + P_J^{g-2}]$$

Consider the estimator

$$var_N(y'_{HT}) = b\sum_{\substack{I,J \in S \\ I<J}}\sum \frac{(\frac{Y_I}{\pi_I} - \frac{Y_J}{\pi_J})^2}{P_I^{g-2} + P_J^{g-2}} = bT_S(say), \tag{2.3}$$

where b is a suitable constant so that

$$\varepsilon[bT_s - V_{SYG}(y'_{HT})] = 0 \qquad \text{for all } s : P(s) > 0$$

This gives

$$b = \frac{2}{n-1}\sum_{I=1}^{N} P_I^{g-1}(1 - nP_I) \tag{2.4}$$

Hence the new variance estimator is

$$var_N(y'_{HT}) = \frac{1}{n-1}\sum_{I=1}^{N} P_I^{g-1}(1 - nP_I)\sum_{\substack{j,i=1 \\ j\neq i}}^{n}\sum\left[\frac{(\frac{Y_I}{\pi_I} - \frac{Y_J}{\pi_J})^2}{P_I^{g-2} + P_J^{g-2}}\right] \tag{2.5}$$

For simple random sampling without replacement (2.5) takes the form as

$$var_N(y'_{HT}) = \frac{N^2}{n}\frac{N-n}{N}\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 = var(y'_{ran})$$

which is consistent with other variance estimators given in this paper.

The estimator $var_N(y'_{HT})$ is model-unbiased and hence design-model-unbiased. The estimator is, however, design-biased and use of this estimator like $var_{KGA}(y'_{HT})[\equiv var_{HB}(y'_{HT})]$, pre-supposes a knowledge of g. It is well known that g lies between 0 and 2 for a wide class of socio-economic surveys (Scott et al, 1978). The value of g can be approximately assessed from a past survey of the similar nature. In what follows we study here the extent of design-bias involved in the variance estimators suggested by Jessen, Kumar-Gupta-Agarwal [$\equiv$ Hanif-Brewer] and new variance estimator suggested in this paper. The stabilities of these variance estimators were also studied using some natural populations given in Table-1.

3. Bias in $var_N(y'_{HT})$

In order to study the extent of bias, some natural populations [given in Table 1] were used ; The samples were drawn by using the Sampford (1967) selection procedure. The different values of g considered were $0.0, 0.5, 1.0, 1.5, 2.0, 2.5$ and 3.0. The quantity

$$\Delta = |[E_p\{var^*(y'_{HT})\}/Var_{SYG}(y'_{HT})] - 1 \mid \times 100$$

was taken as a measure of the relative absolute bias of the estimator $var^*(y'_{HT})$. The values of Δ for the estimators $var_J(y'_{HT})$, $var_{KGA}(y'_{HT})$, $var_N(y'_{HT})$ for these populations for different values of $n = 2, 3, 4$ and for different values of g have been given in Table 2 and 3. $var_N(y'_{HT})$ is seen to be appreciably more biased than $var_{KGA}(y'_{HT})$ for values of $g < 2.0$ for $n = 3, 4$. For higher values of g, $var_N(y'_{HT})$ seem to have the relative less bias of the same magnitude for all values of n. The new estimator as compared with Hanif-Brewer estimator is substantially more biased for the lower (more plausible) range of g-values.

4. Stability of $var_N(y'_{HT})$

In order to study the stability of the variance estimators the parameter

$$\delta = \frac{V_p\{var^*(y'_{HT})\}}{[E_p\{var^*(y'_{HT})\}]^2}$$

was calculated for all the estimators $var_{SYG}(y'_{HT})$, $var_J(y'_{HT})$, $var_{KGA}(y'_{HT})$ and $var_N(y'_{HT})$, where V_p denotes variance with respect to sampling design. The calculations have been presented in Table 4 and 5. It is found that for $g = 0, 0.5, 1.0, 1.5$, $var_N(y'_{HT})$ has a smaller value of δ than all the other estimators for all the values of n considered. For $g = 2$ the stability of Jessen, KGA and New estimator is identical. For $var_N(y'_{HT})$ the value of δ increases as g increases. The value of g does not seem to have any significant effect on any of the estimators considered. This suggests that the proposed estimator can be used for any arbitrary choice of g in [0,2]. Infact the stability of new estimator proposed here is only slightly better. This is disappointing but worth noting. Stability for various values of g for $var_{HB}(y'_{HT})$ does not vary significantly [Table 6].

