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On Testing for Independence Against Right Tail Increasing in Bivariate Models

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Abstract: A random variable Y is right tail increasing (RTI) in X if the failure rate of the conditional distribution of X given $Y > y$ is uniformly smaller than that of the marginal distribution of X for every $y \geq 0$. This concept of positive dependence is not symmetric in X and Y and is stronger than the notion of positive quadrant dependence. In this paper we consider the problem of testing for independence against the alternative that Y is RTI in X. We propose two distribution-free tests and obtain their limiting null distributions. The proposed tests are compared to Kendall's and Spearman's tests in terms of Pitman asymptotic relative efficiency. We have also conducted a Monte Carlo study to compare the powers of these tests.

Key words and phrases: Kendall's test, Spearman's test, Brownian bridge, bivariate exponential distribution, conditional failure rate, weak convergence.

1 Introduction

When two units (or systems) operate in a common environment they are often exposed to "identical" stress and strain. This may result in some pattern of dependence between them. The life times of the units are said to be positively dependent if long life of one unit is associated with long life of the other.

To formalize our discussion, we let X and Y be random variables denoting the lifelengths of two (possibly dependent) aging systems. Let $H(x, y)$ be the joint distribution function of X and Y and $\bar{H}(x, y) = P\{X > x, Y > y\}$. The marginal distribution function of X (resp. Y) is denoted by $F(x)$ (resp. $G(y)$) and the corresponding marginal survival function is defined as $\bar{F}(x) = 1 - F(x)$ (resp. $\bar{G}(y) = 1 - G(y)$). The survival function, $\bar{H}_y(\cdot)$, of the conditional distribution of X given $Y > y$ is defined by

$$\bar{H}_y(x) = \bar{H}(x, y)/\bar{G}(y) = P\{X > x \mid Y > y\} \ . \tag{1.1}$$

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In a landmark paper, Lehmann (1966) gave several nonparametric notions of positive dependence between random variables in terms of their joint and marginal distributions. The most widely studied of them is the notion of *positive quadrant dependence* (PQD) which is defined below.

Definition 1.1: X and Y are PQD if the following equivalent conditions hold

i) $H(x, y) \geq F(x)G(y)\ \forall(x, y)$

ii) $\bar{H}(x, y) \geq \bar{F}(x)\bar{G}(y)\ \forall(x, y)$

and

iii) $\bar{H}_y(x) \geq \bar{F}(x)\ \forall x$ and $\forall y$, (1.2)

where $\bar{H}_y(\cdot)$ is as in (1.1).

Let X_y be a random variable associated with $H_y(\cdot)$ and let "$\overset{st}{\leq}$" denote the univariate stochastic ordering. By (1.2), X and Y are PQD if and only if

$$X_y \overset{st}{\geq} X\ \forall y \geq 0 .$$

The concept of PQD is symmetric in X and Y. In many practical situations asymmetric type of dependence is observed. In such cases the dependence of Y on X may not be the same as that of X on Y. To express skewed dependence, Esary and Proschan (1972) introduced the concept of *right tail increasing* (RTI) which is defined below.

Definition 1.2: Y is RTI in X if

$$P\{Y > y | X > x\} \text{ is increasing in } x \text{ for all } y \geq 0 ,$$

or equivalently if

$$\bar{H}_y(x)/\bar{F}(x) \text{ is increasing in } x \text{ for all } y \geq 0 . \tag{1.3}$$

By comparing (1.2) and (1.3) we see that if Y is RTI in X, then X and Y are PQD and the converse is not necessarily true. This means that the notion of RTI is stronger than the notion of PQD. However, unlike the notion of PQD, the notion of RTI is not symmetric in X and Y.

In the case when the appropriate densities exist, (1.3) is equivalent to

$$r_1(x|Y > y) \le r_1(x)\ \forall x \text{ and } \forall y \ge 0 \ ,$$

where $r_1(x|Y > y)$ is the conditional hazard rate of X given $Y > y$ and $r_1(x)$ is the hazard rate of the marginal distribution of X.

