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Optimality and Construction of Some Rectangular Designs

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Abstract: We obtain a sufficient condition for E-optimality of equireplicate designs. As an application, we prove E-optimality of certain types of three-class PBIBDs based on rectangular association scheme – in short – rectangular designs. These designs turn out to be highly efficient with respect to the A-criterion as well. We also observe that these designs, though themselves not regular graph designs (RGD's), are yet strictly E-better than every competing RGD, whenever $v \geq 26$ and $v \equiv 2$ (mod 4). This provides an infinite series of counter examples to the conjecture of John and Mitchell (1977).

We also present two methods of construction of the rectangular designs. Apart from providing infinitely many examples of the designs proved E-optimal in this paper and in Cheng and Constantine (1986), this construction also provides – as a special case – the first known infinite series of most balanced group divisible designs, which were proved optimal with respect to all type 1 criteria by Cheng (1978).

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1 Introduction

Cheng and Constantine (1986) considered the class of regular generalized line graph designs (which is a subclass of regular graph designs (RGD)) and proved that such designs are E-optimal for block size $k \geq 3$, provided the degree of the underlying graph is $\geq v/2 - 2$ ($v =$ the number of treatments). In Bagchi and Cheng (1991), the restriction on k is removed.

In section 2 of this paper we note that the above result can be generalized without any difficulty to prove the E-optimality of certain equireplicate designs which are not necessarily RGD's provided they satisfy a condition similar to the degree condition mentioned above.

Recall that a rectangular association scheme with m rows and n columns is the scheme with mn treatments arranged in an $m \times n$ array; two treatments are first, second or third associate according as they are in the same row, same column or neither. We shall call a three-class PBIBD a rectangular design if the underlying association scheme is rectangular. We call a rectangular design balanced of type 2 if the underlying scheme has $m = 2$ rows and the design parameters λ_1, λ_2, λ_3 are related by

$$\lambda_1 = \lambda_2 - 2 = \lambda_3 - 1 .$$

A balanced rectangular design of type 2 is in the extended class of E-optimal designs considered in section 2. Clearly these are not RGD's. We study them in more details in section 3 of this paper.

Why type 2 ? This terminology arises because we have decided to call a rectangular design balanced of type 1 if it has $m = 2$ rows and λ_i's satisfying $\lambda_1 = \lambda_2 = \lambda_3 - 1$. In contradistinction to those of type 2, a balanced rectangular design of type 1 is an RGD (in fact a regular generalized line graph design) and hence it is E-optimal by Cheng and Constantine (1988) and Bagchi and Cheng (1991).

John and Mitchell (1977) conjectured that whenever RGD's exist, optimal block designs are found among them. Since then, optimality property of a good many RGD's have been proved [see Shah and Sinha (1989) for a detailed survey]. However, examples of designs other than RGD's better than the best RGD have also been found in Jones and Eccleston (1980) (A-crireion) and in Constantine (1986) (E-criterion). In this paper we provide infinitely many counter examples to the John and Mitchell (1977) conjecture for the E-criterion. [See Theorem 3.2].

However, when $v = 0 \pmod 4$, a balanced rectangular design of type 2 loses to a certain type of three class PBIBD which is an RGD. We call this a double cocktail party graph design. [See Definition 3.4 and Theorem 3.4].

In section 4, we discuss constructional aspects. Two very general difference set methods are discussed, yielding rectangular designs. As special cases we get an infinite series of most balanced group divisible designs of type 1 [See Cheng (1978), for the definition and optimality result], as well as infinite series of balanced rectangular designs of type 1 and type 2.

2 The Main Result

Let $\mathscr{D}(b, k, v)$ denote the collection of all block designs with v treatments and b blocks of size k, where $k < v$. We assume that v divides bk and we denote bk/v by r.

We shall use the following well-known terms and notations in this paper, for the explanation of which we refer to Shah and Sinha (1989) and Raghavarao

(1971): Incidence matrix ($N = N(v \times b)$), C-matrix ($C = C(v \times v)$), partially balanced incomplete block design (PBIBD), association scheme.

For the terms of graph theory, we refer to any standard book on graph theory.

Notation: (i) $\lambda = [r(k-1)/(v-1)]$

(ii) $e = (v-1)(\lambda+1) - r(k-1)$

Here $[x]$ is the largest integer $\leq x$.