5. Conclusions

The following conclusions may be drawn from this empirical study.

1. The suggested estimator $var_N(y'_{HT})$ is design-biased. However, it is seen empirically that the amount of bias involved is fairly large for all the values of g considered. The estimator performs fairly well in comparison with $var_{KGA}(y'_{HT})$. For $g \geq 2.0$ [$g > 2$ is not very realistic] its performance is found to be better than that of $var_{KGA}(y'_{HT})$ for higher values of n.

2. The proposed estimator is empirically seen to be more stable than all the other estimators considered here. The stability of the proposed estimator though

does not depend significantly on the value of g, decreases with increasing value of g. The estimator $var_N(y'_{HT})$ may be used for any value of g between 0 and 3.

Acknowledgement

The authors are indebted to Prof. K.R.W. Brewer for his help throughout the preparation of this paper. Thanks are also extended to the referee for constructive criticism.

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Table 1

Description of the Populations

Sl. no.	Source	Y	X	N	c.v(x)	c.v(y)	ρ
1.	Sampford (1967) p.507	-	-	10	0.4050	0.2169	0.43
2.	Rao (1963) p.207	Corn acreage in 1960	Corn acreage in 1958	14	0.4176	0.3786	0.93
3.	Murthy (1977) p.178	area under wheat	Geographical area	20	0.1882	0.3675	- 0.07
4.	Murthy (1977) p.127	No. of cultivators 1961 census	Cultivated area 1951 census	17	0.5610	0.5784	0.88
5.	Murthy (1977) p.131	Volume of timber	length of timber	13	0.3074	0.5569	0.92
6.	Murthy (1977) p. 228	output	No. of workers	16	0.3422	0.2728	0.86
7.	Jessen (1978) p. 151	No. of fish tagged	Total No. of fish	16	0.5708	0.4739	0.90
8.	Jessen (1978) p. 153	No. of h.h. in 1960	No. of h.h. in 1950	13	0.4715	0.4279	0.52
9.	Konijn (1973) p. 389	Measurement obtained in reinterview	Measurement obtained in first interview	10	0.2020	0.1515	0.77
10.	Konijn (1973) p. 49	Food expen-diture	Total expen-diture	16	0.0781	0.115	0.95
11.	Murthy (1977) p. 398	No. of absentees	No. of workers	18	0.4775	0.5814	0.64
12.	Murthy (1977) p. 339	Area under wheat in 1964	Cultivated area in 1961	15	0.4938	0.6935	0.94
13.	Raj (1972) p. 70	No. of cattle	No. of farms	15	0.4197	0.4144	0.77
14.	Sukhatme (1970) p. 183	Area under wheat (1973)	Area under wheat (1936)	20	0.7231	0.7440	0.98
15.	*Som (1973) p. 225	Average no. of persons per h.h.	Average monthly expenditure on cereals per h.h.	16	0.3510	0.2430	- 0.57

Though X should be taken as average no. of persons per h.h. for population 15, it is clear that the overall conclusion from this study would not be affected if X, Y are taken as shown in this table. Mainly $\rho(X,Y)$ should count.