The Marshall-Olkin bivariate exponential (BVE) distribution is given by

$$\bar{H}(x, y) = \exp\{-\lambda_1 x - \lambda_2 y - \theta \max(x, y)\},\ x, y \ge 0 \ , \tag{1.4}$$

where λ_1, λ_2 and θ are nonnegative parameters. This distribution is not absolutely continuous and has a singular part. It can be shown that if (X, Y) has the BVE distribution of (1.4), then Y is RTI in X.

The absolutely continuous BVE (ACBVE) of Block and Basu (1974) is given by

$$\begin{aligned}\bar{H}(x, y) = {} & \frac{\lambda + \theta}{\lambda} \exp\{-\lambda_1 x - \lambda_2 y - \theta \max(x, y)\} \\ & - \frac{\theta}{\lambda} \exp\{-(\lambda + \theta)\cdot \max(x, y)\},\ x, y \ge 0 \ ,\end{aligned} \tag{1.5}$$

where λ_1, λ_2 and θ are nonnegative parameters and $\lambda = \lambda_1 + \lambda_2$. Assume now (X, Y) has the ACBVE of (1.5). It can be shown that

$$r_1(x|Y > y) = \begin{cases} \left(\dfrac{1}{\lambda_1} - \dfrac{\theta}{\lambda_1(\lambda + \theta)} e^{-\lambda_1(y-x)}\right)^{-1} & \text{for } x < y \\[2ex] (\lambda + \theta)\dfrac{(\lambda_1 + \theta\{1 - e^{-\lambda_2(x-y)}\})}{\lambda + \theta\{1 - e^{-\lambda_2(x-y)}\}} & \text{for } x \ge y \end{cases}$$

which is nonincreasing in y for each x. Hence Y is RTI in X.

In this paper we consider the problem of testing the null hypothesis of independence against the alternative of Y is RTI in X. In Section 2 we propose two test statistics for this problem and derive their asymptotic null distributions. In Section 3 we compare our proposed tests to the tests of Kendall and Spearman in terms of asymptotic relative efficiency. We also conducted a Monte Carlo power comparison of our tests and Spearman's test. The asymptotic theory of the tests of Section 2 is proved in Section 4.

2 The Proposed Tests

Consider the problem of testing the null hypothesis

$$H_o\colon X \text{ and } Y \text{ are independent} \ , \tag{2.1}$$

against the alternative

$$H_1\colon Y \text{ is RTI in } X \ . \tag{2.2}$$

As seen in Section 1, the above problem is equivalent to the problem of testing the null hypothesis

$$H_o\colon H_y(\cdot) = F(\cdot)\ \forall y \geq 0 \tag{2.3}$$

against

$$H_1'\colon \bar{H}_y(x)/\bar{F}(x) \text{ is increasing in } x \text{ for each } y \geq 0 \ . \tag{2.4}$$

By (1.3), H_1, is also equivalent to

$$H_1^*\colon r_1(x|Y > y) \leq r_1(x) \text{ for all } x, y \geq 0 \ . \tag{2.5}$$

Assume, for the moment that $y > 0$ is fixed. The problem of testing

$$H_{o,y}\colon H_y(\cdot) = F(\cdot) \tag{2.6}$$

against

$$H_{1,y}\colon r_1(\cdot|Y > y) \leq r_1(\cdot) \tag{2.7}$$

is like the two-sample problem of testing the equality of two hazard rates (or two DF's) against ordered alternatives. Tests for the latter two-sample problem have been propsed by Kochar (1979, 1981), Joe and Proschan (1984) and Aly (1988), among others. Loosely speaking, the problem of testing H_o of (2.1) (or (2.3)) against H_1 of (2.2) (equivalently against H_1' of (2.4) or H_1^* of (2.5)) is like "testing $H_{o,y}$ of (2.6) against $H_{1,y}$ of (2.7)" for each y. This remark motivated us to propose tests for H_o against H_1 which are based on a family of two-sample tests each corresponging to a fixed value y.