For an equireplicate design d, we define a matrix B as follows.

$$B = (r - \lambda - 1)I + (\lambda + 1)J - NN^T \tag{2.1}$$

where J is the all-one matrix.

Note that each diagonal entry of B is zero and the off-diagonal entries are integers. Further, each row sum of B is e.

Remark: If d is an RGD, then B is nothing but the adjacency matrix of the graph $G(d)$ (and e is its degree) defined in section 1 of Bagchi and Cheng (1991).

Remark: Clearly we have

$$kC_d = \{r(k-1) + \lambda + 1\}I - (\lambda + 1)J + B$$

$$\text{and hence} \quad \mu_i(kC_d) = r(k-1) + \lambda + 1 + \mu_i(B) \tag{2.2}$$

where $\mu_1(M) \leq \mu_2(M) - - \leq \mu_{n-1}(M)$ denote the eigenvalues corresponding to eigenvectors orthogonal to the all-one vector of a symmetric matrix M with constant row sum.

Our main result is:

Theorem 2.0: Let d^* be an equireplicate design satisfying

$$e + 2 \leq v \leq 2e + 4 \ , \tag{2.3}$$

$$e \neq 2 \qquad \text{and} \qquad e + 1 \text{ does not divide } v \tag{2.4}$$

and

$$\mu_1(B) \geq -2 \tag{2.5}$$

Then d^* is E-optimal in $\mathscr{D}(b, k, v)$.

The proof of this result depends heavily on the following two results. For the proof of these see Brouwer, Cohen and Neumaier (1989).

Theorem 2.1 (Doob and Cvetkovic (1979)): If G is a regular graph with $\mu_1(G) > -2$, then either G is a disjoint union of complete graphs or of odd cycles.

Theorem 2.2 (Bussemaker, Cvetkovic and Seidel (1976)): G is a connected regular graph with $\mu_1(G) \geq -2$ if and only if G is in one of the following classes:

(1) the line graphs of connected regular graphs,
(2) the line graphs of connected bipartite semiregular graphs,
(3) the cocktail party graphs,
(4) the 187 exceptional graphs listed in Bussemaker et. al (1976).

In view of these results, it is enough to show the following.

Proposition 2.3: If d is a design in $\mathscr{D}(b, k, v)$ with v and e satisfying (2.3) and (2.4) then

$$\mu_1(kC_d) \leq \mu_1(kC_{d^*}) = r(k-1) + \lambda - 1 = \mu_1 \quad \text{(say)} . \tag{2.6}$$

(Here d^* is as in Theorem 2.0.)

But this is already known in the case when d^* is an RGD, and the same proof applies to yield the above. The case of RGD was done in Cheng and Constantine (1986) and in Bagchi and Cheng (1991).

3 The Study of Balanced Rectangular Designs

First we consider the balanced rectangular designs of type 1. From the definition (given in the introduction), it is clear that its B-matrix is as follows:

$$B_1 = \left[\begin{array}{c|c} J_n - I_n & I_n \\ \hline I_n & J_n - I_n \end{array}\right] \tag{3.1}$$

where $n = v/2$.

This B-matrix is nothing but the adjacency matrix of the line graph of $K_{2,n}$ (see section 3 of Bagchi and Cheng (1991)). Anyhow, B_1 has the following spectrum: $(-2)^{n-1}0^{n-1}(n-2)^1 n^1$ (of which the eigenvalue n corresponds to the all-one vector).

It is interesting to observe that the B-matrix (B_2) of a balanced rectangular design of type 2 has the same spectrum as above, although for B_2, the eigenvalue $n-2$ corresponds to the all-one vector.

$$B_2 = \left[\begin{array}{c|c} J_n - I_n & -I_n \\ \hline -I_n & J_n - I_n \end{array}\right]$$

Now, applying Theorem 2.0, we get the following.

Theorem 3.1: A balanced rectangular design of type 2 is E-optimal within $\mathscr{D}(b, k, v)$ whenever $v \geq 10$.

We shall go further and show the following.

Theorem 3.2: A balanced rectangular design of type 2 is strictly E-better than any RGD in $\mathscr{D}(b, k, v)$ for $v \geq 26$, $v = 2 \pmod 4$.

Remark: In section 4, we present infinitely many examples of balanced rectangular design of type 2 satisfying the parametric conditions of Theorem 3.2.

In order to prove the above theorem, it is enough, in view of (2.2), to prove the following.