Table 2

Percent absolute relative bias of the variance estimator

n = 2

Pop. no.	$var_J(y'_{HT})$	$var_{KGA}(y'_{HT}) \equiv var_{HB}(y'_{HT})$ g=0	0.5	1.0	1.5	2.0	2.5	3.0	$var_N(y'_{HT})$ g=0	0.5	1.0	1.5	2.0	2.5	3.0
1	6.5	1.7	0.4	2.6	4.6	6.5	8.3	9.9	11.7	16.6	16.3	12.5	6.5	0.2	7.4
2	4.1	1.8	1.2	1.3	2.7	4.1	5.6	7.1	9.7	0.6	4.2	5.6	4.1	0.3	8.1
3	0.4	0.3	0.2	0.0	0.2	0.4	0.6	0.8	2.1	0.5	0.4	0.7	0.4	0.6	2.3
4	4.1	4.3	2.3	0.1	2.1	4.1	5.9	7.4	70.2	24.4	2.9	4.9	4.1	3.1	17.1
5	1.0	2.5	1.6	0.7	0.2	1.0	1.8	2.5	8.3	1.5	1.7	2.2	1.0	1.2	4.1
6	1.9	1.3	0.6	0.2	1.1	1.9	2.7	3.5	3.2	1.3	3.4	3.6	1.9	1.7	7.6
7	1.7	8.1	5.7	3.1	0.5	1.7	3.5	4.9	90.3	29.6	3.4	4.1	1.7	5.0	14.3
8	0.5	9.4	6.4	3.7	1.4	0.5	2.0	3.2	81.6	33.6	11.0	1.7	0.5	0.9	4.5
9	8.8	6.5	7.2	7.8	8.3	8.8	9.1	9.4	38.3	32.3	25.3	17.4	8.8	0.4	9.8
10	0.2	0.3	0.3	0.2	0.2	0.2	0.1	0.1	2.0	1.4	0.9	0.5	0.2	0.1	0.2
11	5.1	0.1	1.0	2.3	3.7	5.1	6.4	7.5	2.3	9.3	13.2	11.5	5.1	5.7	22.3
12	2.4	7.9	4.3	1.3	0.8	2.4	3.7	4.8	102.1	29.4	3.4	3.9	2.4	2.5	8.9
13	0.5	4.9	3.7	2.3	0.9	0.5	1.7	2.9	51.5	25.6	10.3	2.4	0.5	0.3	4.2
14	3.4	11.0	6.8	2.5	0.8	3.4	5.4	6.8	600.3	103.3	13.4	5.9	3.4	8.9	31.0
15	3.2	0.2	0.6	1.4	2.3	3.2	4.0	4.8	18.8	17.4	14.0	9.0	3.2	3.4	10.9

Table 2 (Contd.)

Percent absolute relative bias of the variance estimator

n = 3

Pop. no.	$var_J(y'_{HT})$	$var_{KGA}(y'_{HT}) \equiv var_{HB}(y'_{HT})$ g=0	0.5	1.0	1.5	2.0	2.5	3.0	$var_N(y'_{HT})$ g=0	0.5	1.0	1.5	2.0	2.5	3.0
1	10.3	3.7	0.1	3.6	7.1	10.3	13.3	16.0	10.1	16.2	17.2	14.7	10.3	5.3	0.0
2	6.7	2.0	0.1	2.0	4.3	6.7	9.1	11.4	10.5	0.7	4.9	7.2	6.7	3.4	3.0
3	0.6	0.6	0.3	0.0	0.3	0.6	1.0	1.3	2.3	0.6	0.4	0.8	0.6	0.2	1.8
4	6.6	6.9	3.6	0.1	3.4	6.6	9.5	12.0	74.4	26.1	2.9	6.2	6.6	0.8	11.4
5	1.6	4.4	2.8	1.3	0.2	1.6	2.9	4.0	10.2	2.7	1.1	2.2	1.6	0.1	2.5
6	3.0	2.2	1.0	0.3	1.6	3.0	4.3	5.5	4.1	0.8	3.5	4.1	3.0	0.0	5.3
7	2.7	13.5	9.5	5.2	1.0	2.7	5.6	7.9	99.5	34.2	5.5	3.7	2.7	2.7	10.7
8	0.6	16.2	11.2	6.5	2.6	0.6	3.1	5.1	92.8	39.5	14.0	2.9	0.6	0.2	2.3
9	14.2	10.5	11.7	12.7	13.5	14.2	14.8	15.2	41.0	35.6	29.2	22.1	14.2	5.9	2.8
10	0.3	0.5	0.5	0.4	0.3	0.3	0.2	0.1	2.2	1.6	1.0	0.6	0.3	0.0	0.2
11	8.1	0.2	1.6	3.7	5.9	8.1	10.1	11.8	2.4	9.9	14.4	13.4	8.1	1.5	16.5
12	3.8	13.2	7.2	2.4	1.2	3.8	6.0	7.7	112.0	33.0	4.5	4.3	3.8	0.1	5.5
13	0.7	8.2	6.1	3.8	1.5	0.7	2.8	4.7	56.2	28.5	12.0	3.0	0.7	0.8	2.2
14	5.6	17.7	10.9	4.0	1.4	5.6	8.7	11.0	641.8	111.0	15.0	6.4	5.6	5.0	25.0
15	4.9	0.5	0.8	2.1	3.5	4.9	6.3	7.6	18.6	17.6	14.6	10.2	4.9	0.1	7.7