As seen in Joe and Proschan (1984) and Aly (1988) $H_1(H_1'$ or $H_1^*)$ holds if and only if

$$\Delta^*(t, p, y) := p + \bar{p}H_yF^{-1}(t) - H_yF^{-1}(p + \bar{p}t) \geq 0 \ , \tag{2.8}$$

for all $y \geq 0, 0 \leq t, p \leq 1$ with strict inequality for some (t, p, y), where $\bar{p} = 1 - p$.

Define $\Delta(t, p, s) = (1 - s)\Delta^*(t, p, G^{-1}(s))$ and note that (2.8) is equivalent to

$$\Delta(t, p, s) \geq 0 \text{ for all } 0 \leq t, p, s \leq 1$$

with strict inequality for some (t, p, s).

By (1.1), it can be shown that

$$\Delta(t, p, s) = H(F^{-1}(p + \bar{p}t), G^{-1}(s)) - \bar{p}H(F^{-1}(t), G^{-1}(s)) - ps, 0 \leq t, p, s \leq 1 \ . \tag{2.9}$$

Define

$$\begin{aligned}\delta(s) &= \int_0^1\int_0^1 \Delta(t, p, s)dtdp \\ &= -\frac{s}{2} - \int_0^1 \left\{\frac{1}{2} + \ln(1 - u)\right\} H(F^{-1}(u), G^{-1}(s))du \ .\end{aligned} \tag{2.10}$$

Note that $\delta(s) \equiv 0$ under H_o and $\delta(s) \geq 0$ under H_1. Consequently, measures of the deviation from H_o in favor of H_1 can be defined as appropriate functionals of $\delta(\cdot)$. The tests proposed in this article are based on the following two measures

$$K = \sup_{0 \leq s \leq 1} \delta(s) \tag{2.11}$$

and

$$A = \int_0^1 \delta(s)ds + \frac{1}{4} \ . \tag{2.12}$$

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a random sample from $H(\cdot, \cdot)$. The empirical distribution functions $H_n(\cdot, \cdot)$, $F_n(\cdot)$ and $G_n(\cdot)$ are defined by

$$H_n(x, y) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x, Y_i \le y)$$

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x)$$

and

$$G_n(y) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i \le y) ,$$

where $I(A)$ is the indicator function of the event A. Let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ (resp. $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$) be the order statistics corresponding to $X_1, X_2, \ldots, X_n$ (resp. $Y_1, Y_2, \ldots, Y_n$). Let $Y_{[1]}, Y_{[2]}, \ldots, Y_{[n]}$ be the concomitant ordered Y's which are obtained by ordering the pairs $\{(X_i, Y_i), 1 \le i \le n\}$ based on the X variable only.

A natural estimator of $\delta(\cdot)$ of (2.10) is given by

$$\delta_n(s) = -\frac{s}{2} - \int_0^1 \left\{\frac{1}{2} + \ln(1-u)\right\} H_n(F_n^{-1}(u), G_n^{-1}(s))du .$$

It can be proved that

$$\delta_n(s) \simeq \frac{1}{n}\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)\left\{\frac{1}{2} - \ln\left(1-\frac{j}{n}\right)\right\} I(Y_{[j]} \le G_n^{-1}(s)) - \frac{s}{2} . \tag{2.13}$$

Based on $\delta_n(\cdot)$ of (2.13), K of (2.11) and A of (2.12) we propose the following test statistics,

$$K_n = \max\left\{0, \max_{1\le \ell \le n}\left(\frac{1}{n}\sum_{j=1}^{n-1} a_j I(Y_{[j]} \le Y_{(\ell)}) - \frac{\ell}{2n}\right)\right\} ,$$

and

$$A_n = \int_0^1 \delta_n(s)ds + \frac{1}{4}$$

$$= \frac{1}{n}\sum_{j=1}^{n-1} a_j\left(1 - \frac{S_j}{n}\right) ,$$

where $S_j = \mathrm{Rank}(Y_{[j]}) = nG_n(Y_{[j]})$ and

$$a_j = \left(1 - \frac{j}{n}\right)\left\{\frac{1}{2} - \ln\left(1 - \frac{j}{n}\right)\right\}, j = 1, 2, \ldots, n-1 .$$

Large values of K_n and A_n are significant for testing H_0 of (2.1) against H_1 of (2.2). The asymptotic null distributions of K_n and A_n are consequences of the following Theorem which is proved in Section 4.