Proposition 3.3: Suppose G is a regular graph with v vertices, $v \geq 10$, degree $v/2 - 2$ and $\mu_1(G) \geq -2$. Then,

(a) if G is disconnected then there are two possibilities:

  (i) $v = 0 \pmod 4$ and G has two components, each a cocktail party graph on $v/2$ vertices,
  (ii) $v = 10$, one component of G is the complement of a 6-cycle and the other component is K_4;

(b) if G is connected then $v \leq 28$, $v \neq 26$.

Proof: We first consider part (a). Suppose G has m connected components, the size of the ith being v_i, $1 \leq i \leq m$. Then from the hypothesis, $\min_{1 \leq i \leq m} v_i \geq v/2 - 1$. Hence $m = 2$ and there are exactly two possibilities:

(i) $v_1 = v_2 = v/2$.
(ii) $v_1 = v/2 - 1, v_2 = v/2 + 1$.

In case (i) both the components must be a cocktail party graph (the complement of $v/4$ disjoint edges).

In case (ii), the first component (G_1) must be a complete graph, having $\mu_1(G_1) > -2$, and the second component (G_2) is the complement of a graph of degree 2 on $v/2 + 1$ vertices. But then $\bar{G}_2$ (complement of G_2) is a union of cycles, and an n-cycle has eigenvalues $2 \cos \frac{2\pi k}{n}, 0 \le k \le n - 1$. Hence $\mu_1(G_2) \ge -2$ only when $\bar{G}_2$ is a $(v/2 + 1)$ cycle with $v/2 + 1 \le 6$, i.e. $v \le 10$. As $\mu_1(G) = \min(\mu_1(G_1), \mu_1(G_2))$, the result is proved.

To prove part (b) we examine the various classes of graphs listed in Theorem 2.2 to see which ones have degree $v/2 - 2$.

Class (1): G is the line graph $L(G^*)$ of a connected regular graph G^* with v^* vertices and degree e^*. Then G has $v = v^*e^*/2$ vertices and degree $e = 2e^* - 2$. Hence G has $e = v/2 - 2$ if and only if $v^* = 8$, so that $v = 4e^*$, $2 \le e^* \le 7$, and hence $v \le 28$, $v \ne 26$.

Class (2): Let $G = L(G^*)$ where G^* is a connected semiregular bipartite graph. Let the number and degree of vertices of G^* in the i-th part be n_i and e_i respectively ($i = 1, 2$). Now as $L(G^*)$ is regular of degree e and order v, we have $v = n_1e_1 = n_2e_2$ and $e = e_1 + e_2 - 2$. Now $v = 2e + 4$ implies

$$(n_1 - 2)e_1 = 2e_2 \tag{3.2}$$

and

$$(n_2 - 2)e_2 = 2e_1 \ . \tag{3.3}$$

Without loss of generality, $e_1 \ge e_2$, so that (3.2) implies $n_1 \le 4$, and hence $n_1 = 3$ or 4. Now, (3.2) & (3.3) admit the following solutions:-

$$n_1 = 3 \ , \qquad n_2 = 6 \ , \qquad e_1 = 2e_2 \tag{3.4}$$

and

$$n_1 = 4 \ , \qquad n_2 = 4 \ , \qquad e_1 = e_2 \tag{3.5}$$

There are three distinct graphs satisfying (3.4), corresponding to the three admissible values of e_2, viz. 1, 2 and their line graphs have 6, 12 and 18 vertices. The graphs G^* satisfying (3.5) are regular and hence their line graphs have already been considered.

Class (3) of Theorem 2.2 is out since graphs in this class have degree $v - 2 > v/2 - 2$ for all v.

Class (4): All the sporadic graphs of layer 1 in the list of Bussemaker et al (1976) satisfying $e = v/2 - 2$. All these graphs have $v \leq 28$, while none of them has $v = 26$. Hence the proposition is proved.

We find that for large v (viz. for $v > 28$), the only R.G.D. that competes with a balanced rectangular design of type 2 is an RGD whose associated graph is the disjoint union of two cocktail party graphs. So, we study these designs now.

3.4 Definition: Consider $v = 4n$ treatments divided into 2 classes of size $2n$ each. Each class is again divided into n subclasses of size 2 each. A treatment will be called a first associate of the other member of its own subclass, second associate of the remaining members of its own class and third associate of the members of the other class.