Table 2 (Contd.)

Percent absolute relative bias of the variance estimator

n = 4

Pop. no.	$var_J(y'_{HT})$	$var_{KGA}(y'_{HT}) \equiv var_{HB}(y'_{HT})$ g=0	0.5	1.0	1.5	2.0	2.5	3.0	$var_N(y'_{HT})$ g=0	0.5	1.0	1.5	2.0	2.5	3.0
1	11.4	0.3	2.3	5.2	8.3	11.4	14.2	16.2	2.4	10.5	15.8	15.8	11.4	3.1	10.1
2	9.6	3.0	0.2	2.8	0.2	9.6	13.1	16.5	11.6	0.8	5.7	9.0	9.6	7.7	2.9
3	1.0	12.3	9.2	5.8	2.4	1.0	4.1	6.8	62.0	32.2	14.1	3.9	1.0	2.1	0.1
4	8.2	25.4	15.6	5.6	2.2	8.2	12.7	15.9	689.3	119.8	16.8	7.1	8.2	0.5	17.9
5	2.4	1.6	0.6	0.5	1.5	2.4	3.3	4.1	1.2	3.0	3.6	3.3	2.4	1.0	0.9
6	4.3	7.4	1.8	3.8	9.3	4.3	19.0	23.1	6.9	14.7	17.4	16.7	14.3	11.5	8.5
7	3.7	20.3	14.6	8.1	1.8	3.7	8.0	11.4	111.2	40.1	8.3	2.9	3.7	0.1	6.4
8	5.9	4.2	2.1	0.2	2.8	5.9	9.2	12.9	2.6	0.7	0.4	1.0	5.9	0.2	1.2
9	9.6	9.9	5.2	0.1	4.9	9.6	13.7	17.2	79.3	28.0	2.9	7.6	9.6	5.4	4.6
10	2.1	6.8	4.5	2.2	0.0	2.1	4.0	5.8	12.8	4.3	0.2	2.0	2.1	1.1	0.6
11	6.3	1.0	0.7	2.5	4.4	6.3	8.1	9.9	5.2	8.2	9.3	8.6	6.3	2.2	4.2
12	5.8	9.3	5.7	1.8	2.1	5.8	9.0	11.7	64.5	28.4	8.0	2.3	5.8	4.5	1.0
13	0.3	25.5	17.8	10.6	4.6	0.3	4.2	7.3	108.1	47.7	18.4	4.9	0.3	1.4	0.0
14	20.6	15.2	16.9	18.4	19.6	20.6	21.4	22.0	44.1	39.4	33.9	27.6	20.6	13.2	5.6
15	6.8	0.9	0.9	2.8	4.8	6.8	8.7	10.6	18.2	17.6	15.1	11.3	6.8	1.7	4.1