Theorem 2.1: Assume that $H(\cdot, \cdot)$ is continuous and that H_o holds. Then,

$$\sqrt{54n}\delta_n(s) \xrightarrow{\mathscr{D}} B(s) , \tag{2.14}$$

where $B(\cdot)$ is a Brownian Bridge.

Corollary 2.1: Under the conditions of Theorem 2.1, we have

$$\sqrt{54n}K_n \xrightarrow{\mathscr{D}} \sup_{0 \le s \le 1} B(s)$$

and

$$\sqrt{648n}\{A_n - \tfrac{1}{4}\} \xrightarrow{\mathscr{D}} N(0, 1) . \tag{2.15}$$

It is well known that

$$P\left\{\sup_{0 \le s \le 1} B(s) \ge x\right\} = e^{-2x^2}, x \ge 0 .$$

Consequently, using the K_n statistic, we reject H_o in favor of H_1 at approximate level α if $K_n > \left\{-\dfrac{\ln \alpha}{108n}\right\}^{1/2}$. A Monte Carlo study indicated that the convergence in (2.15) is faster when A_n is centered around its exact null mean. It is easy to see that, under H_o, $E(A_n) = \dfrac{n-1}{2n^2}\sum_{j=1}^{n-1} a_j$. Consequently, using the A_n statistic, we reject H_o in favor of H_1 at approximate level α if

$$\frac{\sqrt{162}}{n^{3/2}}\sum_{j=1}^{n-1} a_j(n + 1 - 2S_j) > z_{1-\alpha} ,$$

where $z_{1-\alpha}$ is the $(1-\alpha)^{th}$ quantile of a $N(0, 1)$ rv (i.e., $P\{N(0, 1) \le z_{1-\alpha}\} = 1-\alpha$).

It can be shown that the above two testing procedures are consistent for testing independence against alternatives in H_1.

3 Asymptotic Relative Efficiencies and Power Comparisons

In this section we compare the K_n and A_n tests with the Spearman's rank test statistic

$$\mathscr{S}_n = 1 - 6 \sum_{i=1}^{n} (i - S_i)^2/n(n^2-1) .$$

It is well known that (see for example, Weier and Basu (1980)) the Pitman asymptotic relative efficiency (ARE) of $\mathscr{S}_n$ with respect to the Kendall's τ statistic is equal to one. Recall that $\sqrt{n}(\mathscr{S}_n - E(\mathscr{S}_n)) \xrightarrow{\mathscr{D}}$ a mean zero normal random variable. Under H_o, $E(\mathscr{S}_n) = 0$ and the variance of the limiting normal rv is 1.

First, we compare A_n to $\mathscr{S}_n$ in terms of Pitman's ARE using the following distributions:

a) $\bar{H}_1(x, y) = \bar{F}(x)\bar{G}(y) + \theta\bar{F}(x)\{1 - \bar{F}(x)(1 - \ln \bar{F}(x))\} \cdot \bar{G}(y)G(y), \quad 0 < \theta < 1$

and

b) $\bar{H}_2(x, y)$ as the ACBVE distribution of (1.5) .

Note that both $H_1(\cdot, \cdot)$ and $H_2(\cdot, \cdot)$ belong to H_1. The distribution $H_1(\cdot, \cdot)$ is a Lehmann type alternative in the sense that the power of any rank test against $H_1(\cdot, \cdot)$ is independent of $F(\cdot)$ and $G(\cdot)$.