This is clearly a three class association scheme with the following parameters.

$$n_1 = 1\ , \qquad n_2 = 2n - 2\ , \qquad n_3 = 2n\ ,$$

$$P^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2n-2 & 0 \\ 0 & 0 & 2n \end{bmatrix}, \qquad P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2n-4 & 0 \\ 0 & 0 & 2n \end{bmatrix},$$

$$P^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2n-2 \\ 1 & 2n-2 & 0 \end{bmatrix}$$

(Here we are using the usual notation for the parameters of an association scheme. The (j, k)th entry of the matrix P^i is p^i_{jk}.)

Consider a PBIBD with the association scheme defined above and $\lambda_1 = \lambda_2 + 1 = \lambda_3$. Clearly such a design d is an RGD whose graph $G(d)$ is the disjoint union of two cocktail party graphs. We call such a design a *Double cocktail party design* (DCPD). By Theorem 2.1 in Bagchi and Cheng (1991), we have.

Theorem 3.4: A Double Cocktail Party Design is E-optimal whenever $v \geq 12$.

Remark: For $v > 28$, $v = 0 \pmod 4$, a DCPD is the unique E-optimal RGD in $\mathscr{D}(b, k, v)$.

The E-optimality of a balanced rectangular design of type 1 is shown in Bagchi and Cheng (1991). Here we note something more.

Theorem 3.5: A balanced Rectangular Design of type 1 is the unique E-optimal RGD in $\mathscr{D}(b, k, v)$, whenever $v > 12$.

Proof: A balanced RD of type 1 is an RGD with the associated graph having degree $v/2$. By an analysis similar to that in Proposition 3.3, we observe that the only regular graphs G of order v and degree $v/2$ having $\mu(G) = -2$ are the following:-

(a) The line graph of the cocktail party graph on six vertices.
(b) 5 sporadic graphs with $v = 12$ and one sporadic graph with $v = 8$. Hence the result.

We shall now consider the A-criterion and observe the performance of balanced rectangular designs of both types.

Let RD1 (RD2) denote a balanced rectangular design of type 1 (respectively of type 2).

Theorem 3.6: If an RD2 and a DCPD co-exist, the latter is always strictly A-better.

Proof: Let $a = r(k-1) + \lambda$ (3.6)

and let $$\varphi(d) = \sum_{i=1}^{v-1} (\mu_i(kC_d))^{-1} . \tag{3.7}$$

Then

$$\varphi(\mathrm{RD2}) = (a+n+1)^{-1} + (n-1)(a+1)^{-1} + (n-1)(a-1)^{-1} \tag{3.8}$$

and

$$\varphi(\mathrm{DCPD}) = (a+n-1)^{-1} + n(a+1)^{-1} + (n-2)(a-1)^{-1} \tag{3.9}$$

where $n = v/2$.

[Note that the spectrum of the adjacency matrix of a double cocktail party graph is $(-2)^{n-2}0^n(n-2)^2$, of which the eigenvalue $n-2$ corresponds to the all-one vector. The spectrum of RD2 is given in the para following 3.1. The rest follows from (2.2)].

Hence $\varphi(\mathrm{RD2}) - \varphi(\mathrm{DCPD}) = 2n(2a+n)(a^2-1)^{-1}\{(a+n)^2-1\}^{-1} > 0$ for all a and n.

Theorem 3.7: Both RD1 and RD2 are highly A-efficient designs, the efficiencies of tending to 1 as v tends to ∞.

Proof: First we consider RD1.

$$\varphi(\mathrm{RD1}) = (a^2-1)^{-1}(a+n-1)^{-1}[(v-1)a^2 + 2(n-1)^2 a - 1]$$

where a is as in (3.6) and $n = v/2$. Further since tr $kC_{\mathrm{RD1}} = (v-1)a + n - 1$, the efficiency of RD1 is given by

$$E_1 = 1 - N_1/D_1 \,, \qquad \text{where}$$

$$N_1 = (v-2)[\{(n-1)^2(v-1)^{-1}+1\}a + v(n-1)(v-1)^{-1} \qquad \text{and}$$

$$D_1 = (v-1)a^3 + (n-1)(v-1)a^2 + \{2(n-1)^3(v-1)^{-1} - 1\}a - (n-1)(v-1)^{-1} \,.$$

Clearly E_1 is increasing in a, so that E_1 is minimum at $a = n-1$.