Table 3

Percent absolute relative bias of the variance estimator (Summary table)

	n	$var_J(y'_{HT})$	$var_{KGA}(y'_{HT}) \equiv var_{HB}(y'_{HT})$ g=0	0.5	1.0	1.5	2.0	2.5	3.0	$var_N(y'_{HT})$ g=0	0.5	1.0	1.5	2.0	2.5	3.0
Mean	2	2.9	4.0	2.5	2.0	2.0	2.9	4.0	5.0	72.8	21.8	8.2	5.7	2.9	2.3	10.2
	3	4.6	6.8	4.4	3.2	3.2	4.6	6.5	8.1	78.5	23.9	9.7	5.8	4.6	1.8	6.5
	4	6.2	9.8	6.4	4.5	4.8	6.2	9.4	13.1	72.4	26.3	11.3	8.2	6.2	3.7	4.5
Ist Quartile	2	0.5	0.3	0.4	0.2	0.5	0.5	1.8	2.9	3.2	1.4	2.9	2.2	0.5	0.3	4.2
	3	0.7	0.6	0.5	0.4	1.0	0.7	2.9	4.7	4.1	1.6	2.9	2.0	0.7	0.1	2.2
	4	2.1	1.0	0.7	1.8	1.8	2.1	4.1	6.8	5.2	4.3	3.6	2.9	2.1	1.0	0.9
Median	2	2.4	2.5	1.6	1.4	1.1	2.4	3.7	4.8	18.8	17.4	4.2	4.1	2.4	1.2	8.1
	3	3.8	4.4	2.8	2.4	1.6	3.8	6.0	7.7	18.6	17.6	5.5	3.0	3.8	8.8	3.0
	4	5.8	7.4	4.5	2.8	4.4	5.8	8.7	11.4	18.2	17.6	9.3	7.1	5.8	2.1	4.1
3rd Quartile	2	4.1	7.9	5.7	2.6	2.7	4.1	5.9	7.4	81.6	29.6	13.4	9.0	4.1	3.4	14.3
	3	6.7	13.2	9.6	4.0	4.3	6.7	9.5	11.8	92.8	34.2	14.6	6.4	6.7	3.4	10.7
	4	9.6	15.2	14.6	5.8	6.1	9.6	13.7	16.5	79.3	39.4	16.8	11.3	9.6	5.4	6.4
Lower extreme	2	0.2	0.2	0.0	0.0	0.2	0.2	0.1	0.1	2.0	0.5	0.4	0.5	0.2	0.1	0.2
	3	0.3	0.2	0.1	0.0	0.2	0.3	0.2	0.1	2.2	0.6	0.4	0.6	0.3	0.0	0.0
	4	0.3	0.3	0.2	0.0	0.0	0.3	1.4	1.8	1.2	0.7	0.2	0.1	0.3	0.1	0.0
upper extreme	2	8.8	11.0	7.2	7.8	8.3	8.8	9.1	9.9	600.3	103.3	25.3	17.4	8.8	8.9	31.0
	3	14.2	17.7	11.7	12.7	13.5	14.2	14.8	16.0	641.8	111.0	29.2	22.1	14.2	5.9	25.0
	4	20.6	25.5	17.8	18.4	19.6	20.6	21.4	23.1	689.0	119.8	33.9	27.6	20.6	13.2	17.9