The computation of the Pitman ARE is straightforward but rather quite lengthy and involved. For this reason we will give here the final results and refer the reader to Puri and Sen (1971) for more details. The Pitman ARE of A_n w.r.t. $\mathscr{S}_n$ for $H_1(\cdot, \cdot)$ is equal to 2. In fact, by the results of Shirahata (1974), it can be shown that the A_n test is locally most powerful rank test for testing independence $(\theta = 0)$ against $\theta > 0$ for the alternative $H_1(\cdot, \cdot)$.

The Pitman ARE of A_n w.r.t. $\mathscr{S}_n$ for the alternative $H_2(\cdot, \cdot)$ is given by

$$e_{\lambda_1,\lambda_2}(A_n, \mathscr{S}_n) = e_{\lambda_1,\lambda_2}(A_n)/e_{\lambda_1,\lambda_2}(\mathscr{S}_n) , \tag{3.1}$$

where $e_{\lambda_1,\lambda_2}(A_n) = 648\{a_1 + a_2 + a_3 + a_4\}^2$, $e_{\lambda_1,\lambda_2}(\mathscr{S}_n) = 9\{b_1 + b_2 + b_3\}^2$,

$$a_1 = \frac{1}{2}\left\{\frac{1}{2\lambda} - \frac{3}{4\lambda_1} - \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda(\lambda + \lambda_1)}\right\},$$

$$a_2 = \frac{5}{16\lambda} - \frac{1}{16\lambda_1} - \frac{3}{16\lambda_2} + \frac{\lambda_1}{2\lambda(\lambda + \lambda_1)} - \frac{\lambda_2}{2\lambda(\lambda + \lambda_2)} - \frac{\lambda_1\lambda_2}{4\lambda^2(\lambda + \lambda_1)}$$

$$- \frac{\lambda_1\lambda_2}{4\lambda^2(\lambda + \lambda_2)} - \frac{\lambda_1^2}{4\lambda^3} + \frac{\lambda_1^2}{16\lambda_2\lambda^2} + \frac{\lambda_2\lambda_1^2}{4\lambda^3(\lambda + \lambda_2)},$$

$$a_3 = \frac{1}{2}\left\{-\frac{1}{\lambda} + \frac{7}{8\lambda_1} + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{2(\lambda + \lambda_1)^2}\right\},$$

$$a_4 = \frac{1}{2}\left\{\frac{1}{2\lambda} + \frac{1}{4\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}\right\},$$

$$b_1 = -\frac{5}{8\lambda} + \frac{1}{8\lambda_1\lambda_2} + \frac{\lambda_1\lambda_2}{2\lambda(\lambda + \lambda_1)(\lambda + \lambda_2)} + \frac{\lambda}{(\lambda + \lambda_1)(\lambda + \lambda_2)},$$

$$b_2 = \frac{1}{2}\left\{\frac{1}{\lambda + \lambda_1} - \frac{1}{4\lambda_1} - \frac{1}{2\lambda}\right\},$$

and $b_3 = -a_4$.

Note that in the case $\lambda_1 = \lambda_2 = \lambda_o$, $e_{\lambda_o,\lambda_o}(A_n, \mathscr{S}_n) = 0.7812$ independent of the value of λ_o. In Table 1 below we give $e_{\lambda_1,\lambda_2}(A_n, \mathscr{S}_n)$ of (3.1) for selected values of λ_1 and λ_2. Observe that $e_{\lambda_1,\lambda_2}(A_n, \mathscr{S}_n)$ is not symmetric in λ_1 and λ_2.

Table 1 shows that for fixed λ_2, $e_{\lambda_1,\lambda_2}(A_n, \mathscr{S}_n)$ increases in λ_1 and eventually stabilizes around 1.12. On the other hand, for fixed λ_1, $e_{\lambda_1,\lambda_2}(A_n, \mathscr{S}_n)$ tends to zero as λ_2 increases.