Thus $E_1 \geq 1 - F_1(n)$, where

$$F_1(n) = \frac{2(n-1)(n^2+2n)}{2(2n-1)^2(N-1)^2 + 2(n-1)^3 - 4n + 1}$$

which is clearly decreasing in n and tends to 0 as n tends to ∞.

Finally we consider small values of v.

$$v = 10\colon \quad E_1 \geq 1 - F_1(5) = 0.70 \,,$$

$$v = 12\colon \quad E_1 \geq 1 - F_1(6) = 0.92 \,.$$

Next we consider RD2. Using (3.8), we get

$$E_2 = 1 - N_2/D_2 \,,$$

where N_2/D_2 is also a decreasing function of a so that $E_2 \geq 1 - F_2(n)$, where $F_2(n)$ is the value of N_2/D_2 at $a = n + 1$. But

$$F_2(n) = \frac{(n-1)5n^2}{(n+1)^2(5n^2 - 6n + 2) - n}$$

is decreasing in n and hence tends to 0 as n tends to ∞.

Finally, for small values of v, we have

$$v = 10: \quad E_2 \geq 0.84 \ ,$$

$$v = 12: \quad E_2 \geq 0.86 \ .$$

In view of Theorem 3.6, we have:-

Corollary 3.9: The A-efficiency of DCPD tends to 1 as v tends to ∞.

Thus with regard to A-criterion also, both RD1 and RD2 perform quite well. Now since the comparison is with a BIBD, which cannot co-exist with these designs, their actual performance is even better! However, we still do not know what the A-optimal design in $\mathscr{D}(b, k, v)$ is, and thus cannot calculate the exact efficiency. Therefore we attempt to have an idea about their actual performances by comparing them with a small class $\mathscr{C} = \mathscr{C}(b, v, k)$ of competing designs, hoping that $\mathscr{C}$ contains an A-optimal design. A reasonable choice for $\mathscr{C}$ is the class of E-optimal RGDs. In view of Theorem 3.2 and Theorem 3.5 our task in greatly reduced, having to consider only the cases $v = 8$ and 12 for RD1 and $10 \leq v \leq 28$, $v \equiv 0 \pmod 2$, for RD2. In order to avoid the existence question, we choose $k = 2$. Further, to simplify matters, we take $\lambda = 0$ (see Notation 2.1), so that $r = v - 1 - e$. Thus the blocks of an RGD d are nothing but the edges of the complement $\bar{G}(d)$ of the graph $G(d)$. [See Remark 2.2 and the comments following (1.2) in Bagchi and Cheng (1990)] Note that this is a connected design, although $G(d)$ may be a disconnected graph.

Our task is further reduced by the painstaking work in Bussemaker et al (1976), where the eigenvalues of the sporadic graphs are listed. They also tell us that certain line graphs are co-spectral with certain sporadic graphs. Thus we had only to find the eigenvalues of the line graphs of two regular graphs with 8 vertices and degree 3.

Observations on RD1: While proving Theorem 3.5 we noted that when $v = 8$, RD1 has one competitor in the E-sense. It is the RGD corresponding to the sporadic graph #185 in Bussemaker et al and it is A-better than RD1. For

$v = 12$, there are five sporadic graphs (#166–170, op. cit.) and one line graph in $\mathscr{C}$ and RD1 is A-better than each of them.

Observations on RD2: We present our findings in Table 3.1 below. There we use the following notation. $\bar{d}$ denotes the RGD in $\mathscr{C}$ with the minimum φ-value and $G(\bar{d})$ is the associated graph. DCP denotes the disjoint union of two equal cocktail party graphs. $K_{m,n}$ is the complete bipartite graph with parts of size m & n. Finally, sporadic graphs are referred by their serial number (#) in Table 9.1 of Bussesmaker et al. (1976).

Table 3.1.

No. of treatments (v)	Degree (e)	φ(RD2)	$\varphi(\bar{d})$	$G(\bar{d})$
10	3	1.4548	1.4148	4
12	4	1.5298	1.50	DCP
14	5	1.5863	1.5710	17, 18, 24, 25, 26, 28.
16	6	1.6306	1.6125	DCP
18	7	1.6662	1.6611	70 & $L(K_{m,n})$
20	8	1.6955	1.6833	DCP
22	9	1.7200	1.7209	$135 \leq \# \leq 147$
24	10	1.7408	1.7320	DCP
28	12	1.7744	1.7679	DCP

Remarks:

1 We have only presented the minimum value of the φ-value of the RGDs. However, we observed that the range of the φ-values for fixed v and e is very small, not more that 0.01, showing that these designs are for all practical purposes equally efficient.
2 Whenever a DCPD exists, it has the minimum φ-value.
3 For $v = 22$, RD2 has smaller φ-value (although the difference is very small) than the best RGD in $\mathscr{C}$.
4 The strongly regular graphs do not have any special status in the present study, in the sense that none of them give rise to an A-optimal design in the restricted class. Note that strongly regular competitors of RD2 do exist. For instance, within the range $10 \leq v \leq 28$ of Table 3.1, we have the triangular graph $T(8)$, the latin square graph $L_2(4)$ as well as the complement of $T(5)$.

The above observations lead us to make the following conjectures.

Conjecture 1: A balanced rectangular design of type 1 is A-optimal when $v \geq 12$.

Conjecture 2: A double cocktail party design is A-optimal whenever $v \geq 12$.

Conjecture 3: A balanced rectangular design of type 2 is A-optimal whenever $v \geq 22$, $v = 2 \bmod 4$.

4 Construction

We provide two difference set methods of construction based on finite fields.

Let s be a prime power and let p be a divisor of $s - 1$. Let F denote the Galois field of order s and F^* denote its multiplicative group. Let H be the subgroup of F^* of order p. In both constructions, $V = \{0, 1\} \times F$ will be the set of treatments. ($\times$ is cartesean product.) For notational simplicity, we write (i, A) in place of $\{i\} \times A$ for $i \in \{0, 1\}$, $A \subseteq F$. In terms of these notations, the base blocks in the two methods of construction are as follows:-

(a) *In the first construction, we use two base blocks* B_1 & B_2 given by $B_1 = (0, H_1) \cup (1, H_2)$ and $B_2 = (0, H_1) \cup (1, H_1)$, where H_1, H_2 are any two *distinct cosets* of H in F^*.

(b) *In the second construction, we use a single base block* B given by $B = B_1 \cup \{(0, 0), (1, 0)\}$ where B_1 is as in the first construction.

To generate all the blocks of our design from the given base blocks, we use the group G of affine transformations of F. That is, G is the group of all $s(s - 1)$ permutations of F of the form $x \mapsto ax + b$ ($a \in F^*$, $b \in F$). This group G acts on the treatment set V as follows:- $g \in G$ sends $(i, x) \in V$ to $(i, g(x)) \in V$. In both constructions, the blocks are the distinct images of the base block(s) under this action of G.

Note that the stabiliser in G of each base block is the subgroup G_0 of order p consisting of the permutations $x \mapsto ax$ ($a \in H$). Therefore each base block has sq images under G, where $q = (s - 1)/p$. Hence the first construction yields a design with $2sq$ blocks and the second yields one with sq blocks.

It is easy to verify that both the constructions yield rectangular designs with two rows (underlying association scheme: the treatments (i, a), (i, b) are first associates for $i = 0, 1$; $(0, a)$ & $(1, a)$ are second – and $(0, a)$ & $(1, b)$ are third associates, where $a \neq b \in F$).

The parameters of the rectangular designs obtained:-

First Construction: $v = 2s, \quad b = 2sq, \quad r = 2pq, \quad k = 2p,$

$$\lambda_1 = 2(p - 1), \quad \lambda_2 = pq, \quad \lambda_3 = 2p - 1.$$

Second Construction: $v = 2s, \quad b = sq, \quad r = pq + q, \quad k = 2p + 2,$
$\lambda_1 = p + 1, \quad \lambda_2 = q \quad \text{and} \quad \lambda_3 = p + 2.$

Special Cases:

a) $q = 2$ (i.e. $s = 2p + 1$) in the first construction. The design is balanced rectangular of type 2.
   Examples: s any odd prime power.
b) $q = p + 2$ (i.e., $s = (p + 1)^2$) in the second construction. The design is a most balanced group divisible design.
   Examples: $s = 9, 16, 25, 49, 64, \ldots$.
c) $q = p + 1$ (i.e., $s = p^2 + p + 1$) in the second construction. The design is a balanced rectangular design of type 1.
   Examples: $s = 3, 7, 13, 31, 43, \ldots$.
d) $q = p + 3$ (i.e., $s = p^2 + 3p + 1$) in the second construction. The design is a balanced rectangular design of type 2.
   Examples: $s = 5, 11, 19, 29, 41, \ldots$.

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