Table 4

Stability of the Variance estimators *

n = 2

Pop no.	*SYG*	*Jessen*	*KGA ≡ HB*	New g = 0	0.5	1.0	1.5	2.0	2.5	3.0
1	2.2	2.0	2.0	1.1	1.2	1.4	1.7	2.0	2.3	2.4
2	1.4	1.3	1.3	1.2	1.2	1.2	1.2	1.3	1.4	1.5
3	1.6	1.6	1.6	1.4	1.5	1.5	1.6	1.6	1.6	1.7
4	1.7	1.5	1.5	1.4	1.3	1.2	1.3	1.5	1.9	2.6
5	1.6	1.6	1.6	1.4	1.4	1.4	1.5	1.6	1.7	1.7
6	1.4	1.3	1.3	1.3	1.2	1.2	1.2	1.3	1.5	1.8
7	2.4	2.2	2.2	1.4	1.5	1.4	1.7	2.2	2.7	3.1
8	1.9	2.0	2.0	1.8	1.9	1.9	2.0	2.0	2.0	2.0
9	6.5	6.5	6.5	6.6	6.6	6.6	6.6	6.5	6.5	6.6
10	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6
11	3.5	3.2	3.2	1.6	1.7	2.1	2.6	3.2	3.8	4.7
12	1.7	1.6	1.6	1.4	1.4	1.4	1.4	1.6	1.9	2.1
13	1.4	1.4	1.4	1.9	1.7	1.5	1.4	1.4	1.4	1.5
14	2.7	2.2	2.2	2.5	1.8	1.4	1.4	2.2	3.8	8.0
15	2.5	2.5	2.5	1.4	1.7	2.0	2.3	2.5	2.6	2.8

Table 4 (Contd.)

Stability of the Variance estimators *

n = 3

Pop. no.	*SYG*	*Jessen*	*KGA ≡ HB*	New g = 0	0.5	1.0	1.5	2.0	2.5	3.0
1	1.2	1.0	1.0	0.4	0.5	0.6	0.8	1.0	1.2	1.3
2	0.6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.6	0.6
3	0.7	0.7	0.7	0.6	0.7	0.7	0.7	0.7	0.8	0.8
4	0.9	0.7	0.7	0.6	0.5	0.5	0.6	0.7	0.9	1.2
5	0.8	0.7	0.7	0.7	0.6	0.6	0.7	0.7	0.8	0.8
6	0.6	0.6	0.5	0.5	0.5	0.5	0.5	0.5	0.6	0.8
7	1.4	1.1	1.1	0.8	0.7	0.7	0.8	1.1	1.4	1.7
8	1.0	1.0	1.0	0.9	1.0	1.0	1.0	1.0	1.0	1.0
9	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0
10	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
11	2.1	1.8	1.8	0.7	0.9	1.2	1.5	1.8	2.1	2.6
12	0.8	0.7	0.7	0.6	0.6	0.6	0.6	0.7	0.9	1.0
13	0.6	0.6	0.6	0.8	0.7	0.6	0.6	0.6	0.6	0.6
14	1.6	1.2	1.2	1.0	0.7	0.6	0.7	1.2	2.0	3.8
15	2.5	1.3	1.3	0.7	0.8	1.0	1.2	1.3	1.4	1.5

Table 4 (Contd.)

Stability of the Variance estimators

n = 4

pop no.	SYG	Jessen	KGA ≡ HB	New g = 0	0.5	1.0	1.5	2.0	2.5	3.0
1	1.5	1.2	1.2	0.4	0.6	0.8	1.0	1.2	1.5	1.7
2	0.4	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
3	0.4	0.3	0.3	0.5	0.4	0.4	0.3	0.3	0.3	0.4
4	1.3	0.8	0.8	0.5	0.4	0.3	0.4	0.8	1.3	2.4
5	0.6	0.6	0.6	0.5	0.5	0.6	0.6	0.6	0.6	0.6
6	0.5	0.6	0.6	0.2	0.3	0.4	0.5	0.6	0.8	0.9
7	1.1	0.7	0.7	0.5	0.4	0.4	0.5	0.7	1.0	1.2
8	0.8	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.5
9	0.7	0.4	0.4	0.3	0.3	0.3	0.4	0.4	0.5	0.7
10	2.8	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.5
11	0.5	0.4	0.4	0.3	0.3	0.3	0.4	0.4	0.6	0.7
12	0.9	0.7	0.7	1.0	0.9	0.8	0.7	0.7	0.7	0.8
13	0.2	0.6	0.6	0.5	0.6	0.6	0.6	0.6	0.6	0.6
14	0.5	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8
15	1.0	0.9	0.9	0.4	0.5	0.7	0.8	0.9	1.0	1.1

* Variation in values of measure of stability for various value of g for $var_{HB}(y'_{HT})$ are very insignificant (see Table 6).