The asymptotic distribution of the K_n statistic is not normal. For this reason its performance can not be compared to other tests in terms of Pitman ARE. We conducted a Monte Carlo simulation study to compare the powers of $\mathscr{S}_n$, A_n and

Table 1. ARE of A_n w.r.t. $\mathscr{S}_n$ for the ACBVE distribution of (1.5)

λ_1	$e_{\lambda_1,0.1}(A_n, \mathscr{S}_n)$	λ_2	$e_{0.1,\lambda_2}(A_n, \mathscr{S}_n)$
0.1	0.7812	0.1	0.7812
1	0.9408	1	0.07409
10	1.1108	10	0.00098
20	1.1131	20	0.00024
30	1.1151	30	0.00010
40	1.1165	40	0.00007

Table 2. Monte Carlo Estimates of Powers for the Marshall-Olkin Bivariate Exponential Distribution with $\lambda_2 = 0.1$, $\lambda_1 = 10, 20(20)100$ and $\theta = 0.2\lambda_1$

	λ_1	S_n	A_n	K_n
	10	.1630	.1215	.1230
	20	.1680	.1140	.1125
$n = 10$	40	.1565	.1285	.1185
	60	.1525	.1195	.1090
	80	.1550	.1230	.1085
	100	.1635	.1170	.1140
	10	.2250	.1835	.1845
	20	.2340	.1925	.1885
$n = 20$	40	.2170	.1825	.1755
	60	.2315	.1705	.1690
	80	.2300	.1845	.1720
	100	.2305	.1785	.1820
	10	.4245	.4500	.4345
	20	.4300	.4430	.4360
$n = 50$	40	.4520	.4675	.4705
	60	.4420	.4655	.4525
	80	.4435	.4735	.4690
	100	.4280	.4570	.4490
	10	.7945	.7290	.8565
	20	.7345	.7880	.8645
$n = 100$	40	.7620	.8005	.8680
	60	.7385	.8010	.8750
	80	.7295	.7910	.8670
	100	.7505	.8105	.8745

K_n. In this study we employed 2,000 independent random samples of sizes 10, 20, 50 and 100 from the BVE distribution of Marshall and Olkin of (1.4). The significance level used in this study is $\alpha = 0.05$ and the critical values used were obtained by simulation. Part of this study is reported in Table 2 above in which $\lambda_2 = 0.1$, $\lambda_1 = 10, 20(20)100$ and $\theta = 0.2\lambda_1$.

Table 2 suggests that for small samples, $\mathscr{S}_n$ performs better than both A_n and K_n. For large samples, K_n is distinctly much better than both $\mathscr{S}_n$ and A_n. For moderate samples ($n \simeq 50$), both A_n and K_n are slightly more powerful than $\mathscr{S}_n$.

In addition to the power results discussed above we have also considered the case $\lambda_2 = 0.1$, $\lambda_1 = 10, 20(20)100$ and $\theta = 0.1\lambda_1$. These results, which are not reported here, show that the powers of the three tests are more or less the same, but are significantly lower than their corresponding values of Table 2.

4 Asymptotic Theory

Let the empirical distribution functions $H_n(\cdot, \cdot)$, $F_n(\cdot)$ and $G_n(\cdot)$ be as defined following (2.12). Define

$$L(t, s) = H(F^{-1}(t), G^{-1}(s)) \ ,$$

$$L_n(t, s) = H_n((F^{-1}(t), G^{-1}(s)) \ ,$$

$$\alpha_n(t, s) = n^{1/2}\{L_n(t, s) - L(t, s)\} \ ,$$

$$U_n(y) = FF_n^{-1}(y), u_n(y) = n^{1/2}(U_n(y) - y) \ ,$$

$$V_n(y) = GG_n^{-1}(y), v_n(y) = n^{1/2}(V_n(y) - y) \ ,$$

and

$$\gamma_n(t, p, s) = n^{1/2}\{\Delta_n(t, p, s) - \Delta(t, p, s)\} \ ,$$

where $\Delta(t, p, s)$ is as in (2.9) and

$$\Delta_n(t, p, s) = H_n(F_n^{-1}(p + \bar{p}t), G_n^{-1}(s)) - \bar{p}H_n(F_n^{-1}(t), G_n^{-1}(s)) - ps,$$
$$0 \leq t, p, s \leq 1 \ .$$

Next, we define two Gaussian processes which will be needed in the sequel (cf. Csörgö (1984) for more details). A Brownian bridge $B(\cdot, \cdot)$ on $[0, 1] \times [0, 1]$ is a real valued mean zero separable Gaussian process with continuous sample paths and $EB(x_1, y_1)B(x_2, y_2) = (x_1 \wedge x_2)(y_1 \wedge y_2) - x_1x_2y_1y_2$, $0 \leq x_1, x_2, y_1, y_2 \leq 1$. A Brownian Bridge $B(\cdot)$ on $[0, 1]$ is a real valued mean zero separable Gaussian process with continuous sample paths and $EB(x_1)B(x_2) = (x_1 \wedge x_2) - x_1x_2$, $0 \leq x_1, x_2 \leq 1$. Note that

$$B(x, 1) \stackrel{\mathscr{D}}{=} B(1, x) \stackrel{\mathscr{D}}{=} B(x), 0 \leq x \leq 1 \ .$$

By the Theorem of Tusnády (1977), there exists a sequence of Brownian bridges $\{B_n(\cdot, \cdot)\}_{n=1}^{\infty}$ such that under H_o,

$$\sup_{0 \leq t, s \leq 1} |\alpha_n(t, s) - B_n(t, s)| \stackrel{\text{a.s.}}{=} O(n^{-1/2} \log^2 n) \ . \tag{4.1}$$

Define,

$$\alpha_{1n}(t) = n^{1/2}(F_nF^{-1}(t) - t)$$

and

$$\alpha_{2n}(s) = n^{1/2}(G_n G^{-1}(s) - s) \ .$$

It follows from (4.1),

$$\sup_{0 \le t \le 1} |\alpha_{1n}(t) - B_n(t, 1)| \overset{\text{a.s.}}{=} O(n^{-1/2} \log^2 n) \tag{4.2}$$

and

$$\sup_{0 \le s \le 1} |\alpha_{2n}(s) - B_n(1, s)| \overset{\text{a.s.}}{=} O(n^{-1/2} \log^2 n) \ . \tag{4.3}$$

By applying the Bahadur-Kiefer result (Bahadur (1966) and Kiefer (1970)) and by (4.2) and (4.3) we obtain

$$\sup_{0 \le t \le 1} |u_n(t) + B_n(t, 1)| \overset{\text{a.s.}}{=} O(r(n)) \ , \tag{4.4}$$

and

$$\sup_{0 \le s \le 1} |v_n(s) + B_n(1, s)| \overset{\text{a.s.}}{=} O(r(n)) \ , \tag{4.5}$$

where $r(n) = n^{-1/4}(\log^{1/2} n)(\log \log n)^{1/4}$.

The following Theorem is the main result of this section.

Theorem 4.1: Assume that H_o holds true and $H(\cdot, \cdot)$ is continuous. Then, there exists a sequence of Brownian bridges $\{B_n(\cdot, \cdot)\}_{n=1}^{\infty}$ such that

$$\sup_{0 \le t, p, s \le 1} |\gamma_n(t, p, s) - \Gamma(t, p, s; B_n)| \overset{P}{=} o(1) \ , \tag{4.6}$$

where

$$\Gamma(t, p, s; B) = \Gamma_1(p + \bar{p}t, s; B) - \bar{p}\Gamma_1(t, s; B)$$

and

$$\Gamma_1(t, s; B) = B(t, s) - sB(t, 1) - tB(1, s) \ . \tag{4.7}$$

Proof: It is easy to see that

$$\gamma_n(t, p, s) = \gamma_{1n}(p + \bar{p}t, s) - \bar{p}\gamma_{1n}(t, s) \ , \tag{4.8}$$

where

$$\gamma_{1n}(t, s) = \alpha_n(U_n(t), V_n(s)) + n^{1/2}\{L(U_n(t), V_n(s)) - L(t, s)\} \ .$$