Table 5

Stability of the Variance estimators (Summary Table) *

1		SYG	Jessen	KGA ≡ HB	New g = 0	0.5	1.0	1.5	2.0	2.5	3.0
	2	2.3	2.2	2.2	1.9	1.8	1.9	2.0	2.2	2.4	2.9
Mean	3	1.3	1.1	1.1	0.9	0.9	0.9	1.0	1.1	1.3	1.5
	4	0.9	0.7	0.7	0.6	0.6	0.6	0.7	0.7	0.9	1.0
Ist	2	1.6	1.5	1.5	1.4	1.3	1.4	1.4	1.5	1.6	1.7
Quartile	3	0.7	0.7	0.7	0.6	0.5	0.6	0.6	0.7	0.7	0.8
	4	0.5	0.4	0.4	0.3	0.3	0.3	0.4	0.4	0.4	0.5
	2	1.7	1.6	1.6	1.4	1.5	1.4	1.6	1.6	1.9	2.1
Median	3	0.9	0.7	0.7	0.7	0.7	0.6	0.7	0.7	0.9	1.0
	4	0.7	0.6	0.6	0.4	0.4	0.4	0.5	0.6	0.6	0.7
	2	2.5	2.2	2.2	1.9	1.7	1.9	2.0	2.2	2.7	3.1
3rd	3	1.4	1.2	1.2	0.8	0.8	1.0	1.0	1.2	1.4	1.7
Quartile	4	1.1	0.8	0.8	0.5	0.6	0.8	0.7	0.8	1.0	1.2
	2	1.4	1.3	1.3	1.1	1.2	1.2	1.2	1.3	1.1	1.5
Lower	3	0.6	0.5	0.5	0.4	0.5	0.5	0.5	0.5	0.6	0.6
extreme	4	0.2	0.2	0.3	0.2	0.3	0.3	0.3	0.3	0.3	0.3
	2	6.5	6.5	6.5	6.6	6.6	6.6	6.6	6.5	6.5	8.0
upper	3	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0
extreme	4	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8

* Stability of var_{KGA} (} var_{HB}) does not vary significantly with g as may be seen from table 6.

Table 6

Stability of $var_{HB}(y'_{HT}) \equiv var_{KGA}(y'_{KGA})$

n = 2

Pop. no.	g = 0.0	0.5	1.0	1.5	2.0	2.5	3.0
1.	2.018507	2.018505	2.018506	2.018506	2.018507	2.018507	2.018505
2.	1.025620	1.025621	1.025620	1.025620	1.025620	1.025620	1.025621
3.	1.601954	1.601958	1.601957	1.601958	1.601955	1.601956	1.601956
4.	1.545977	1.545978	1.545977	1.545977	1.545976	1.545977	1.545976
5.	1.575560	1.575558	1.575560	1.575557	1.575557	1.575557	1.575558
6.	1.636603	1.636602	1.636604	1.636604	1.636605	1.636605	1.636605
7.	2.244465	2.244466	2.244466	2.244465	2.244466	2.244467	2.244466
8.	1.963425	1.963427	1.963426	1.963427	1.963426	1.963428	1.963425
9.	6.533597	6.533596	6.533594	6.533597	6.533596	6.533597	6.533596
10.	1.627174	1.627173	1.627172	1.627173	1.627173	1.627173	1.627173
11.	3.201249	3.201246	3.201248	3.201247	3.201247	3.201248	3.201244
12.	1.566458	1.566458	1.566458	1.566460	1.566458	1.566459	1.566460
13.	1.364494	1.364491	1.364492	1.364492	1.364492	1.364493	1.364492
14.	2.207424	2.207426	2.207428	2.207430	2.207425	2.207428	2.207425