Consequently, (4.6) will follow from (4.8) if we show under the conditions of Theorem 4.1 that

$$\sup_{0 \le t, s \le 1} |\gamma_{1n}(t, s) - \Gamma_1(t, s; B_n)| \stackrel{P}{=} o(1) \ , \tag{4.9}$$

where $\Gamma_1(t, s; B)$ is as in (4.7).

Assume the conditions of Theorem 4.1. To prove (4.9) we note first that

$$\gamma_{1n}(t, s) = \alpha_n(U_n(t), V_n(s)) + su_n(t) + tv_n(s) + n^{-1/2}u_n(t)v_n(s) \tag{4.10}$$

It is well known that

$$\sup_{0 \le t \le 1} |U_n(t) - t| \stackrel{\text{a.s.}}{=} O(n^{-1/2}(\log\log n)^{1/2}) \tag{4.11}$$

and

$$\sup_{0 \le s \le 1} |V_n(s) - s| \stackrel{\text{a.s.}}{=} O(n^{-1/2}(\log\log n)^{1/2}) \ . \tag{4.12}$$

By (4.10)–(4.12) we obtain

$$\sup_{0 \le t, s \le 1} |\gamma_{1n}(t, s) - \alpha_n(U_n(t), V_n(s)) - su_n(t) - tv_n(s)| \stackrel{\text{a.s.}}{=} O(n^{-1/2} \log\log n) \ . \tag{4.13}$$

Let $\{B_n(\cdot, \cdot)\}_{n=1}^{\infty}$ be as in (4.1). By (4.1), we obtain

$$\sup_{0 \le t, s \le 1} |\alpha_n(U_n(t), V_n(s)) - B_n(U_n(t), V_n(s))| \stackrel{\text{a.s.}}{=} O(n^{-1/2} \log^2 n) \ .$$

By (4.11) and (4.12) and the almost sure continuity of $B_n(\cdot,\cdot)$ for each n, we obtain

$$\sup_{0\le t,s\le 1} |\alpha_n(U_n(t), V_n(s)) - B_n(t,s)| \overset{P}{=} o(1) . \tag{4.14}$$

By (4.4), (4.5), (4.13) and (4.14) we get (4.9). This completes the proof of Theorem 4.1.

Proof of Theorem 2.1: Assume the conditions of Theorem 4.1. Recall that

$$\delta_n(s) = \int_0^1\int_0^1 \Delta_n(t,p,s)dtdp$$

and, under H_0,

$$\sqrt{54n}\delta_n(s) = \sqrt{54}\int_0^1\int_0^1 \gamma_n(t,p,s)dtdp .$$

Consequently, by (4.6), we have

$$\sqrt{54n}\delta_n(s) \overset{\mathscr{D}}{\to} \sqrt{54}\int_0^1\int_0^1 \Gamma(t,p,s;B)dtdp , \tag{4.15}$$

where $\Gamma(t,p,s;B)$ is as in (4.6) and $B(\cdot,\cdot)$ is a Brownian bridge.
It can be shown that

$$\begin{aligned}\sqrt{54}\int_0^1\int_0^1 \Gamma(t,p,s;B)dtdp &= -\sqrt{54}\Bigg(\frac{1}{2}\mathbf{B}(1,s)\\ &\quad + \int_0^1\left\{\frac{1}{2} + \log(1-t)\right\}\{B(t,s) - sB(t,1)\}dt\Bigg)\\ &\overset{\mathscr{D}}{=} \mathbf{B}(s) ,\end{aligned}$$

where $\mathbf{B}(\cdot)$ is a Brownian bridge. This result combined with (4.15) implies (2.14).

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